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Results on convergence in hybrid systems via detectability and an invariance principle*

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Abstract— **Two invariance principles for generalized hybrid systems are presented. One version involves the use of a nonincreasing function, like in the original work of LaSalle. The other version involves "meagreness" conditions. These principles characterize asymptotic convergence of bounded hybrid trajectories to weakly invariant sets. A detectability property is used to locate a set in which the Ω-limit set of a trajectory is contained. Next, it is shown how the invariance principles can be used to certify asymptotic stability in hybrid systems. Lyapunov and Krasovskii theorems for hybrid systems are included.**

I. Introduction

Hybrid systems theory has been a very active research field in the recent decades due to the large number of technological advances that require mathematical models allowing interactions between discrete and continuous dynamics. Having state trajectories that can evolve continuously (flow) and/or discretely (jump), hybrid systems permit modeling and simulation of complex systems in a wide range of applications. Several different hybrid solution concepts and models have appeared in the literature. See, for example, the work of Tavernini [23], Michel and Hu [18], Lygeros et. al. [16], [17], Aubin et. al. [1], and van der Schaft and Schumacher [25]. The recent work by Goebel and Teel [8] and by Goebel et. al. in [7] (related to the concurrent results by Collins [6]), propose a novel concept of solution and a generalized hybrid system model.

One of the most important tools for convergence analysis in dynamical systems is the invariance principle presented by LaSalle [12], [13]. In his work, LaSalle proposed an invariance principle for differential and difference equations as an extension of Lyapunov's theorem for convergence of bounded solutions. Byrnes and Martin [4] presented a version stating that bounded solutions converge to the largest invariant set contained in the set of points where an integrable output function is zero. A subsequent result by Ryan [20] extends this integral invariance principle to differential inclusions. A recent result by Logeman and Ryan [15] extends LaSalle's invariance principle for differential inclusions using the notion of meagre functions, alongside a generalization of Barbalat's Lemma. For systems with discontinuous right-hand side, invariance principles based on LaSalle's principle were presented by Shevitz and Paden [22] and Bacciotti and Ceragioli [2] for solutions in the sense of Filippov, and by Bacciotti and Ceragioli [3] for Caratheodory solutions. For hybrid systems, in [17], Lygeros et al. present an extension of LaSalle's invariance principle for nonblocking, deterministic, and continuous hybrid systems. In [9], Hespanha proposes an invariance principle for switched linear systems under a specific family of switching signals. Results therein are further extended to a class of nonlinear systems in [10].

In this paper, we consider generalized hybrid systems given by sets of hybrid trajectories. These generalized hybrid systems can be specialized to the ones considered in [7] and [8] (see also [6].) Our purpose is to provide sufficient conditions for convergence of bounded hybrid trajectories. For this, we propose two invariance principles for general hybrid systems that generalize the principles established in [17],[9], and [10], and resemble the original one formulated by LaSalle. The first invariance principle requires a nonincreasing function and involves conditions on all trajectories that remain in a given set. The other invariance principle relaxes the assumptions, considering a pair of auxiliary functions satisfying a meagre-limsup condition only for a single hybrid trajectory. These conditions appear to be the weakest previously used in invariance principles for continuous and discrete time systems.

We also invoke observability and detectability for convergence, and we relate this approach to the invariance principles. When coupled with stability, our convergence results give new sufficient conditions for asymptotic stability in generalized hybrid systems. Special cases include versions of Lyapunov's basic theorem and Krasovskii's extension [11] for hybrid systems.

For our results, the key property of the general hybrid systems is "upper semicontinuity" of solutions. Using the tools of set-valued analysis and graphical convergence, it is shown in [8] that, under mild conditions, any sequence of solutions to a hybrid system has a convergent subsequence, the limit of which is still a solution. Such a property is present not just for the set of all solutions of hybrid systems but also for various special classes of solutions.

For the formal proofs and examples of the results we refer the reader to the journal version of this paper [21].

II. Sets of hybrid trajectories

Throughout this paper we work with hybrid systems given as sets of hybrid trajectories satisfying certain properties. In most applications, but not all, the set of hybrid trajectories corresponds to all solutions of certain generator equations (or inclusions.) We start by defining hybrid trajectories and their domains.

We write $\mathbb{R}_{\geq 0}$ for $[0, +\infty)$ and $\mathbb{N}_{\geq 0}$ for $\{0, 1, 2, ...\}$.

Definition 2.1 (hybrid time domain): A subset $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_{\geq 0}$ is a compact hybrid time domain if

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$$\mathcal{D} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \le t_1 \le t_2 ... \le t_J$. It is a hybrid time domain if for all $(T, J) \in \mathcal{D}$, $\mathcal{D} \cap ([0, T] \times \{0, 1, ...J\})$ is a compact hybrid domain.

Equivalently, $\mathcal{D}$ is a hybrid time domain if $\mathcal{D}$ is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the "last" interval possibly of the form $[t_j, T)$ with T finite or $T = +\infty$.

Definition 2.2 (hybrid trajectory): A hybrid trajectory is a pair $(x, \operatorname{dom} x)$ consisting of a hybrid time domain $\operatorname{dom} x$ and a function x defined on $\operatorname{dom} x$ that is continuous in t on $\operatorname{dom} x \cap (\mathbb{R}_{\ge 0} \times \{j\})$ for each $j \in \mathbb{N}_{\ge 0}$.

We will usually not mention $\operatorname{dom} x$ explicitly, and understand that with each hybrid trajectory comes a hybrid time domain $\operatorname{dom} x$. Alternatively, one could think of a hybrid trajectory as a set-valued mapping from $\mathbb{R}_{\ge 0} \times \mathbb{N}_{\ge 0}$ whose domain is a hybrid time domain. In what follows, $\operatorname{rge} x$ will denote the range of the hybrid trajectory x, i.e. $\operatorname{rge} x = x(\operatorname{dom} x)$.

A hybrid trajectory x is called *nontrivial* if $\operatorname{dom} x$ contains at least one point different from $(0, 0)$, *complete* if $\operatorname{dom} x$ is unbounded, and *Zeno* if it is complete but the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\ge 0}$ is bounded. Given a set of hybrid trajectories $\mathcal{S}$, a trajectory $x \in \mathcal{S}$ is called *maximal* (with respect to $\mathcal{S}$) if there does not exist $x' \in \mathcal{S}$ such that x is a truncation of x' to some proper subset of $\operatorname{dom} x'$. $\mathcal{S}(x^0)$ denotes the set of all hybrid trajectories in $\mathcal{S}$ with $x(0, 0) = x^0$. We will restrict the state space in which the trajectories can evolve to an open set O. We will call a sequence $\{x_i\}_{i=1}^{\infty}$ of hybrid trajectories *locally eventually bounded* with respect to O if for any $m > 0$, there exists $i_0 > 0$ and a compact set $\mathcal{K} \subset O$ such that for all $i > i_0$, all $(t, j) \in \operatorname{dom} x_i$ with $t + j < m$, $x_i(t, j) \in \mathcal{K}$. Finally, x is *precompact* if it is complete and $\overline{\operatorname{rge} x} \subset O$ is compact.

In what follows, we will rely on a nonclassical notion of convergence, namely convergence in the graphical sense. A sequence of (set-valued) mappings $\{M_i\}_{i=1}^{\infty}$ converges graphically to a mapping M if the graphs $\operatorname{gph} M_i$ converge to $\operatorname{gph} M$ as sets (for a mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, the graph $\operatorname{gph} M$ is $\{(a, b) \in \mathbb{R}^m \times \mathbb{R}^n \ : \ b \in M(a)\}$.) For details on set convergence, see Chapter 3 in [19]. Below, we specialize the concept of graphical convergence to hybrid trajectories.

We note that a general property of set convergence implies that from any locally eventually bounded sequence of elements of $\mathcal{S}$, a graphically convergent subsequence can be picked. A sequence of hybrid trajectories $\{x_i\}_{i=1}^{\infty}$ converges graphically to a hybrid trajectory x if

(a) for any $(t, j) \in \operatorname{dom} x$ there exists a sequence $(t_i, j_i) \in \operatorname{dom} x_i$ such that $\lim_{i \to \infty} x_i(t_i, j_i) = x(t, j)$,
(b) for any convergent sequence $(t_i, j_i) \in \operatorname{dom} x_i$ such that $\lim_{i \to \infty} x_i(t_i, j_i)$ exists, the limit equals $x(t, j)$ where $(t, j) = \lim_{i \to \infty} (t_i, j_i)$.

A given sequence of hybrid trajectories does not need to converge graphically, and even when it does, the limit does not need to be a hybrid trajectory itself. Throughout this paper we will consider sets of hybrid trajectories $\mathcal{S}$ for which each locally eventually bounded sequence of trajectories that converges graphically has a limit that belongs to $\mathcal{S}$. Below, we make this assumption explicit.

Assumption 2.3 (Basic Assumption): The set $\mathcal{S}$ of hybrid trajectories and the open set $O \subset \mathbb{R}^n$ satisfy

(B1) $\operatorname{rge} x \subset O$ for all $x \in \mathcal{S}$,
(B2) for any $x \in \mathcal{S}$ and any $(\bar{t}, \bar{j}) \in \operatorname{dom} x$ we have $\bar{x} \in \mathcal{S}$, where $\bar{x}(t, j) = x(t + \bar{t}, j + \bar{j})$ for all $(t, j) \in \operatorname{dom} \bar{x}$,
(B3) for any locally eventually bounded (with respect to O) sequence $\{x_i\}_{i=1}^{\infty}$ of elements of $\mathcal{S}$ that converges graphically, the limit is an element of $\mathcal{S}$.

The set of solutions, denoted $\mathcal{S}_{\mathcal{H}}$, of the hybrid system $\mathcal{H}$ with state space O given by

$$\begin{array}{rcll} \dot{x} & \in & F(x) & x \in C \\ x^+ & \in & G(x) & x \in D \end{array} \qquad (1)$$

satisfy the Basic Assumption when, using the solution concept of [8], [7] or [6], the data (F, G, C, D) satisfies the assumptions therein. Also, the following subsets of $\mathcal{S}_{\mathcal{H}}$ satisfy the Basic Assumption.

(a) For any $\delta \ge 0$, the set of all $x \in \mathcal{S}_{\mathcal{H}}$ for which $t_{j+1} - t_j \ge \delta$ for all $j = 1, 2, ...$ (the times t_j come from Definition 2.1.), i.e., the set of all solutions for which jumps are separated by at least δ.
(b) For any closed set $S \subset \mathbb{R}_{\ge 0} \times \mathbb{N}_{\ge 0}$, the set of all $x \in \mathcal{S}_{\mathcal{H}}$ such that $\operatorname{dom} x \subset S$. Special cases here include: $S = \{0\} \times \mathbb{N}_{\ge 0}$ corresponding to all solutions to the difference inclusion $x^+ \in G(x)$, $S = \mathbb{R}_{\ge 0} \times \{0\}$ corresponding to all solutions to the differential inclusion $\dot{x} \in F(x)$, and $S = \mathbb{R}_{\ge 0} \times \{0, 1, ..J\}$ corresponding to all solutions with at most J jumps.
(c) For any continuous function $V : O \to \mathbb{R}$, and any fixed $j \in \mathbb{N}_{\ge 0}$, the set of all $x \in \mathcal{S}_{\mathcal{H}}$ such that, if $(t, j-1), (t, j) \in \operatorname{dom} x$ then $V(x(t, j)) \le V(x(t, j-1))$. (The set of all x such that, if x has a j-th jump, then V does not increase during that jump.)

In contrast, the following subsets of $\mathcal{S}_{\mathcal{H}}$ in general may not meet the Basic Assumption.

(d) The set of all x for which $t_{j+1} - t_j > 0$. Indeed, a sequence of such solutions can converge graphically to an instantaneous Zeno solution; consider, for example, the bouncing ball. While this system satisfies the Basic Assumption (see [7]), the subset of its trajectories such that the times between jumps are positive does not. This is because the graphical limit of such solutions includes the instantaneous Zeno solution from zero height. The latter is not an element of the subset.
(e) The set of all x that has exactly J jumps (or at least J jumps.) Indeed, it is easy to construct a system having a graphically convergent sequence $\{x_i\}_{i=1}^{\infty}$ of solutions such that all J jumps occur at time i. The graphical limit will have no jumps.

III. Weak invariance and Ω-limit sets

In this section we define invariance for the set of hybrid trajectories $\mathcal{S}$ and we extend results from differential/difference equations.

Definition 3.1 (weak invariance): The set $\mathcal{M} \subset O$ is

(a) *weakly forward invariant* (with respect to $\mathcal{S}$) if for each $x^0 \in \mathcal{M}$, there exists at least one complete trajectory $x \in \mathcal{S}(x^0)$ with $x(t,j) \in \mathcal{M}$ for all $(t,j) \in \operatorname{dom} x$;
(b) *weakly backward invariant* (with respect to $\mathcal{S}$) if for each $q \in \mathcal{M}$, $N > 0$, there exist $x^0 \in \mathcal{M}$ and at least one trajectory $x \in \mathcal{S}(x^0)$ such that for some $(t^*, j^*) \in \operatorname{dom} x$, $t^* + j^* \geq N$, we have $x(t^*, j^*) = q$ and $x(t,j) \in \mathcal{M}$ for all $(t,j) \preceq (t^*, j^*)$, $(t,j) \in \operatorname{dom} x$;
(c) *weakly invariant* (with respect to $\mathcal{S}$) if it is both weakly forward invariant and weakly backward invariant.

Requiring completeness in forward invariance and arbitrarily large $N > 0$ in backward invariance leads to the "smallest" possible invariant sets. To verify the forward invariance though, it is sufficient to test every point x^0 of $\mathcal{M}$ for the existence of a trajectory x starting at x^0 such that $x(t,j) \in \mathcal{M}$ for all $t + j \leq 1$, $(t,j) \in \operatorname{dom} x$.

Given a hybrid trajectory $x \in \mathcal{S}$, a sequence $\{(t_i, j_i)\}_{i=1}^{\infty}$ of points in $\operatorname{dom} x$ is *unbounded* if the sequence of $t_i + j_i$'s is unbounded, and *increasing* if for $i = 1, 2, \ldots$, $(t_i, j_i) \preceq (t_{i+1}, j_{i+1})$ in the natural ordering on $\operatorname{dom} x$ (equivalently, $t_i + j_i \leq t_{i+1} + j_{i+1}$.)

Definition 3.2 (ω-limit point): For a complete hybrid trajectory $x \in \mathcal{S}$, its *ω-limit* set, denoted $\Omega(x)$, is the set of all points $x^* \in \mathbb{R}^n$ for which there exists an increasing and unbounded sequence $\{(t_i, j_i)\}_{i=1}^{\infty}$ in $\operatorname{dom} x$ so that $\lim_{i\to\infty} x(t_i, j_i) = x^*$.

Lemma 3.3: (ω-limit set properties) If $x \in \mathcal{S}$ is a precompact hybrid trajectory then its ω-limit set $\Omega(x)$ is nonempty, compact, and weakly invariant. Moreover, the trajectory x approaches $\Omega(x)$, which is the smallest closed set approached by x.

IV. AN INVARIANCE PRINCIPLE INVOLVING A NONINCREASING FUNCTION

The invariance principles we formulate in this section will rely on properties of certain functions not only on the range of the trajectory in question, but also on the neighborhood of its range. Below, given a hybrid trajectory x, $t(j)$ will denote the largest time t such that $(t,j) \in \operatorname{dom} x$, while $j(t)$ will denote the smallest index j such that $(t,j) \in \operatorname{dom} x$. The notation $f^{-1}(r)$ will stand for the r-level set of f on $\operatorname{dom} f$, i.e. $f^{-1}(r) := \{z \in \operatorname{dom} f \mid f(z) = r\}$.

A. Sets of hybrid trajectories

Theorem 4.1: (V invariance principle) Suppose that there exist a continuous function $V : O \to \mathbb{R}$, a set $\mathcal{U} \subset O$, and functions $u_c, u_d : O \to [-\infty, +\infty]$ such that for any hybrid trajectory $y \in \mathcal{S}$ with $\operatorname{rge} y \subset \mathcal{U}$, $u_c(y(t,j)) \leq 0$, $u_d(y(t,j)) \leq 0$ for all $(t,j) \in \operatorname{dom} y$ and

$$V(x(t',j')) - V(x(t,j)) \leq \int_t^{t'} u_c(x(t,j(t)))\,dt + \sum_{i=j}^{j'-1} u_d(x(t(j),j)) \quad (2)$$

for any $(t,j), (t',j') \in \operatorname{dom} y$, $(t,j) \preceq (t',j')$.

Let $x \in \mathcal{S}$ be a precompact hybrid trajectory such that for some $(T,J) \in \operatorname{dom} x$, $\overline{\{x(t,j) \mid (t,j) \in \operatorname{dom} x, (T,J) \preceq (t,j)\}} \subset \mathcal{U}$. Then x approaches the largest weakly invariant subset of $V^{-1}(r) \cap \mathcal{U} \cap \left(\overline{u_c^{-1}(0)} \cup u_d^{-1}(0)\right)$, for some $r \in V(\mathcal{U})$.

Corollary 4.2: Under the assumptions of Theorem 4.1,
(a) if x is Zeno, then it approaches the largest weakly invariant subset of $V^{-1}(r) \cap \mathcal{U} \cap u_d^{-1}(0)$, for some $r \in V(\mathcal{U})$;
(b) if x is such that for some $\gamma > 0$ and for every j such that $(t_j, j), (t_{j+1}, j) \in \operatorname{dom} x$, $t_{j+1} - t_j \geq \gamma$, then x approaches the largest weakly invariant subset of $V^{-1}(r) \cap \mathcal{U} \cap \overline{u_c^{-1}(0)}$, for some $r \in V(\mathcal{U})$.

B. Hybrid Systems

We now consider the hybrid systems $\mathcal{H}$ defined in [8]. The functions $u_c(x)$ and $u_d(x)$ of the previous section will be constructed from a Lyapunov-like function V and will be denoted by $u_C(x)$ and $u_D(x)$, respectively. One will be determined by the "derivative" of V at x in directions belonging to $F(x)$; the other by the difference between V at x and at points belonging to $G(x)$. These functions will be used to bound the increment of V as in (2).

We begin by formulating this infinitesimal inequality. Let $V : O \to \mathbb{R}$ be continuous on O and locally Lipschitz on a neighborhood of C. Let x be any solution to the hybrid system, and let $(\underline{t}, \underline{j}), (\overline{t}, \overline{j}) \in \operatorname{dom} x$ be such that $(\underline{t}, \underline{j}) \preceq (\overline{t}, \overline{j})$. The increment $V(x(\overline{t}, \overline{j})) - V(x(\underline{t}, \underline{j}))$ between them must take into account the "continuous increment" due to the integration of the time derivative of $V(x(t,j))$ and the "discrete increment" due to the difference in V before and after the jump. Consequently, we have

$$\begin{aligned} V(x(\overline{t}, \overline{j})) - V(x(\underline{t}, \underline{j})) &= \int_{\underline{t}}^{\overline{t}} \frac{d}{dt} V(x(t, j(t)))\,dt \\ &\quad + \sum_{j=\underline{j}}^{\overline{j}-1} \left[V(x(t(j), j+1)) - V(x(t(j), j))\right]. \end{aligned} \quad (3)$$

The integral above expresses the desired quantity as $t \mapsto V(x(t, j(t)))$ is locally Lipschitz and absolutely continuous on every interval on which $t \mapsto j(t)$ is constant.

The function $u_C : O \to [-\infty, +\infty)$ must satisfy $\frac{d}{dt} V(x(t, j(t))) \leq u_C(x(t, j(t)))$ for almost every t. When V is locally Lipschitz, u_C can be constructed using the generalized Clarke directional derivative of V. It turns out that, for this case, u_C is upper semicontinuous. A better bound arises in the case that V is nonpathological (see [24] for details.) In this case, the bounding function may not be upper semicontinuous.

To bound the "discrete contribution" to the increase in V in (3), we will use the following quantity:

$$u_D(x) = \max_{x^+ \in G(x)} \{V(x^+) - V(x)\} \quad (4)$$

for $x \in D$ and $u_D(x) = -\infty$ for $x \notin D$. Even without any regularity on V, one gets the bound $V(x(t_{j+1}, j+1)) - V(x(t_{j+1}, j)) \leq u_D(x(t_{j+1}, j))$ for any solution to the hybrid system.

Now we state the hybrid invariance principle.

Corollary 4.3: (hybrid V invariance principle) Given a hybrid system $\mathcal{H}$, let $V : O \to \mathbb{R}$ be continuous on O

and locally Lipschitz on a neighborhood of C. Suppose that $\mathcal{U} \subset O$ is nonempty and such that $u_C(z) \leq 0, \;\; u_D(z) \leq 0$ for all $z \in \mathcal{U}$. Let $x \in \mathcal{S}_{\mathcal{H}}$ be precompact with $\overline{\operatorname{rge} x} \subset \mathcal{U}$. Then, for some constant $r \in V(\mathcal{U})$, x approaches the largest weakly invariant set in $V^{-1}(r) \cap \mathcal{U} \cap \left(\overline{u_C^{-1}(0)} \cup u_D^{-1}(0)\right)$.

V. A MEAGRE-LIMSUP INVARIANCE PRINCIPLE

In the result below we use the concept of a weakly meagre function. A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is *weakly meagre* if $\lim_{n\to\infty}(\inf_{t\in I_n} |f(t)|) = 0$ for every family $\{I_n \mid n \in \mathbb{N}\}$ of nonempty and pairwise disjoint closed intervals I_n in $\mathbb{R}_{\geq 0}$ with $\inf_{n\in\mathbb{N}} \mu(I_n) > 0$. Here, μ stands for the Lebesgue measure. Weak meagreness was used previously in [15] to formulate extensions of Barbalat's lemma and resulting invariance principles. Following [15], we state that f is weakly meagre if for some $\tau > 0$,

$$\lim_{M\to+\infty} \int_M^{M+\tau} |f(t)|\, dt = 0. \tag{5}$$

In particular, any L^1 function is weakly meagre.

A. Sets of hybrid trajectories

Lemma 5.1: (meagre-limsup conditions) Let $x \in \mathcal{S}$ be a precompact hybrid trajectory. Suppose that for some set $\mathcal{U}$ with $\operatorname{rge} x \subset \mathcal{U}$ there exist functions $\ell_c, \ell_d : \mathcal{U} \to [0, +\infty]$ that satisfy the meagre-limsup conditions given by

(a) *if the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is unbounded then $t \mapsto \ell_c(x(t, j(t)))$ is weakly meagre,*
(b) *if the projection of $\operatorname{dom} x$ onto $\mathbb{N}_{\geq 0}$ is unbounded then $\limsup_{j\to\infty} \ell_d(x(t(j), j)) = 0$.*

Then $\Omega(x) \subset E_{x,\ell_c} \cup E_{x,\ell_d}$, where E_{x,ℓ_c} is defined as
$\{z \in \overline{\operatorname{rge} x} \,|\, \exists z_i \to z, z_i \in \operatorname{rge} x, \liminf_{i\to\infty} \ell_c(z_i) = 0\}$
and E_{x,ℓ_d} is defined as
$\{z \in \overline{\operatorname{rge} x} \mid \exists z_i \to z, z_i \in \operatorname{rge} x, \liminf_{i\to\infty} \ell_d(z_i) = 0\}$.

Remark 5.2: In Lemma 5.1, E_{x,ℓ_c} is a subset of $\{z \in \overline{\operatorname{rge} x} \mid \underline{\ell_c}(z) = 0\}$, where $\underline{\ell_c}$ is the lower semicontinuous closure of ℓ_c. (Given a set $\mathcal{U}$ and a function $\ell : \mathcal{U} \to [-\infty, +\infty]$, its *lower semicontinuous closure* $\underline{\ell} : \overline{\mathcal{U}} \to [-\infty, +\infty]$, is the greatest lower semicontinuous function defined on $\overline{\mathcal{U}}$, bounded above by ℓ on $\mathcal{U}$. Equivalently, for any $x \in \overline{\mathcal{U}}$, $\underline{\ell}(x) = \liminf_{x_i\to x} \ell(x_i)$. In this terminology, $E_{x,\ell}$ is the zero-level set of the lower semicontinuous closure of the function ℓ truncated to $\operatorname{rge} x$.) In particular, if both ℓ_c and ℓ_d are lower semicontinuous, and $\overline{\operatorname{rge} x} \subset \mathcal{U}$, then the conclusion of Lemma 5.1 implies that $\Omega(x)$ is a subset of $\{z \in \overline{\operatorname{rge} x} \mid \ell_c(z) = 0\} \cup \{z \in \overline{\operatorname{rge} x} \mid \ell_d(z) = 0\}$. However, if the assumption that ℓ_c, ℓ_d are nonnegative was weakened to say that they are nonnegative only on $\operatorname{rge} x$, the last conclusion above may fail.

Remark 5.3: Let $x \in \mathcal{S}$ be a precompact hybrid trajectory for which there exist functions $u_c, u_d : O \to [-\infty, 0]$ and $V : O \to \mathbb{R}$ such that (2) holds for all $(t, j), (t', j') \in \operatorname{dom} x$ such that $(t, j) \preceq (t', j')$. Then $\ell_c = -u_c$, $\ell_d = -u_d$ satisfy conditions (a) and (b) of Theorem 5.1. In fact, there exists a constant $M > 0$ for which

$$\int_0^T \ell_c(x(t, j(t)))\, dt < M, \quad \sum_{j=0}^{J} \ell_d(x(t(j), j)) < M, \tag{6}$$

for any $(T, J) \in \operatorname{dom} x$.

Based on the previous remark, the next result shows that when a function V with the right properties exists, the conditions (a) and (b) of Lemma 5.1 are guaranteed.

Corollary 5.4: Let $x \in \mathcal{S}$ be a precompact hybrid trajectory. Suppose that there exists a continuous function $V : O \to \mathbb{R}$, and functions $u_c, u_d : O \to [-\infty, +\infty]$ such that for some $(T, J) \in \operatorname{dom} x$, $u_c(x(t, j)) \leq 0, \;\; u_d(x(t, j)) \leq 0$ for all $(t, j) \in \operatorname{dom} x$, $(T, J) \preceq (t, j)$, and (2) holds for any $(t, j), (t', j') \in \operatorname{dom} x$ such that $(T, J) \preceq (t, j) \preceq (t', j')$. Then $\Omega(x) \subset E^{x,u_c} \cup E^{x,u_d}$, where E^{x,u_c} is defined as
$\{z \in \overline{\operatorname{rge} x} \mid \exists z_i \to z, z_i \in \operatorname{rge} x, \limsup_{i\to\infty} u_c(z_i) = 0\}$
and E^{x,u_d} is defined as
$\{z \in \overline{\operatorname{rge} x} \mid \exists z_i \to z, z_i \in \operatorname{rge} x, \limsup_{i\to\infty} u_d(z_i) = 0\}$.

Results for continuous time only or for discrete time only can be easily recovered from Lemma 5.1.

Corollary 5.5: Let $x \in \mathcal{S}$ be precompact.

(a) *If the projection of $\operatorname{dom} x$ onto $\mathbb{N}_{\geq 0}$ is bounded and there exists a function $\ell_c : \operatorname{rge} x \to [0, +\infty]$ such that $t \mapsto \ell_c(x(t, j(t)))$ is weakly meagre, then $\Omega(x) \subset E_{x,\ell_c}$.*
(b) *If the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is bounded and there exists a function $\ell_d : \operatorname{rge} x \to [0, +\infty]$ such that $\limsup_{j\to\infty} \ell_d(x(t(j), j)) = 0$, then $\Omega(x) \subset E_{x,\ell_d}$.*

It also turns out that if multiple instantaneous jumps can occur "only on the zero-level set of ℓ_d" (for a hybrid system $\mathcal{H}$, this is equivalent to $\ell_d\,(G(D) \cap D) = 0$) and the hybrid trajectory of $\mathcal{S}$ is precompact, then only (a) of the meagre-limsup conditions needs to be checked to draw the conclusion of Lemma 5.1. The reason for this is that under such assumption on the jumps, on each compact set away from the zero level set of ℓ_d, times between the jumps are uniformly bounded below (by a positive constant.)

Corollary 5.6: Given the function $\ell_d : O \to \mathbb{R}_{\geq 0}$, assume that for all $\tilde{x} \in \mathcal{S}$, if $(t, j-1), (t, j), (t, j+1) \in \operatorname{dom} \tilde{x}$, then $\ell_d(\tilde{x}(t, j)) = 0$. Let $x \in \mathcal{S}$ be precompact. Suppose that there exists a function $\ell_c : \operatorname{rge} x \to [0, +\infty]$ such that condition (a) of the meagre-limsup conditions holds. Then conclusion of Lemma 5.1 holds.

If, for the hybrid trajectory, the elapsed time between jumps is uniformly positive then only (a) of the meagre-limsup conditions needs to be checked to draw the conclusion of Lemma 5.1.

Corollary 5.7: Let $x \in \mathcal{S}$ be a complete hybrid trajectory of $\mathcal{S}$ such that $t_{j+1} - t_j \geq \gamma > 0$ for all $j = 1, 2, \ldots$. Suppose that x is precompact and that there exists a function $\ell_c : \operatorname{rge} x \to [0, +\infty]$ such that condition (a) of the meagre-limsup conditions holds. Then $\Omega(x) \subset E_{x,\ell_c}$.

Based on the results stated so far in this section, various invariance principles can be stated. For example, in light of Remark 5.2, we have the following result.

Corollary 5.8: (meagre-limsup invariance principle) Let $x \in \mathcal{S}$ be a precompact hybrid trajectory. Suppose that *for some set $\mathcal{U} \subset O$ such that* $\operatorname{rge} x \subset \mathcal{U}$*, there exist functions* $\ell_c, \ell_d : \mathcal{U} \to [0, +\infty]$ *for which the meagre-limsup conditions hold. Then x converges to the largest weakly invariant subset of* $\{z \in \overline{\mathcal{U}} \mid \underline{\ell_c}(z) = 0\} \cup \{z \in \overline{\mathcal{U}} \mid \underline{\ell_d}(z) = 0\}$*. If* $\overline{\operatorname{rge} x} \subset \mathcal{U}$ *and* ℓ_c, ℓ_d *are lower semicontinuous, then all the closure operations above can be removed.*

Remark 5.9: For hybrid systems, the natural counterparts of ℓ_c, ℓ_d are the functions $-u_C$ and $-u_D$. One can show that the results in Section IV-B can be rewritten replacing the zero-level sets of u_C, u_D by E^{x,u_c}, E^{x,u_c}, respectively.

One difference between Theorem 4.1 and Corollary 5.8 is that in the latter, properties of the functions ℓ_c, ℓ_d only on the range of the hybrid trajectory x in question are relevant. In the former, properties of u_c, u_d (counterparts of ℓ_c, ℓ_d) and also V holding for other trajectories (in particular, for the trajectories verifying forward invariance of $\Omega(x)$) are assumed, and the conclusions of Theorem 4.1 do use these properties.

VI. Locating weakly invariant sets using observability, or stability and detectability

Now we extend results on stability and convergence, and the implications of detectability, from differential equations to sets of hybrid trajectories.

In the literature of differential equations, detectability is the property that when the output is held to zero, the limit as $t \to \infty$ of the norm of the state equals zero. Our generalization of detectability for hybrid trajectories relaxes also the limit condition, replacing it by a lower limit.

Definition 6.1 (detectability): Given sets $\mathcal{A}, \mathcal{K} \subset O$, the distance to $\mathcal{A}$ is *detectable* on $\mathcal{K}$ for the set of trajectories $\mathcal{S}$ if for every complete trajectory $x \in \mathcal{S}$ such that $\operatorname{rge} x \subset \mathcal{K}$ we have $\liminf_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$.

As discussed in [14], this detectability condition can be understood as the trajectory x having a ω-limit point at $\mathcal{A}$.

Definition 6.2 (relative stability): Given sets $\mathcal{A}, \mathcal{K} \subset O$, $\mathcal{A}$ is *stable relative to* $\mathcal{K}$ for the set of trajectories $\mathcal{S}$ if for any compact $\mathcal{K}' \subset \operatorname{int} \mathcal{K}$ there exists $\delta > 0$ such that any trajectory $x \in \mathcal{S}(x^0)$ with $\operatorname{rge} x \subset \mathcal{K}$ and $x^0 \in (\mathcal{A} + \delta\mathbb{B})$ satisfies $\operatorname{rge} x \subset \mathcal{K}'$.

Stability of $\mathcal{A}$ is the same as stability relative to O. When detectability (as in Definition 6.1) is combined with relative stability, the usual detectability is recovered.

Lemma 6.3: (detectability and relative stability) Let $\mathcal{A}, \mathcal{K} \subset O$ be compact. Suppose that the distance to $\mathcal{A}$ is detectable on $\mathcal{K}$ and $\mathcal{A}$ is stable relative to $\mathcal{K}$. Then each complete trajectory $x \in \mathcal{S}$ with $\operatorname{rge} x \subset \mathcal{K}$ *converges to $\mathcal{A}$.*

Theorem 6.4: (detectability and invariance principle) Let $\mathcal{A}, \mathcal{K} \subset O$ be compact and suppose that $\mathcal{A}$ is stable relative to $\mathcal{K}$. Then the following statements are equivalent:

1) *The distance to $\mathcal{A}$ is detectable on $\mathcal{K}$.*
2) *The largest weakly invariant set in $\mathcal{K}$ is a subset of $\mathcal{A}$.*

The detectability assumption on a stable compact attractor allow us to conclude uniform convergence respect to trajectories that stay in a compact set of initial conditions.

Theorem 6.5: (uniform convergence) Let $\mathcal{A}, \mathcal{K} \subset O$ be compact. Suppose that $\mathcal{A}$ is stable relatively to $\mathcal{K}$ and the distance to $\mathcal{A}$ is detectable on $\mathcal{K}$. Then for each $\epsilon > 0$ there exists $M > 0$ such that for each complete trajectory $x \in \mathcal{S}$ with $\operatorname{rge} x \subset \mathcal{K}$ *we have $|x(t,j)|_{\mathcal{A}} \leq \epsilon$ for all $(t,j) \in \operatorname{dom} x$, $t + j \geq M$.*

Remark 6.6: If for a certain (output) function $h : O \to \mathbb{R}^k$, $\mathcal{K} = h^{-1}(0)$, we say that the the distance to $\mathcal{A}$ is detectable through the (output) h. Also note that the natural notion of observability (for every nontrivial trajectory $x \in \mathcal{S}$ such that $\operatorname{rge} x \subset \mathcal{K}$ we have $\operatorname{rge} x \subset \mathcal{A}$) and the related results can be easily recovered from our definition of detectability.

VII. Asymptotic Stability

A. Definitions and a $\mathcal{KLL}$-characterization

The previous results rely on the convergence property (B3) of the Basic Assumption. For results on uniform convergence without a priori restriction of the trajectories to a compact set, we need an additional condition. Besides the Basic Assumption, from now on, we assume:

(B4) any sequence $\{x_i\}_{i=1}^{\infty}$ of hybrid trajectories in $\mathcal{S}$ for which initial points $x_i(0,0)$ converge to a point x^0 where every solution $x \in \mathcal{S}(x^0)$ is complete, is locally eventually bounded.

For solutions of hybrid systems, this property requires local boundedness of G, see Theorem 4.4 in [8]. In light of other growth properties of G, and the fact that G maps to O, its local boundedness is equivalent to *local boundedness with respect to O*: for any compact $\mathcal{K} \subset O$ there exists a compact $\mathcal{K}' \subset O$ such that $G(\mathcal{K}) \subset \mathcal{K}'$.

Definition 7.1 (relative attractivity): Given sets $\mathcal{A}, \mathcal{K} \subset O$, $\mathcal{A}$ is *attractive relative to* $\mathcal{K}$ for the set of trajectories $\mathcal{S}$ if there exists $\rho > 0$ such that for any $x^0 \in \mathcal{A} + \delta\mathbb{B}$, each trajectory $x \in \mathcal{S}(x^0)$ with $\operatorname{rge} x \subset \mathcal{K}$ is complete and satisfies $\lim_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$.

Attractivity of $\mathcal{A}$ is the same as attractivity relative to O. We denote by $\mathcal{B}_{\mathcal{A}}$ the basin of attraction of a compact attractor $\mathcal{A}$, i.e. the set of all points x^0 for which $\mathcal{S}(x^0)$ is nonempty, each $x \in \mathcal{S}(x^0)$ is complete and such that $\lim_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$. One could also define a *relative basin of attraction* of $\mathcal{A}$ relative to $\mathcal{K}$, as the set of all points $x^0 \in \mathcal{K}$ for which each trajectory $x \in \mathcal{S}(x^0)$ with $\operatorname{rge} x \subset \mathcal{K}$ is complete and satisfies $\lim_{t+j\to\infty} |x(t,j)|_{\mathcal{A}} = 0$. The set $\mathcal{A}$ is said to be asymptotically stable (relative to $\mathcal{K}$) if it is both stable and attractive (relative to $\mathcal{K}$.)

Given an open set $X \subset O$ and a compact set $\mathcal{A} \subset X$, a proper indicator $\omega : X \to \mathbb{R}_{\geq 0}$ for $\mathcal{A}$ on X is a continuous function that is positive definite with respect to $\mathcal{A}$ and proper with respect to X. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{N}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KLL}$ if it is continuous, $\beta(\cdot, t, j)$ is zero at zero and nondecreasing, $\beta(s, \cdot, j)$ and $\beta(s, t, \cdot)$ are nonincreasing and converge to zero as the argument becomes unbounded. We say that the set of hybrid trajectories $\mathcal{S}$ is forward complete on $\mathcal{K}$ if for every $x^0 \in \mathcal{K}$, every $x \in \mathcal{S}(x^0)$ is complete.

Definition 7.2: ($\mathcal{KLL}$ stability with one measure) Let $\omega : \mathcal{K} \to \mathbb{R}_{\geq 0}$ be continuous. The set of hybrid trajectories $\mathcal{S}$ is said to be $\mathcal{KLL}$-stable with respect to ω if it is forward complete on $\mathcal{K}$ and there exists $\beta \in \mathcal{KLL}$ such that, for each $x^0 \in \mathcal{K}$, all solutions $x \in \mathcal{S}(x^0)$ satisfy

$$\omega(x(t,j)) \leq \beta(\omega(x^0), t, j) \quad \text{for each } (t,j) \in \operatorname{dom} x. \quad (7)$$

We define $\mathcal{R}$ as the range set of trajectories of a given set $\mathcal{S}$, i.e. $\mathcal{R} := \{\operatorname{rge} x \mid x \in \mathcal{S}\}$.

Theorem 7.3: (asymptotic stability implies $\mathcal{KLL}$ stab.) Suppose that, for the set of trajectories $\mathcal{S}$, the compact set $\mathcal{A}$ is locally asymptotically stable with basin of attraction $\mathcal{B}_{\mathcal{A}}$. Then, for each open set X such that $\mathcal{B}_{\mathcal{A}} = X \cap \mathcal{R}$ and for each proper indicator for $\mathcal{A}$ on X, denoted by ω, there exists $\beta \in \mathcal{KLL}$ such that for every $x^0 \in \mathcal{B}_{\mathcal{A}}$, every $x \in \mathcal{S}(x^0)$ satisfies $\omega(x(t,j)) \leq \beta(\omega(x^0), t, j)$ for each $(t,j) \in \operatorname{dom} x$.

For results on the existence of smooth Lyapunov functions for asymptotically stable compact attractors see the work by Cai et. al. [5].

B. Lyapunov and Krasovskii theorems for hybrid systems

In what follows, we assume that the hybrid system $\mathcal{H}$ given by (1) satisfies assumptions A0-A3 in [8], and furthermore, that for any point $x^0 \in C \cup D$, there exists a nontrivial solution $x \in \mathcal{S}_{\mathcal{H}}(x^0)$. Explicit conditions on C, D, F, and G that guarantee such existence are given in Proposition 2.1 of [8]. Here, we only mention that the conditions are satisfied if $C \cup D = O$. A particular consequence of existence of nontrivial solutions from every point of $C \cup D$ is that any maximal solution to $\mathcal{H}$ is either complete or eventually leaves any compact subset of O. So, any $x \in \mathcal{S}(C \cup D)$ (any maximal solution to $\mathcal{H}$) which is bounded with respect to O (i.e. for some compact $\mathcal{K} \subset O$, $\operatorname{rge} x \subset \mathcal{K}$) is complete, and hence, precompact. Below, this fact will be needed in showing attractivity.

Theorem 7.4: (hybrid Krasovskii) Given a hybrid system $\mathcal{H}$, suppose that

- ($\star$) *$\mathcal{A} \subset O$ is compact, $\mathcal{U} \subset O$ is a neighborhood of $\mathcal{A}$, $V : O \to \mathbb{R}_{\geq 0}$ is continuous on O, locally Lipschitz on a neighborhood of C, and positive definite with respect to $\mathcal{A}$, and u_C and u_D satisfy $u_C(z) \leq 0$, $u_D(z) \leq 0$ for all $z \in \mathcal{U}$.*

Then $\mathcal{A}$ is stable. Suppose additionally that

- ($\star\star$) *there exists $c > 0$ such that for all $c' \in (0, c)$ the largest weakly invariant subset of $V^{-1}(c') \cap \{z \in \mathcal{U} \mid u_C(z) = u_D(z) = 0\}$ is empty.*

Then $\mathcal{A}$ is locally asymptotically stable.

Corollary 7.5: (hybrid Lyapunov) For a hybrid system $\mathcal{H}$, suppose that ($\star$) of Theorem 7.4 holds, and that furthermore, $u_C(z) < 0$, $u_D(z) < 0$ for all $z \in \mathcal{U} \setminus \mathcal{A}$. Then $\mathcal{A}$ is locally asymptotically stable.

The following result states that when u_C (respectively, u_D) is negative in points near a compact attractor and instantaneous Zeno solutions (respectively, complete continuous solutions) converge to the attractor, then the compact attractor is asymptotically stable.

Theorem 7.6: For the hybrid system $\mathcal{H}$, suppose that ($\star$) of Theorem 7.4 holds. Suppose that

- (a) *$u_C(z) < 0$ for each $z \in \mathcal{U} \setminus \mathcal{A}$ (respectively, $u_D(z) < 0$ for each $z \in \mathcal{U} \setminus \mathcal{A}$);*
- (b) *any instantaneous Zeno solution x to $\mathcal{H}$ with $\operatorname{rge} x \subset \mathcal{U}$ converges to $\mathcal{A}$ (respectively, any complete continuous solution x to $\mathcal{H}$ with $\operatorname{rge} x \subset \mathcal{U}$ converges to $\mathcal{A}$).*

Then $\mathcal{A}$ is locally asymptotically stable.

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