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Fuzzy Model-based Control for Dynamic Variable Structure Systems

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***Abstract*— This paper presents a dynamic variable structure system and its controller design conditions. The dynamic variable structure system, which is a class of hybrid dynamic systems, consists of plural subsystems which are switched by switching conditions. A key feature of the dynamic variable structure system is that each subsystem can have different degrees of freedom. In this paper, we employ switching fuzzy models to represent the nonlinear subsystem's dynamics, switching conditions and conservation laws with respect to states, momentum and/or energy of the system. We derive controller design conditions for dynamic variable structure systems.**

I. INTRODUCTION

Recently, hybrid systems and switching systems, whose control objects are switched depending on some kind of events, have been discussed in a lot of literature [1], [2], [3], [4]. Most of them deal with control objects whose subsystems' degrees of freedom do not change before and after switching. However, when we consider a constrained system like a knee or an elbow joint, it may be more natural and efficient to consider that the system has plural subsystems and each subsystem has different degrees of freedom.

In this paper, we propose a dynamic variable structure system and derive its controller design conditions. The dynamic variable structure system consists of plural subsystems which are switched based on switching conditions. A key feature of the dynamic variable structure system is that each subsystem can have different degrees of freedom. However, controllers designed independently for each subsystem cannot always stabilize the dynamic variable structure system although each controller can stabilize the corresponding subsystem. To stabilize the system, it is important to consider not only subsystems' stability but also switching conditions and conservation laws of states, momentum and/or energy of the system. In this paper we employ switching fuzzy models [5], [6], [7], [8] to represent nonlinear subsystems' dynamics, switching conditions and conservation laws. To stabilize dynamic variable structure systems, we derive controller design conditions which are taking switching conditions and conservations laws into account.

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II. DYNAMIC VARIABLE STRUCTURE SYSTEMS

We define the general form of dynamic variable structure systems as follows:

$$\text{Subsystem } \boldsymbol{S}_i: \quad \dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}_i(\boldsymbol{x}_i(t), \boldsymbol{u}_i(t)) \tag{1}$$

$$\text{Switching condition}: \quad \xi_{ij}(\dot{\boldsymbol{x}}_i(t), \boldsymbol{x}_i(t), \boldsymbol{u}_i(t)) \le 0 \tag{2}$$

$$\text{Conservation law}: \quad \boldsymbol{T}_{ij}(\boldsymbol{x}_j(t)) = \boldsymbol{g}_{ij}(\boldsymbol{x}_i(t)) \tag{3}$$

where,

$$\begin{aligned}
&i = 1, 2, \cdots, \gamma,\ j = 1, 2, \cdots, \gamma,\ i \ne j \\
&\boldsymbol{x}_i(t) = \left[\begin{array}{cccc} x_{i1}(t) & x_{i2}(t) & \cdots & x_{in_i}(t) \end{array}\right]^T \\
&\boldsymbol{u}_i(t) = \left[\begin{array}{cccc} u_{i1}(t) & u_{i2}(t) & \cdots & u_{im_i}(t) \end{array}\right]^T \\
&n_i \le n,\ m_i \le m, \\
&n = \max_i n_i < \infty,\ m = \max_i m_i < \infty
\end{aligned}$$

A dynamic variable structure system has γ subsystems $\boldsymbol{S}_i$, where $i = 1, 2, \cdots, \gamma$, which switch one subsystem to other subsystem depending on states and inputs. Each subsystem can have different degrees of freedom (the number of states and inputs). The switching from Subsystem $\boldsymbol{S}_i$ to $\boldsymbol{S}_j$ occurs based on switching conditions. The switching can occur not only when the system is at a static equilibrium state but also when the system is in a dynamic motion. Based on the conservation laws of states, momentum and/or energy, a part of states of a subsystem before switching is taken over to a subsystem after switching. We assume that the following condition is satisfied when Subsystem $\boldsymbol{S}_i$ is switched to $\boldsymbol{S}_j$.

$$\sum_{\nu=1}^{n_i}\sum_{\sigma=1}^{n_j} e_{ij\nu\sigma} \ne 0 \tag{4}$$

$$e_{ij\nu\sigma} = \begin{cases} 1 & x_{i\nu} \text{ and } x_{j\sigma} \\ & \quad \text{are essentially same variables} \\ 0 & x_{i\nu} \text{ and } x_{j\sigma} \\ & \quad \text{are not same variables} \end{cases}$$

We show an example to demonstrate that considering only each subsystem's stability cannot guarantee stability of a dynamic variable structure system.

[Example 1] Consider the following simple system.

$$\text{Subsystems } \boldsymbol{S}_1 \text{ and } \boldsymbol{S}_2: \begin{array}{l} \dot{\boldsymbol{x}}_1(t) = \boldsymbol{A}_1\boldsymbol{x}_1(t) + \boldsymbol{B}_1\boldsymbol{u}_1(t), \\ \dot{\boldsymbol{x}}_2(t) = \boldsymbol{A}_2\boldsymbol{x}_2(t) + \boldsymbol{B}_2\boldsymbol{u}_2(t), \end{array}$$

$$\text{Switching conditions}: \begin{array}{l} \xi_{12}(\boldsymbol{x}_1(t)) = \boldsymbol{\Xi}_{12}\boldsymbol{x}_1(t) = 0, \\ \xi_{21}(\boldsymbol{x}_2(t)) = \boldsymbol{\Xi}_{21}\boldsymbol{x}_2(t) = 0, \end{array}$$

$$\text{Conservation laws}: \begin{array}{l} \boldsymbol{x}_2(t) = \boldsymbol{G}_{12}\boldsymbol{x}_1(t), \\ \boldsymbol{x}_1(t) = \boldsymbol{G}_{21}\boldsymbol{x}_2(t), \end{array}$$

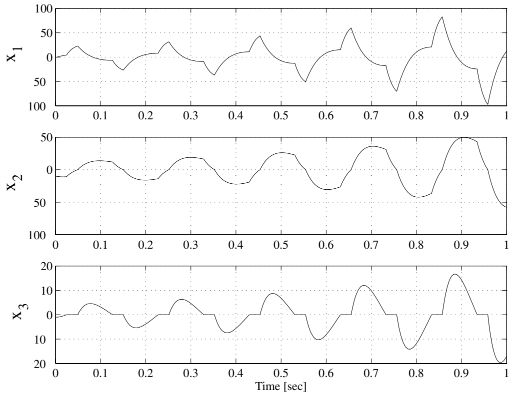

Fig. 1. Control result without considering switching

where

$$\boldsymbol{x}_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}, \ \boldsymbol{x}_2(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \\ x_{23}(t) \end{bmatrix},$$

$$\boldsymbol{A}_1 = \begin{bmatrix} -16 & -11 \\ -10 & 22 \end{bmatrix}, \ \boldsymbol{B}_1 = \begin{bmatrix} 11 \\ 8 \end{bmatrix},$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 3 & 0 & 4 \\ 32 & 9 & 4 \\ 26 & 0 & 0 \end{bmatrix}, \ \boldsymbol{B}_2 = \begin{bmatrix} 0 & 11 \\ 0 & 0 \\ 8 & 1 \end{bmatrix},$$

$$\boldsymbol{\Xi}_{12} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ \boldsymbol{\Xi}_{21} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{G}_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{G}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$x_{11}(t)$ and $x_{21}(t)$, $x_{12}(t)$ and $x_{22}(t)$ are essentially same variables, respectively. When the switching conditions are satisfied, Subsystems $\boldsymbol{S}_1$ or $\boldsymbol{S}_2$ switch to $\boldsymbol{S}_2$ or $\boldsymbol{S}_1$, respectively. Initial states after switching are determined based on the conservation laws.

For each subsystem, we independently design the LQR controller $\boldsymbol{u}_i(t) = -\boldsymbol{F}_i\boldsymbol{x}_i(t)$ without considering switching conditions and conservation laws. The designed controllers are also switched according to subsystems. Fig. 1 shows a simulation result. The controllers designed independently for each subsystem cannot always stabilize the dynamic variable structure system although each controller can stabilize the corresponding subsystem.

III. Stability analysis of dynamic variable structure systems

We showed the general form of dynamic variable structure systems in Section II. In this section, we consider the following special case of the dynamic variable structure systems.

Subsystem $\boldsymbol{S}_i$:

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}_{i1}(\boldsymbol{x}_i(t)) + \boldsymbol{f}_{i2}(\boldsymbol{x}_i(t))\boldsymbol{u}_i(t) \quad (5)$$

Switching condition : $\boldsymbol{C}_i\dot{\boldsymbol{x}}_i(t) + \xi_{ij}(\boldsymbol{x}_i(t)) = 0$ (6)

Conservation law : $\boldsymbol{x}_j(t) = \boldsymbol{g}_{ij}(\boldsymbol{x}_i(t))$ (7)

In this paper, we utilize a switching fuzzy model [5], [6] to represent nonlinear subsystems, switching conditions and conservation laws. The switching fuzzy model is constructed by dividing state space into quadrants which are based on state variable $\boldsymbol{x}_i(t)$. In many cases, a switching plane of a dynamic variable structure system corresponds to that of quadrants, that is, that of switching fuzzy models. Even though the switching planes do not correspond, we can make them correspond by performing coordinate transformations. Therefore, the switching fuzzy model is useful for representing dynamic variable structure systems.

A. Switching fuzzy model-based control

In this section, we show a switching fuzzy model, a switching fuzzy controller and controller design conditions proposed in [5], [6], [7], [8]. By applying sector nonlinearity [5], [6] to the subsystem (5), the following switching fuzzy model can be obtained.

$$\dot{\boldsymbol{x}}_i(t) = \sum_{q_i=1}^{Q_i}\sum_{k=1}^{r_i} v_{iq_i}(\boldsymbol{x}_i(t))h_{iq_ik}(\boldsymbol{x}_i(t)) \times (\boldsymbol{A}_{iq_ik}\boldsymbol{x}_i(t) + \boldsymbol{B}_{iq_ik}\boldsymbol{u}_i(t)) \quad (8)$$

where

$$v_{iq_i}(\boldsymbol{x}_i(t)) = \begin{cases} 1, & \boldsymbol{x}_i(t) \in \text{Region } q_i, \\ 0, & \boldsymbol{x}_i(t) \notin \text{Region } q_i, \end{cases}$$

$\boldsymbol{A}_{iq_ik} \in R^{n_i \times n_i}$, $\boldsymbol{B}_{iq_ik} \in R^{n^i \times m^i}$, $h_{iq_ik}(\boldsymbol{x}_i(t))$ is a membership function. Q_i is the number of regions. Each region corresponds to quadrants. r_i is the number of fuzzy model rules. The switching fuzzy model has local Takagi-Sugeno fuzzy model [12] in each region and can represent the nonlinear system (5) by switching local models. q_ith region can be shown as follows:

$$\text{Region } q_i: \ R_{iq_i}(s_{i1q_i}, s_{i2q_i}, \cdots, s_{in_iq_i}),$$
$$s_{i\nu q_i} = \begin{cases} 1 & x_{i\nu}(t) \geq 0 \\ 0 & x_{i\nu}(t) < 0, \ \nu = 1, 2, \cdots, n_i. \end{cases}$$

To stabilize the fuzzy model (8), we employ the so-called parallel distributed compensation (PDC) [9], [10]

$$\boldsymbol{u}_i(t) = -\sum_{q_i=1}^{Q_i}\sum_{k=1}^{r_i} v_{iq_i}(\boldsymbol{x}_i(t))h_{iq_ik}(\boldsymbol{x}_i(t)) \times \tilde{\boldsymbol{F}}_{iq_ik}\boldsymbol{E}_{iq_i}\hat{\boldsymbol{E}}_i\boldsymbol{x}_i(t) \quad (9)$$

where $\tilde{\boldsymbol{F}}_{iq_ik} \in R^{m_i \times 2n_i}$ is a feedback gain. $\boldsymbol{E}_{iq_i} \in R^{2n_i \times 2n_i}$ is the known non-singular matrix (For more details, see [8]). $\hat{\boldsymbol{E}}_i = [\boldsymbol{I}_{n_i} \ \boldsymbol{0}_{n_i}]^T \in R^{2n_i \times n_i}$.

Theorem 1: [8] The switching fuzzy model (8) is asymptotically stabilizable by the controller (9) if there exist matrices $\boldsymbol{X}_i \in R^{2n_i \times 2n_i}$ and $\boldsymbol{M}_{iq_ik} \in R^{m_i \times 2n_i}$ satisfying

(10), (11) and (12).

$$\begin{aligned}
&\boldsymbol{X}_i > \boldsymbol{0}, &(10)\\
&\boldsymbol{E}_{iq_i}\tilde{\boldsymbol{A}}_{iq_ik}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i + (\boldsymbol{E}_{iq_i}\tilde{\boldsymbol{A}}_{iq_ik}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i)^T\\
&\quad -\boldsymbol{E}_{iq_i}\tilde{\boldsymbol{B}}_{iq_ik}\boldsymbol{M}_{iq_il} - (\boldsymbol{E}_{iq_i}\tilde{\boldsymbol{B}}_{iq_ik}\boldsymbol{M}_{iq_il})^T < \boldsymbol{0}, &(11)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall i, q_i,\\
&\boldsymbol{E}_{iq_i}\tilde{\boldsymbol{A}}_{iq_ik}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i + (\boldsymbol{E}_{iq_i}\tilde{\boldsymbol{A}}_{iq_ik}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i)^T\\
&\quad \boldsymbol{E}_{iq_i}\tilde{\boldsymbol{A}}_{iq_il}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i + (\boldsymbol{E}_{iq_i}\tilde{\boldsymbol{A}}_{iq_il}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i)^T\\
&\quad -\boldsymbol{E}_{iq_i}\tilde{\boldsymbol{B}}_{iq_ik}\boldsymbol{M}_{iq_il} - (\boldsymbol{E}_{iq_i}\tilde{\boldsymbol{B}}_{iq_ik}\boldsymbol{M}_{iq_il})^T\\
&\quad -\boldsymbol{E}_{iq_i}\tilde{\boldsymbol{B}}_{iq_il}\boldsymbol{M}_{iq_ik} - (\boldsymbol{E}_{iq_i}\tilde{\boldsymbol{B}}_{iq_il}\boldsymbol{M}_{iq_ik})^T < \boldsymbol{0}, &(12)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall i, q_i, k < l,
\end{aligned}$$

where $\tilde{\boldsymbol{F}}_{iq_ik} = \boldsymbol{M}_{iq_ik}\boldsymbol{X}_i^{-1}$,

$$\tilde{\boldsymbol{A}}_{iq_ik} = \begin{bmatrix} \boldsymbol{A}_{iq_ik} & \boldsymbol{0} \\ \boldsymbol{0} & -\alpha\boldsymbol{I}_{n_i} \end{bmatrix},\ \tilde{\boldsymbol{B}}_{iq_ik} = \begin{bmatrix} \boldsymbol{B}_{iq_ik} \\ \boldsymbol{0} \end{bmatrix},$$

and α is an arbitrary positive value.

Remark 1: The switching fuzzy model switches regions depending on state variables as well as the dynamic variable structure system. However, since the switching fuzzy model cannot deal with the change of degrees of freedom, we newly propose dynamic variable structure systems.

B. Controller design with conservation laws

As shown in Example 1, even though we design controllers which can stabilize the corresponding subsystems, the designed controllers do not always stabilize dynamic variable structure systems. In this section, we derive controller design conditions based on the conservation law (7) to stabilize dynamic variable structure systems. By applying sector nonlinearity to (7), the conservation law (7) can be represented as follows:

$$\boldsymbol{x}_j(t) = \sum_{q_i=1}^{Q_i}\sum_{k=1}^{\rho_i} v_{iq_i}(\boldsymbol{x}_i(t))\phi_{iq_ik}(\boldsymbol{x}_i(t))\boldsymbol{G}_{ijq_ik}\boldsymbol{x}_i(t) \quad (13)$$

where $\phi_{iq_ik}(\boldsymbol{x}_i(t))$ is a membership function. ρ_i is the number of fuzzy model rules. $\boldsymbol{G}_{ijq_ik} \in R^{n_j \times n_i}$. We consider the following switching functions as candidates of Lyapunov functions for Subsystems $\boldsymbol{S}_i$ and $\boldsymbol{S}_j$, respectively.

$$\begin{aligned}
V_i(\boldsymbol{x}_i(t)) &= \sum_{q_i=1}^{Q_i} v_{iq_i}(\boldsymbol{x}_i(t))\,\boldsymbol{x}_i^T(t)\\
&\qquad \times\hat{\boldsymbol{E}}_i^T\boldsymbol{E}_{iq_i}^T\boldsymbol{P}_i\boldsymbol{E}_{iq_i}\hat{\boldsymbol{E}}_i\boldsymbol{x}_i(t) \quad (14)\\
V_j(\boldsymbol{x}_j(t)) &= \sum_{q_j=1}^{Q_j} v_{jq_j}(\boldsymbol{x}_j(t))\,\boldsymbol{x}_j(t)^T\\
&\qquad \times\hat{\boldsymbol{E}}_j^T\boldsymbol{E}_{jq_j}^T\boldsymbol{P}_j\boldsymbol{E}_{jq_j}\hat{\boldsymbol{E}}_j\boldsymbol{x}_j(t) \quad (15)
\end{aligned}$$

The whole system can be eventually stabilized if Lyapunov functions always decrease such as (16) when the switching from Subsystem $\boldsymbol{S}_i$ to $\boldsymbol{S}_j$ occurs.

$$V_i(\boldsymbol{x}_i(t)) > V_j(\boldsymbol{x}_j(t)) \quad (16)$$

Theorem 2: Lyapunov functions (14) and (15) satisfy (16) if there exist $\boldsymbol{X}_i = \boldsymbol{P}_i^{-1}$, $\boldsymbol{X}_j = \boldsymbol{P}_j^{-1}$ satisfying (17), (18) and (19).

$$\begin{aligned}
&\boldsymbol{X}_i > \boldsymbol{0}, &(17)\\
&\boldsymbol{X}_j > \boldsymbol{0}, &(18)\\
&\begin{bmatrix} \boldsymbol{X}_i & \left(\boldsymbol{E}_{jq_j}\tilde{\boldsymbol{G}}_{ijq_ik}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i\right)^T \\ \boldsymbol{E}_{jq_j}\tilde{\boldsymbol{G}}_{ijq_ik}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i & \boldsymbol{X}_j \end{bmatrix} > \boldsymbol{0},\\
&\qquad\qquad\qquad\qquad \forall q_i, q_j, k, &(19)
\end{aligned}$$

where

$$\tilde{\boldsymbol{G}}_{ijq_ik} = \begin{bmatrix} \boldsymbol{G}_{ijq_ik} & \boldsymbol{0}_{n_j\times n_i} \\ \boldsymbol{0}_{n_j\times n_i} & \boldsymbol{0}_{n_j\times n_i} \end{bmatrix}$$

(proof) From (13), (14), (15) and (16),

$$\begin{aligned}
&V_i(\boldsymbol{x}_i(t)) - V_j(\boldsymbol{x}_j(t))\\
&= \sum_{q_i=1}^{Q_i} v_{iq_i}(\boldsymbol{x}_i(t))\,\boldsymbol{x}_i^T(t)\hat{\boldsymbol{E}}_i^T\boldsymbol{E}_{iq_i}^T\boldsymbol{P}_i\boldsymbol{E}_{iq_i}\hat{\boldsymbol{E}}_i\boldsymbol{x}_i(t)\\
&\quad -\sum_{q_i=1}^{Q_i}\sum_{q_j=1}^{Q_j}\sum_{k_1=1}^{\rho_i}\sum_{k_2=1}^{\rho_i} v_{iq_i}(\boldsymbol{x}_i(t))\\
&\quad \times v_{jq_j}(\boldsymbol{x}_j(t))\phi_{iq_ik_1}(\boldsymbol{x}_i(t))\phi_{iq_ik_2}(\boldsymbol{x}_i(t))\\
&\quad \times\boldsymbol{x}_i^T(t)\boldsymbol{G}_{ijq_ik_2}^T\hat{\boldsymbol{E}}_j^T\boldsymbol{E}_{jq_j}^T\boldsymbol{P}_j\boldsymbol{E}_{jq_j}\hat{\boldsymbol{E}}_j\\
&\quad \times\boldsymbol{G}_{ijq_ik_1}\boldsymbol{x}_i(t)\\
&= \sum_{q_i=1}^{Q_i}\sum_{q_j=1}^{Q_j}\sum_{k_1=1}^{\rho_i}\sum_{k_2=1}^{\rho_i} v_{iq_i}(\boldsymbol{x}_i(t))\\
&\quad \times v_{jq_j}(\boldsymbol{x}_j(t))\phi_{iq_ik_1}(\boldsymbol{x}_i(t))\phi_{iq_ik_2}(\boldsymbol{x}_i(t))\\
&\quad \times\boldsymbol{x}_i^T(t)\hat{\boldsymbol{E}}_i^T\Big(\boldsymbol{E}_{iq_i}^T\boldsymbol{P}_i\boldsymbol{E}_{iq_i}\\
&\qquad -\tilde{\boldsymbol{G}}_{ijq_ik_2}^T\boldsymbol{E}_{jq_j}^T\boldsymbol{P}_j\boldsymbol{E}_{jq_j}\tilde{\boldsymbol{G}}_{ijq_ik_1}\Big)\hat{\boldsymbol{E}}_i\boldsymbol{x}_i(t)\\
&\geq \sum_{q_i=1}^{Q_i}\sum_{q_j=1}^{Q_j}\sum_{k=1}^{\rho_i} v_{iq_i}(\boldsymbol{x}_i(t))\\
&\quad \times v_{jq_j}(\boldsymbol{x}_j(t))\phi_{iq_ik}^2(\boldsymbol{x}_i(t))\\
&\quad \times\boldsymbol{x}_i^T(t)\hat{\boldsymbol{E}}_i^T\Big(\boldsymbol{E}_{iq_i}^T\boldsymbol{P}_i\boldsymbol{E}_{iq_i}\\
&\qquad -\tilde{\boldsymbol{G}}_{ijq_ik}^T\boldsymbol{E}_{jq_j}^T\boldsymbol{P}_j\boldsymbol{E}_{jq_j}\tilde{\boldsymbol{G}}_{ijq_ik}\Big)\hat{\boldsymbol{E}}_i\boldsymbol{x}_i(t)\\
&> 0 \qquad\qquad\qquad\qquad\qquad\qquad (20)
\end{aligned}$$

Inequality (20) is satisfied when the following condition is satisfied.

$$\boldsymbol{E}_{iq_i}^T\boldsymbol{P}_i\boldsymbol{E}_{iq_i} - \tilde{\boldsymbol{G}}_{ijq_ik}^T\boldsymbol{E}_{jq_j}^T\boldsymbol{P}_j\boldsymbol{E}_{jq_j}\tilde{\boldsymbol{G}}_{ijq_ik} > \boldsymbol{0} \quad (21)$$

By multiplying the inequality on the left by $\boldsymbol{X}_i\boldsymbol{E}_{iq_i}^{-T}$ and on the right by $\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i$, the following inequality is obtained.

$$\boldsymbol{X}_i - \boldsymbol{X}_i\boldsymbol{E}_{iq_i}^{-T}\tilde{\boldsymbol{G}}_{ijq_ik}^T\boldsymbol{E}_{jq_j}^T\boldsymbol{P}_j\boldsymbol{E}_{jq_j}\tilde{\boldsymbol{G}}_{ijq_ik}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i > \boldsymbol{0} \quad (22)$$

By applying Schur complement to (22), we can obtain (19).
(Q.E.D.)

Remark 2: Note that, even though subsystems switch regardless of switching conditions, Lyapunov functions (14) and (15) always satisfy (16) by using $\boldsymbol{X}_i$ and $\boldsymbol{X}_j$ satisfying Theorem 2. In other words, Theorem 2 includes switching condition (6).

From Theorems 1 and 2 and Remark 2, we can obtain the following theorem.

Theorem 3: The dynamic variable structure system (5) is stabilizable by the controller (9) if there exist matrices $\boldsymbol{X}_i$ and $\boldsymbol{M}_{iq_ik}$ satisfying (11), (12), (17), (18) and (19) simultaneously.

Remark 3: Note that conditions (11), (12), (17), (18) and (19) are given in terms of linear matrix inequalities (LMIs) with respect to matrices $\boldsymbol{X}_i$ and $\boldsymbol{M}_{iq_ik}$. Therefore, we can effectively design the controller by computer software like MATLAB.

C. Controller design with switching conditions

In this section, we consider switching conditions. As mentioned in Remark 2, Theorem 2 includes switching condition (6). However, essentially, the conservation laws should be applied only on the area satisfying the switching condition. In this point of view, Theorem 2 is conservative. In this section, we derive relaxed controller design conditions by utilizing the switching condition (6).

The switching condition (6) can be represented as follows by using a switching fuzzy model.

$$\begin{aligned}
0 &= \boldsymbol{C}_i\dot{\boldsymbol{x}}_i(t) + \xi_{ij}(\boldsymbol{x}_i(t)) \\
&= \sum_{q_i=1}^{Q_i}\sum_{k=1}^{r_i} v_{iq_i}(\boldsymbol{x}_i(t)) h_{iq_ik}(\boldsymbol{x}_i(t)) \\
&\qquad \times \boldsymbol{C}_i\left(\boldsymbol{A}_{iq_ik}\boldsymbol{x}_i(t) + \boldsymbol{B}_{iq_ik}\boldsymbol{u}_i(t)\right) \\
&\quad + \sum_{q_i=1}^{Q_i}\sum_{k=1}^{\lambda_i} v_{iq_i}(\boldsymbol{x}_i(t)) w_{iq_ik}(\boldsymbol{x}_i(t)) \boldsymbol{\Xi}_{ijq_ik}\boldsymbol{x}_i(t) \\
&= \sum_{q_i=1}^{Q_i}\sum_{k=1}^{r_i}\sum_{l=1}^{r_i} v_{iq_i}(\boldsymbol{x}_i(t)) h_{iq_ik}(\boldsymbol{x}_i(t)) h_{iq_il}(\boldsymbol{x}_i(t)) \\
&\qquad \times \boldsymbol{C}_i\left(\boldsymbol{A}_{iq_ik} - \boldsymbol{B}_{iq_ik}\tilde{\boldsymbol{F}}_{iq_il}\boldsymbol{E}_{iq_i}\hat{\boldsymbol{E}}_i\right)\boldsymbol{x}_i(t) \\
&\quad + \sum_{q_i=1}^{Q_i}\sum_{k=1}^{\lambda_i} v_{iq_i}(\boldsymbol{x}_i(t)) w_{iq_ik}(\boldsymbol{x}_i(t)) \boldsymbol{\Xi}_{ijq_ik}\boldsymbol{x}_i(t) \quad (23)
\end{aligned}$$

where $w_{iq_ik}(\boldsymbol{x}_i(t))$ is a membership function, λ_i is the number of fuzzy model rules, $\boldsymbol{\Xi}_{ijq_ik} \in R^{1\times n_i}$. In addition, we consider the following switching function.

$$y_i(\boldsymbol{x}_i(t)) = \sum_{q_i=1}^{Q_i} v_{iq_i}(\boldsymbol{x}_i(t))\boldsymbol{Y}_{iq_i}\boldsymbol{E}_{iq_i}\hat{\boldsymbol{E}}_i\boldsymbol{x}_i(t) \quad (24)$$

where $\boldsymbol{Y}_{iq_i} \in R^{1\times 2n_i}$. On the area satisfying the switching condition $\boldsymbol{x}_i(t) \in \chi_i$ ($\chi_i = \{\boldsymbol{x}_i(t) | \boldsymbol{C}_i\dot{\boldsymbol{x}}_i(t) + \xi_{ij}(\boldsymbol{x}_i(t)) = 0\}$), the following condition is satisfied.

$$\begin{aligned}
&{y_i}^T(\boldsymbol{x}_i(t))\left(\boldsymbol{C}_i\dot{\boldsymbol{x}}_i(t) + \xi_{ij}(\boldsymbol{x}_i(t))\right) \\
&+(\boldsymbol{C}_i\dot{\boldsymbol{x}}_i(t) + \xi_{ij}(\boldsymbol{x}_i(t)))^T y_i(\boldsymbol{x}_i(t)) = 0 \quad (25)
\end{aligned}$$

If the following condition is satisfied, both switching conditions and conservation laws are satisfied on the area satisfying the switching conditions.

$$\begin{aligned}
&V_i(\boldsymbol{x}_i(t)) - V_j(\boldsymbol{x}_j(t)) \\
&\quad +{y_i}^T(\boldsymbol{x}_i(t))\left(\boldsymbol{C}_i\dot{\boldsymbol{x}}_i(t) + \xi_{ij}(\boldsymbol{x}_i(t))\right) \\
&\quad +(\boldsymbol{C}_i\dot{\boldsymbol{x}}_i(t) + \xi_{ij}(\boldsymbol{x}_i(t)))^T y_i(\boldsymbol{x}_i(t)) > 0 \quad (26)
\end{aligned}$$

Theorem 4: If there exist $\boldsymbol{X}_i$, $\boldsymbol{X}_j$ $\boldsymbol{M}_{iq_ik}$ and $\boldsymbol{Z}_{iq_i}$ satisfying (27), (28) and (29), Inequality (26) is satisfied at any point $\boldsymbol{x}_i(t) \in \chi_i$.

$$\boldsymbol{X}_i > \boldsymbol{0}, \quad (27)$$

$$\boldsymbol{X}_j > \boldsymbol{0}, \quad (28)$$

$$\begin{bmatrix}
\begin{pmatrix}
\boldsymbol{X}_i \\
+\boldsymbol{Z}_{iq_i}^T\tilde{\boldsymbol{\Xi}}_{ijq_ik}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i \\
+(\boldsymbol{Z}_{iq_i}^T\tilde{\boldsymbol{\Xi}}_{ijq_ik}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i)^T \\
+\boldsymbol{Z}_{iq_i}^T\tilde{\boldsymbol{\mathcal{A}}}_{iq_ik}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i \\
+(\boldsymbol{Z}_{iq_i}^T\tilde{\boldsymbol{\mathcal{A}}}_{iq_ik}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i)^T \\
-\boldsymbol{Z}_{iq_i}^T\tilde{\boldsymbol{\mathcal{B}}}_{iq_ik}\boldsymbol{M}_{iq_ik} \\
-(\boldsymbol{Z}_{iq_i}^T\tilde{\boldsymbol{\mathcal{B}}}_{iq_ik}\boldsymbol{M}_{iq_ik})^T
\end{pmatrix} & * \\
\boldsymbol{E}_{jq_j}\tilde{\boldsymbol{G}}_{ijq_ik}\boldsymbol{E}_{iq_i}^{-1}\boldsymbol{X}_i & \boldsymbol{X}_j
\end{bmatrix} > \boldsymbol{0}, \quad (29)$$

$$\forall i, j, q_i, q_j, k,$$

where $\boldsymbol{Y}_{iq_i} = \boldsymbol{Z}_{iq_i}\boldsymbol{P}_i$

$$\tilde{\boldsymbol{\Xi}}_{ijq_ik} = \left[\boldsymbol{\Xi}_{ijq_ik}\ \boldsymbol{0}_{1\times n_i}\right],$$

$$\tilde{\boldsymbol{\mathcal{A}}}_{iq_ik} = \begin{bmatrix} \boldsymbol{C}_i\boldsymbol{A}_{iq_ik} & \boldsymbol{0} \\ \boldsymbol{0} & -\alpha\boldsymbol{I}_{n_i} \end{bmatrix},$$

$$\tilde{\boldsymbol{\mathcal{B}}}_{iq_ik} = \begin{bmatrix} \boldsymbol{C}_i\boldsymbol{B}_{iq_ik} \\ \boldsymbol{0} \end{bmatrix},$$

and the symbol $*$ denotes the transposed matrices for symmetric positions.

(proof) This is similar to the proof of Theorem 2.

Theorem 5: The dynamic variable structure system (5) is stabilizable by the controller (9) if there exist $\boldsymbol{X}_i$, $\boldsymbol{X}_j$ $\boldsymbol{M}_{iq_ik}$ and $\boldsymbol{Z}_{iq_i}$ satisfying (11), (12), (27), (28) and (29).

Remark 4: Unfortunately, the condition (29) is a bilinear matrix inequality (BMI), not an LMI. Therefore, to design the controller, we need to utilize some kind of techniques to solve BMI problems.

IV. EXAMPLE

For the system in Example 1, we design controllers using Theorem 3. By solving Theorem 3, we can obtain the

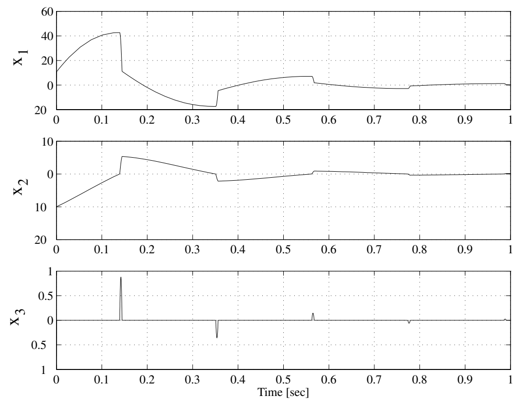

Fig. 2. Control result with considering switching

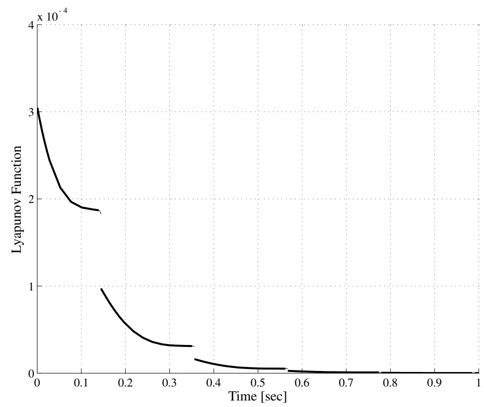

Fig. 3. Lyapunov function

following positive definite matrices and feedback gains.

$$\hat{\boldsymbol{E}}_1^T \boldsymbol{E}_1^T \boldsymbol{P}_1 \boldsymbol{E}_1^T \hat{\boldsymbol{E}}_1^T = \begin{bmatrix} 0.103 & 0 \\ 0 & 2.949 \end{bmatrix} \times 10^{-6}$$

$$\hat{\boldsymbol{E}}_2^T \boldsymbol{E}_2^T \boldsymbol{P}_2 \boldsymbol{E}_2^T \hat{\boldsymbol{E}}_2^T = \begin{bmatrix} 0.103 & 0 & 0 \\ 0 & 5.926 & 0.544 \\ 0 & 0.544 & 0.172 \end{bmatrix} \times 10^{-6}$$

$$\tilde{\boldsymbol{F}}_1 \boldsymbol{E}_1 \tilde{\boldsymbol{E}}_1 = \begin{bmatrix} -1.433 & 3.362 \end{bmatrix}$$

$$\tilde{\boldsymbol{F}}_2 \boldsymbol{E}_2 \tilde{\boldsymbol{E}}_2 = \begin{bmatrix} 1.267 & -2.515 & -0.441 \\ 0.301 & 175.405 & 18.164 \end{bmatrix}$$

Figures 2 and 3 show simulation results. The controller satisfying conservation laws can stabilize the dynamic variable structure system. As shown in Figure 3, the Lyapunov function is discontinuous, but monotonically decrease.

V. Conclusions

In this paper, we have proposed a dynamic variable structure system whose subsystems' degrees of freedom can change before and after switching. We have shown that only considering each subsystem's stability cannot guarantee stability of the dynamic variable structure system. To stabilize the system, we have derived controller design conditions which can achieve not only stability of each subsystem but also switching conditions and conservation laws.

A human elbow joint is regarded as 2-link manipulator while bending and 1-link manipulator while stretching. Therefore, elbow motions can be represented as a dynamic variable structure system. Our future work is to design controllers to stabilize such kind of elbow motions.

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