First order reset elements and the Clegg integrator revisited

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Abstract— We revisit a class of reset control systems containing first order reset elements (FORE) and Clegg integrators and propose a new class of models for these systems. The proposed model generalizes the models available in the literature and we illustrate, using the Clegg integrator, that it is more appropriate for describing the behavior of reset systems. Then, we state computable sufficient conditions for $\mathcal{L}_2$ stability of the new class of models. Our results are based on LMIs and they exploit quadratic and piecewise quadratic Lyapunov functions. Finally, a result on stabilization of linear minimum phase systems with relative degree one using high gain FOREs is stated. We present two examples to illustrate our results. In particular, we show that for some systems a FORE can achieve lower $\mathcal{L}_2$ gain than the underlying linear controller without resets.

I. INTRODUCTION

Reset controllers were proposed for the first time by Clegg in 1958 [8] with the aim of providing more flexibility in linear controller designs and of potentially removing fundamental performance limitations of linear controllers. The first systematic designs for reset controllers were reported in the 1970's [17], [13] and there has been a renewed interest in this class of systems in the late 1990's [5], [1], [3], [4], [6], [7], [10], [12], [11], [14], [19].

A reset controller is a linear controller whose output is reset to zero whenever its input and output satisfy an appropriate algebraic relationship. For instance, in [4] and the references cited therein a class of reset controllers was considered where the output of the controller is reset to zero whenever its input is equal to zero. The Clegg integrator introduced in [8] acts like a linear integrator whenever its input and output have the same sign and it resets its output to zero otherwise (see Section II). Consequently, its describing function has the same magnitude plot as the linear integrator but it has $51.9°$ less phase lag than the linear integrator. This feature of the Clegg integrator was used for the first time in [17] to provide a systematic procedure for controller design exploiting this device. Subsequently, a new reset device called the *first order reset element (FORE)* was introduced in [13] and a controller design procedure based on FOREs was proposed. The design procedure was based on linear frequency domain techniques for robust

stabilization. These early results on reset control systems are summarized in a recent paper [6].

The first example that clearly illustrated advantages of reset over linear controllers was presented in [2] where a reset controller was designed to achieve design specifications that are impossible to achieve by any linear controller. First attempts to rigorously analyze stability of reset systems with Clegg integrators can be found in [14], [12]. In particular, an integral quadratic constraint was proposed in [12] to analyze stability of a class of reset systems. However, the proposed condition was conservative as it was independent of reset times. Stability analysis of reset system consisting of a second order plant and a FORE was conducted in [7]. However, the same approach could not be used to analyze higher order reset systems. Stability analysis of general reset systems can be found in [4] (see also [11]) where Lyapunov based conditions for asymptotic stability of general reset systems were presented. Moreover, the authors proposed computable conditions for quadratic stability based on linear matrix inequalities (LMIs). Bounded-input bounded-state stability of general reset systems was obtained as a consequence of quadratic stability. Finally, an internal model principle was proved for tracking of and disturbance rejection.

Recently, in [18] we have presented Lyapunov like conditions for $\mathcal{L}_2$ stability and exponential stability of general reset systems. Our results apply to a more general class of models than those considered in the literature (see [4]); in particular, resets can occur on more complicated sets than those considered in [4]. Using the Clegg integrator we show that this class of models is indeed natural to consider when analyzing reset systems (see Section III). Moreover, our conditions involve locally Lipschitz Lyapunov functions as opposed to continuously differentiable ones considered in [4]. In particular, this allows us to consider piecewise quadratic Lyapunov functions in verifying exponential or $\mathcal{L}_2$ stability of reset systems. We also prove a new result that shows that any minimum phase relative degree one SISO linear plant can be $\mathcal{L}_2$ stabilized using a high gain FORE. Using examples we show that piecewise quadratic functions allow to analyze systems with unstable FOREs and, in general, corresponding to exponentially unstable linear closed-loops (these situations cannot be effectively addressed by using quadratic Lyapunov functions).

The paper is organized as follows. In Section II we discuss the physical model of the Clegg integrator and in Section III we comment on our hybrid notation, inspired by [9]. In Section IV we establish properties of SISO reset systems via quadratic and piecewise quadratic Lyapunov functions and state a high gain result for closed-loop systems with FOREs. Finally, in Section V we illustrate the proposed methodology on two examples. All the statements will be reported without proofs due to space constraints.

Notation: We use the notation $(x, y) = [x^T \ y^T]^T$. Given a

state variable x of a system with jumps, throughout the paper we will denote its derivative with respect to time (which is defined almost everywhere) by $\dot{x}$. Moreover, at jump times, we will denote the value of the state after the jump by x^+ and the value of the state before the jump simply by x. See also Section III-A for more details.

II. A RESET MODEL OF THE CLEGG INTEGRATOR

In 1958, J.C. Clegg published a paper [8] where he proposed a modification to the existing analog control schemes to reduce the phase lag induced by a linear integrator. When using infinite gain operational amplifiers, it is well known that the linear integrator can be implemented using a resistor on the input path and a capacitor on the feedback path of the circuit. The corresponding input/output relation of the linear integrator can be written in the time domain as $\dot{x} = -\frac{1}{RC}v$.
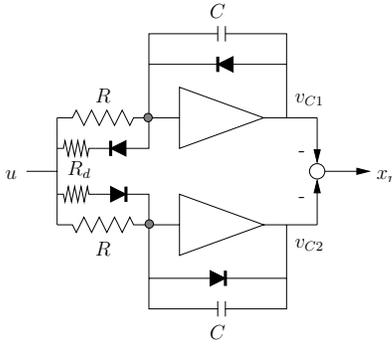

Fig. 1. The "Clegg integrator".

The modification proposed by Clegg corresponds to the scheme of Figure 1 (which is reported here from [8] with a sign inversion at the output, for convenience of exposition). The Clegg integrator dynamics can be described as follows. First note that by the infinite gain assumption of the operational amplifier, the input voltages (marked by gray dots on the figure) are always zero. Then, the two capacitors' voltages satisfy $v_{C1}(t) \leq 0$ and $v_{C2}(t) \geq 0$ for all times (otherwise the infinite current flowing in the diodes would instantaneously discharge the capacitor). Moreover, when $v(t) < 0$, regardless of the preceding voltage stored in the upper capacitor, the current flowing in the two diodes and through the upper R_d will (almost) instantaneously impose $v_{C1}(t) = 0$. However, when $v(t) \geq 0$, the upper circuit will correspond to the linear integrator because the diodes will both be open (being subjected to a non-positive voltage). Similarly for the lower circuit, if $v(t) > 0$, we will have $v_{C2}(t) = 0$ and if $v(t) \leq 0$ the circuit will integrate. Since, as commented above, $v_{C1}(t) \leq 0$ and $v_{C2}(t) \geq 0$ for all times, given $x_r(t) := -v_{C1}(t) - v_{C2}(t)$, the integrating and reset conditions for both circuits can be written as

$$\begin{cases} \dot{x}_r = \dfrac{1}{RC}v, & \text{if } x_r v \geq 0, \\ x_r^+ = 0, & \text{if } x_r v \leq 0. \end{cases} \quad (1)$$

Remark 1: Further insight on equation (1) can be gained by observing that v and x_r can never have opposite signs. Indeed, if $v > 0$, then $v_{C2}^+ = 0$, and since $v_{C1} \leq 0$ for all times, $v x_r \geq 0$. Similarly for the case $v < 0$. On the other hand, whenever $v \neq 0$, there will always be one circuit

integrating (the upper one if $v > 0$ and the lower one if $v < 0$) and the other circuit at zero. $\circ$

III. MODELING ISSUES FOR THE CLEGG INTEGRATOR

In this section we will comment on how the model (1) should be interpreted and represented to guarantee important properties of the arising reset dynamical system.

A. A hybrid representation for solutions

The results given in this paper are based on the theoretical results reported in [18]. Before even addressing the problem of characterizing useful properties of the solutions to reset linear systems, it is mandatory to adopt a mathematical representation for such solutions which is capable of guaranteeing fundamental properties, such as existence of solutions for all times and robustness of the stability results. A first step in this direction corresponds to suitably defining the domain of the solution for the reset system. In particular, the domain of the solution x_r to the Clegg integrator of Figure 1 will be taken to be the Cartesian product of two domains: one related to the elapsed time and the other one related to the number of jumps that occurred since the initial time. This type of mathematical framework for hybrid systems was introduced in [9] and used in [18] to prove the main results that this paper appeals to.

According to the notion of solutions commented above, the Clegg integrator dynamics (1) can be written as

$$\begin{cases} \dot{x}_r(t,j) = \dfrac{1}{RC}v(t), & \text{if } x_r(t,j)v(t) \geq 0, \\ x_r(t_{j+1}, j+1) = 0, & \text{if } x_r(t_{j+1},j)v(t_{j+1}) \leq 0, \end{cases} \quad (2)$$

where $x_r(t,j)$ represents the state of the Clegg integrator at time t, assuming that j resets have occurred before time t. Note that with this notation, if t_{j+1} represents the time at which the j-th reset occurred, then $x_r(t_{j+1}, j)$ represents the integrator value before the jump and $x_r(t_{j+1}, j+1)$ represents the integrator value after the jump.
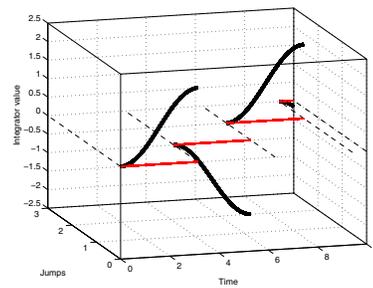
Fig. 2. Response to a sinusoidal input in a hybrid domain.

As an example, Figure 2 represents the Clegg integrator state response with this notion of solution when the input v is selected as a sine wave with unit frequency. The dashed horizontal lines represent the reset times t_j and the red horizontal lines are the hybrid domain of the solution $x_r(t,j)$. Throughout the rest of the paper we will always use the shorthand notation $\dot{x}$ for $\dot{x}(t,j)$ and x^+ for $x(t_{j+1}, j+1)$.

B. Requiring closed flow and jump sets

One way to understand the hybrid model (2) for the Clegg integrator is to call its first equation the "flow" equation and its second equation the "jump" equation. The two conditions at the right hand side become then the "flow" condition and the "jump" condition. At any time t, a solution of the hybrid system (2) will then flow or jump depending on whether its value at that time belongs to the so-called "jump set" (namely, the set of states for which the jump condition is true) or it belongs to the "flow set" or even both. Let's consider the simple example of a Clegg integrator connected to an integrating plant:

$$v = r - y, \quad \dot{y} = x_r + d, \tag{3}$$

where r and d ar reference and disturbance inputs. For this feedback interconnection, the flow condition corresponds to $x_r y \leq 0$ and the jump condition corresponds to $x_r y \geq 0$. In particular, when looking at the (y, x_r) plane as a representation for the closed-loop state, the jump set corresponds to the first and third quadrants and the flow set corresponds to the second and fourth quadrants. Note that the vertical and horizontal axes belong to both the jump and the flow sets.

According to the notation in [18], we denote the flow set by $\mathcal{C}$ and the jump set by $\mathcal{D}$. For the system (3), we can use the following representation for these two sets:

$$
\begin{aligned}
\mathcal{C} &:= \left\{ (y, x_r) : \begin{bmatrix} x_r \\ y \end{bmatrix}^T M \begin{bmatrix} x_r \\ y \end{bmatrix} \geq 0 \right\}, \\
\mathcal{D} &:= \left\{ (y, x_r) : \begin{bmatrix} x_r \\ y \end{bmatrix}^T M \begin{bmatrix} x_r \\ y \end{bmatrix} \leq 0 \right\},
\end{aligned}
\tag{4}
$$

where $M := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. It is important to emphasize that both the jump and flow sets defined in (4) are closed. This condition is necessary for the theoretical developments in [9] to apply. The results in [9] (which are also used in [18]) allow us to establish robust asymptotic stability results for the closed-loop between linear plants and reset controlling elements. Therefore we will assume from now on that the sets $\mathcal{C}$ and $\mathcal{D}$ are closed.

Remark 2: (Uniqueness of solutions) Asking that the sets $\mathcal{C}$ and $\mathcal{D}$ be closed implies that there are regions of the state space belonging to both sets. Therefore, solutions may jump or flow in these regions, so that the solutions to the arising reset linear systems may be non unique. Non uniqueness becomes a necessary notion when wanting to establish robust results for the reset system, as a matter of fact, when the feedback system is affected by (arbitrarily small) noise, the state could be pushed in several different directions and different solutions may correspond to different noise selections. These and other robustness issues are addressed and solved in the hybrid framework that we adopt here and in the stability results that we will rely on in this paper. ∘

It should be emphasized that several recent papers (see, e.g., [11], [4] and references therein) have addressed the problem of feedback systems with Clegg integrators and, more generally, feedback systems with First Order Reset Elements (FORE). In that work, the Clegg integrator dynamics is captured by a different set of equations, namely

$$
\begin{cases}
\dot{x}_r = \dfrac{1}{RC} v, & \text{if } v \neq 0, \\
x_r^+ = 0, & \text{if } v = 0.
\end{cases}
\tag{5}
$$

This representation is however prone to problems and we will not adopt it here for several reasons:

1. The model (5) does not represent the behavior of the Clegg integrator circuit for all initial conditions. Assume that $x_r < 0$ and v is a constant positive signal. Since the input and output of the circuit in Figure 1 can never have opposite signs (see Remark 1), the actual circuit will reset x_r instantaneously to zero. However, the model (5) will never reset x_r to zero because $v \neq 0$ for all times and the response will be flowing for all times. On the other hand, the model (1) will jump instantaneously to zero and generate one solution that flows for all positive times. This solution corresponds to the actual response of the circuit.

2. In (5), the flow set $\mathcal{C}$ corresponds to almost all the state space so that the closure of $\mathcal{C}$ coincides with the whole state space. It is therefore possible to generate arbitrarily small noise that will prevent the state from jumping on all trajectories. This lack of robustness is solved in (1) where closed jump and flow sets are employed (see [9] for more details on robustness concepts in the hybrid domains). Note that the lack of robustness of the model (5) can be even experienced in simulation. For example, when implementing the closed-loop (3) in Simulink using (5), the Clegg integrator state is never reset to zero due to the time discretization performed by the simulator.

3. The model (5) will impose a flow condition in regions of the state space where the Clegg integrator circuit would jump. Consequently, for establishing useful stability results, the model (5) would require to check suitable inequalities also in regions of the state space that do not require it, whereas our representation (1) wouldn't.

C. Imposing temporal regularization

One of the problems accurately modeled in [9] corresponds to the so-called "Zeno solutions" where the state jumps infinitely many times in a bounded time interval (so that the corresponding hybrid domain is unbounded in the jump direction thereby remaining bounded in the flow direction). While this peculiar situation is well modeled mathematically in [9], it is not reasonable to implement a control system that could generate solutions of this kind because unmodeled effects would come into place so that the actual solution exhibits an unpredictable behavior. To avoid the presence of Zeno solutions, reset control systems are augmented here with an extra jump rule which imposes that after any jump a pre-designed time interval ρ has to expire before a subsequent jump is allowed. This type of rule has been used, e.g., in [16] and was therein called "temporal regularization". The same idea was also used in the context of reset systems in [7, §2]. We will therefore use that notation also here. As an example, the Clegg integrator augmented with temporal regularization corresponds to generalizing equation (2) as follows:

$$
\left.
\begin{aligned}
\dot{\tau} &= 1, \\
\dot{x}_r &= \frac{1}{RC} v
\end{aligned}
\right\}
\quad \text{if } x_r v \geq 0 \text{ or } \tau \leq \rho,
\tag{6a}
$$

$$
\left.
\begin{aligned}
\tau^+ &= 0, \\
x_r^+ &= 0
\end{aligned}
\right\}
\quad \text{if } x_r v \leq 0 \text{ and } \tau \geq \rho,
\tag{6b}
$$

The stability and performance results that we will report next will all refer to reset elements which are modified as

in (6) where ρ is a sufficiently small positive number. Note that even simulation packages like Simulink will not be able to compute the solution forward in time if the simulation scheme is not implemented with some kind of temporal regularization.

IV. HYBRID MODELING AND PROPERTIES OF SISO RESET SYSTEMS

A. Models for FORE control loops

Consider a strictly proper SISO linear plant whose dynamics is described by

$$\mathcal{P} \begin{cases} \dot{x}_p = A_p x_p + B_{pu} u + B_{pd} d, \\ y = C_p x_p, \end{cases} \tag{7}$$

where u is the control input, d is a disturbance input and y is the measured plant output (A_p, B_{pu}, B_{pd} and C_p are matrices of appropriate dimensions). For the plant (7), assume that a FORE control system is designed as follows:

$$\text{FORE} \begin{cases} \dot{x}_r = \lambda_r x_r + e, & \text{if } ex_r \geq 0 \\ x_r^+ = 0, & \text{if } ex_r \leq 0, \end{cases} \tag{8}$$

$$\text{Interconnection} \begin{cases} u = kx_r, \\ e = r - y \end{cases} \tag{9}$$

where $r \in \mathbb{R}$ is a reference signal. Moreover, k denotes the loop gain and $\lambda_r \in \mathbb{R}$ denotes the time constant of the FORE. Note that λ_r can be any number (including positive ones) while k should be positive. For example, choosing $k = 1$ and $\lambda_r = 0$ corresponds to implementing in the FORE the Clegg integrator commented in Section II.

The overall closed-loop system augmented with the temporal regularization can then be described by the following equations:

$$\begin{cases} \dot{\tau} = 1, \\ \dot{x} = Ax + B_d d + B_r r, & \text{if } x^T M x \geq 0 \text{ or } \tau \leq \rho, \\ \tau^+ = 0, \\ x^+ = A_r x, & \text{if } x^T M x \leq 0 \text{ and } \tau \geq \rho, \\ y = Cx \end{cases} \tag{10}$$

where A denotes the flow matrix, A_r denotes the reset matrix and M characterizes the flow and the jump sets (note that these two sets have their boundaries in common). Based on the values in (7), (8) and (9), the matrices in (10) are

$$A = \begin{bmatrix} A_p & B_{pu}k \\ -C_p & \lambda_r \end{bmatrix}, B_d = \begin{bmatrix} B_{pd} \\ 0 \end{bmatrix}, B_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$A_r = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, M = \begin{bmatrix} 0 & -C_p^T \\ -C_p & 0 \end{bmatrix}, \tag{11}$$
$$C = \begin{bmatrix} C_p & 0 \end{bmatrix}.$$

B. Lyapunov-based results

In [18] Lyapunov-based results for a general class of reset systems have been given. These results allow us to establish exponential stability of the closed-loop and $\mathcal{L}_2$ performance properties. The main result of [18] can be written as follows for the special reset control system in (10), when only focusing on second order homogeneous Lyapunov functions.

Theorem 1: [18] Consider the reset control system (10) with the matrix selection (11) and pick any arbitrarily small number $\epsilon > 0$. Select either $w = r$ and $B_w = B_r$ or $w = d$ and $B_w = B_d$. Assume that there exists a locally Lipschitz

function $V(x) := x^T P(x) x$, and strictly positive constants a_1, a_2, γ and ε, such that
1) $a_1 |x|^2 \leq V(x) \leq a_2 |x|^2$ for all $x \in \mathbb{R}^n$,
2) $P(\lambda x) = P(x) = P^T(x) > 0$ for all $x \in \mathbb{R}^n$ and for all $\lambda \in \mathbb{R}$,
3) $\frac{\partial V(x)}{\partial x}(Ax + B_w w) + \varepsilon |x|^2 + \frac{1}{\gamma}|y|^2 - \gamma |w|^2 < 0$, for almost all x such that $x^T(M + \epsilon I)x \geq 0$,
4) $V(A_r x) - V(x) \leq 0$ for all x such that $x^T M x \leq 0$.

Then there exists a small enough $\rho^* > 0$ such that for any fixed $\rho \in (0, \rho^*)$, the FORE control system (10) is exponentially stable and has a finite $\mathcal{L}_2$ gain from w to y which is smaller than γ.

Remark 3: The condition at item 2 corresponds to requiring that the Lyapunov function is homogeneous of degree two. The condition at item 3 corresponds to requiring that in a set that is slightly larger than the flow set the Lyapunov function is a disturbance attenuation Lyapunov function for the input w and the output y. The condition at item 4 corresponds to requiring that the Lyapunov function does not increase along resets. As compared to the main result in [18], Theorem 1 does not explicitly require that immediately after the resets the closed-loop state belongs to the flow set. Indeed, since resets will always drive the FORE state to zero, the state after reset will necessary belong to the flow set (by the structure of M) and no extra requirement is needed on the resetting strategy. ○

C. LMI formulations of the Lyapunov results

The result given in Theorem 1 can lead to a convex formulation if the homogeneous of degree two Lyapunov function is restricted to a smaller class of functions. We concentrate next on Lyapunov functions that are quadratic or piecewise quadratic and show how the conditions of Theorem 1 transform into convex constraints formulated as linear matrix inequalities (LMIs).

Theorem 2: (Quadratic Lyapunov conditions) Consider the reset control system (10) with the matrix selection (11). Select either $w = r$ and $B_w = B_r$ or $w = d$ and $B_w = B_d$.

If the following linear matrix inequalities in the variables $P = P^T > 0$, τ_F, $\tau_R \geq 0$, $\gamma > 0$ are feasible:

$$\begin{bmatrix} A^T P + PA + \tau_F M & PB_w & C^T \\ \star & -\gamma I & 0 \\ \star & \star & -\gamma I \end{bmatrix} < 0, \tag{12}$$
$$A_r^T P A_r - P - \tau_R M \leq 0,$$

Then there exists a small enough $\rho^* > 0$ such that for any fixed $\rho \in (0, \rho^*)$, the FORE control system (10) is exponentially stable and has a finite $\mathcal{L}_2$ gain from w to y which is smaller than γ.

The LMI results in Theorem 2 have severe limitations mainly arising from the conservativeness associated with the use of quadratic Lyapunov functions. Indeed, it is possible to show that the conditions (12) are never feasible if the FORE element is not exponentially stable. The most natural relaxation of the quadratic conditions consists in piecewise quadratic conditions where several quadratic functions are selected in different sectors of the state space and patched together to form a unique piecewise quadratic function. In general, piecewise quadratic relaxations of convex quadratic conditions lead to non convex formulations that don't correspond to LMIs. However, in our special case, it is

possible to write convex conditions by exploiting the special structure of the flow and jump sets. The corresponding result is reported next.

Theorem 3: (Piecewise quadratic Lyapunov conditions) Consider the reset control system (10) with the matrix selection (11). Assume (without loss of generality) that the plant (7) is in observability canonical form (so that $C_p = [\,0 \; \cdots \; 0 \; 1\,]$). Select either $w = r$ and $B_w = B_r$ or $w = d$ and $B_w = B_d$. Choose any $N \geq 2$ and define θ_i, $i = 0, \ldots, N$ such that $0 = \theta_0 < \theta_1 < \cdots < \theta_N = \dfrac{\pi}{2}$ (for example, in our case studies we select $\theta_i = \frac{i}{N}\frac{\pi}{2}$). Define the angle vectors $\Theta_i \in \mathbb{R}^n$ as

$$\Theta_i = [\; 0_{1 \times n-2} \quad \sin(\theta_i) \quad \cos(\theta_i) \;]^T, \quad i = 0, \ldots, N,$$

and their orthogonal matrices $\Theta_{i\perp}$ (so that $\Theta_{i\perp}^T \Theta_i = 0$) as

$$\Theta_{i\perp} := \begin{bmatrix} I & 0 & 0 \\ 0 & \cos(\theta_i) & -\sin(\theta_i) \end{bmatrix}^T, \quad i = 0, \ldots, N.$$

Define also the sector matrices $S_i = S_i^T \in \mathbb{R}^{n \times n}$ as

$$\begin{aligned} S_0 &:= \Theta_0 \Theta_N^T + \Theta_N \Theta_0^T \\ S_i &:= -(\Theta_i \Theta_{i-1}^T + \Theta_{i-1} \Theta_i^T), \quad i = 1, \ldots, N. \end{aligned}$$

If the following linear matrix inequalities in the variables $P_i = P_i^T > 0$, $\tau_{Fi} \geq 0$, $i = 1, \ldots, N$, $\hat{P} = \hat{P}^T > 0$, $\tau_J \geq 0$, $\gamma > 0$ are feasible:

$$\begin{bmatrix} A^T P_i + P_i A + \tau_{Fi} S_i & P_i B_w & C^T \\ \star & -\gamma I & 0 \\ \star & \star & -\gamma I \end{bmatrix} < 0, \quad (13a)$$
$$i = 1, \ldots, N,$$

$$A_r^T \hat{P} A_r - \hat{P} + \tau_J S_0 \leq 0 \tag{13b}$$
$$\Theta_{i\perp}^T (P_i - P_{i+1}) \Theta_{i\perp} = 0, \quad i = 1, \ldots, N-1, \tag{13c}$$
$$\Theta_{0\perp}^T (P_1 - \hat{P}) \Theta_{0\perp} = 0 \tag{13d}$$
$$\Theta_{N\perp}^T (P_N - \hat{P}) \Theta_{N\perp} = 0 \tag{13e}$$

then there exists a small enough $\rho^* > 0$ such that for any fixed $\rho \in (0, \rho^*)$, the FORE control system (10) is exponentially stable and has a finite $\mathcal{L}_2$ gain from w to y which is smaller than γ.

Remark 4: An interpretation of the LMIs (13) will be useful. The piecewise quadratic Lyapunov function arising from Theorem 3 is obtained by patching together N quadratic functions (characterized by the matrices $P_1, \ldots, P_N$) defined in the flow set and one quadratic function (characterized by the matrix $\hat{P}$) in the jump set. According to this selection, conditions (13a) ensure that the proposed Lyapunov function is a disturbance attenuation Lyapunov function in the flow set and condition (13b) ensures that the Lyapunov function is not increasing upon jumps within the jump set. The remaining conditions ensure that the patched Lyapunov function is continuous, namely conditions (13c) ensure that it is continuous on the patching surfaces inside the flow set and conditions (13d), (13e) ensure that it is continuous on the patching surfaces at the boundary of the flow and the jump sets. ∘

Remark 5: For implementation purposes, it is useful to point out that the LMI constraints (13) can be implemented with a good level of approximation by only using strict linear matrix inequalities. In particular, once a very small

tolerance ε has been fixed, the non strict LMI (13b) can be replaced by the strict LMI $A_r^T \hat{P} A_r - \hat{P} + \tau_J S_0 < \varepsilon I$, and the equality constraints (13c) can be replaced by the LMIs

$$\begin{bmatrix} -\varepsilon I & \Theta_{i\perp}^T (P_i - P_{i+1}) \Theta_{i\perp} \\ \star & -\varepsilon I \end{bmatrix} < 0,$$

(and similarly for (13d) and (13e)). The arising solutions will satisfy the LMIs (13) up to a very small tolerance (related to the selection of ε).

Note that given any solution to (13), all the variables can be scaled by an arbitrary constant while preserving its feasibility. It is therefore useful to also impose $P_i > I$, $i = 1, \ldots, N$ (without loss of generality) to obtain reasonable values in the entries of the matrices P_i. ∘

D. A high gain stability result

In this section, using a simple Lyapunov construction for a FORE controlling an integrator (which is a planar system) and a small gain result, we establish that any SISO minimum phase plant with relative degree one can be stabilized using an appropriate high gain FORE. Indeed, note that if the plant is minimum phase and relative degree one, there exists a nonsingular change of coordinates so that we can write the model (7) as follows [15, Remark 4.3.1]:

$$\begin{aligned} \dot{z} &= Az + By + B_d d \\ \dot{y} &= Cy + Dz + Eu + E_d d, \end{aligned}$$

where $y \in \mathbb{R}$ and $u \in \mathbb{R}$ are respectively the plant output and input and A is Hurwitz. Then the following holds:

Theorem 4: Consider the closed loop system consisting of (14), (14), (8) and (9), with $r \equiv 0$ and A is Hurwitz. Then, there exists $k^* > 0$ such that for all $k \geq k^*$, the closed loop system is exponentially stable and $\mathcal{L}_2$ stable from d to y.

V. EXAMPLES

In this section we will show how the LMI formulations given in Section IV-C can be used to establish useful stability and performance properties of several reset control systems involving FORE or Clegg integrators. We first address the most classical example, already introduced in Section III-B of a Clegg integrator connected to an integrator, and then address the case of a FORE connected to an integrator.

Example 1: (A Clegg integrator controlling an integrator plant) One of the simplest reset systems considered in the literature corresponds to the control system where a Clegg integrator is connected in feedback with an integrating plant. Studying the stability of this simple closed-loop by Lyapunov tools is already a challenging task to accomplish which was addressed and solved recently in [12], [14]. The equations of the closed-loop system can be written as

$$\left\{ \begin{array}{ll} \left. \begin{array}{l} \dot{y} = x_r + d \\ \dot{x}_r = r - y, \end{array} \right\} & \text{if } x_r(r - y) \geq 0, \\ x_r^+ = 0, & \text{if } x_r(r - y) \leq 0. \end{array} \right. \tag{14}$$

In this section, not only do we establish stability of the closed-loop system but we also give a tight estimate on the $\mathcal{L}_2$ gain of the system from the input d to the output y, by employing the LMI-based techniques of Section IV-C. In light of the stability properties established, e.g., in
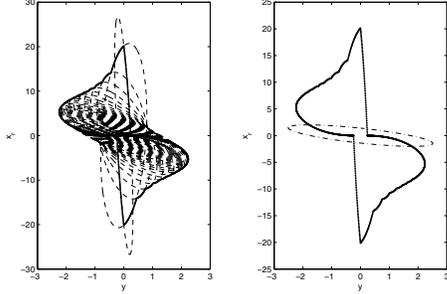
Fig. 3. Example 2. Left: level sets of the 51 quadratic Lyapunov functions used for the case $N = 50$. Right: level set of the arising piecewise quadratic Lyapunov function (bold) and of the quadratic Lyapunov function from Theorem 2.

[12], [14], we may expect to get an estimate of the $\mathcal{L}_2$ gain from d to y using the quadratic Lyapunov functions proposed in Theorem 2. However, the LMI constraints therein proposed turn out to be non feasible for this particular problem. Indeed, even for such a simple closed-loop system, a piecewise quadratic Lyapunov function is necessary to obtain an estimate of the $\mathcal{L}_2$ gain. When using the LMIs of Theorem 3, it is necessary to use at least $N = 2$ to prove the closed-loop exponential stability. Moreover, as N increases, tighter and tighter bounds are obtained for the $\mathcal{L}_2$ gain of the system. The following table reports some of the values obtained by increasing the number of regions.

N	2	3	4	8	15	50
gain	2.834	1.819	1.377	0.914	0.884	0.87

○

Example 2: (A FORE controlling an integrator) It has been discussed in several papers (see, e.g., [11], [2]) that reset control systems can overcome certain limitations of linear control systems. In particular, for an integrating plant, it is shown in [11] that the reset controller can achieve arbitrarily large rising time of the closed-loop while guaranteeing zero overshoot. This is proved to be non achievable for linear control systems (see [11] for details). In [11, §9.2.1], a FORE with time constant $\tau = 1$ connected in unit negative feedback to an integrator is used to illustrate this fact. According to our notation, the closed-loop system is described by the following dynamics

$$
\left\{
\begin{array}{l}
\dot{y} = x_r + d \\
\dot{x}_r = \dfrac{1}{\tau}(r - y - x_r), \\
x_r^+ = 0,
\end{array}
\right\}
\quad
\begin{array}{l}
\text{if } x_r(r - y) \geq 0, \\
\\
\text{if } x_r(r - y) \leq 0.
\end{array}
\quad (15)
$$

For this example, it is of interest to compare the $\mathcal{L}_2$ gain of the reset control system to the $\mathcal{L}_2$ gain characterizing the closed-loop without resets. In particular, the $\mathcal{L}_2$ gain of the linear closed-loop is 1.468 and the estimate arising from the quadratic Lyapunov construction of Theorem 2, corresponding to 1.84, is not good enough to show that the $\mathcal{L}_2$ gain of the reset control system is improved as compared to the linear case. However, determining a bound using the piecewise quadratic quadratic construction of Theorem 3 leads to a less conservative estimate of 1.18 (determined using 51 quadratic Lyapunov functions), which is able to predict the improved performance of the reset control system.

Figure 3 shows on the left the level sets of the quadratic

functions involved in the piecewise quadratic construction, and on the right, a level set of the patched piecewise quadratic Lyapunov function (bold) compared to the level set of the quadratic Lyapunov function establishing the 1.84 gain estimate. Note that for this example the optimal piecewise quadratic Lyapunov function is nonconvex again.

It is useful to mention that it not necessary for the FORE element in (15) to be exponentially stable. Indeed, it is possible to show that any real selection of τ enforces closed-loop stability. For example, picking $\tau = -1$ in (15), we obtain a reset closed-loop which would be exponentially unstable without resets. On the other hand, the reset closed-loop is exponentially stable, even though closed-loop exponential stability cannot be established by the quadratic construction of Theorem 2 in this case. However, quite surprisingly, the $\mathcal{L}_2$ gain estimate established by the piecewise quadratic construction of Theorem 3 corresponds to 0.71 and is highly improved with respect to the case with $\tau = 1$ (see [18] for further insight on the use of unstable FOREs). ○

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