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Differential Geometric Structures in Vehicle Lane Keeping and Roll Mitigation

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***Abstract*— This paper uses a geometric framework to create a control scheme designed to prevent road departure and vehicle roll-over. The system is based on making the closed-loop dynamics analogous to a dissipative mechanical system with a degenerate kinetic energy (a kinetic energy that is zero for some nonzero velocities), using the concept of metric compatible affine connections. Some propositions are given concerning the possibility of implementing the desired closed loop dynamics when the system is underactuated. They are applied to the task of preventing vehicle roll-over and road departure, where the system is underactuated due to the lack of direct actuation of the roll mode. An example from simulation is presented for illustration.**

I. Introduction

It is well known that the dynamics of a mechanical system are closely related to the geometric structures arising from the Riemannian metric defined by its kinetic energy. A convenient feature of dissipative mechanical systems is that the total energy (kinetic energy plus potential energy) is never increasing [5], [7]. This has led to efforts to implement control laws that would make the closed-loop system behave like mechanical systems whose kinetic and potential energies have desirable characteristics [1].

This paper presents an application of this paradigm to a system designed to prevent roll-over and lane departure in road vehicles. The idea is to use the concept of metric compatible connections to make the system dissipate a quantity which can be interpreted as the energy of the error.

In the usual notion of energy for mechanical systems, where the total energy is the sum of the kinetic energy and the potential energy, the kinetic energy is defined by the mass matrix, which is a symmetric, positive definite bilinear form, or a Riemannian metric [4]. The strategy for control adopted here is to use the inputs to implement a closed-loop system which dissipates what can be considered the energy of the error, which is the sum of a kinetic energy-like term and a potential energy-like term. In lane keeping and roll-over prevention, however, the energy of the error should be zero when the vehicle is traveling straight down the center of the lane, even with non-zero velocity. This means that the kinetic energy-like term should be based on a positive semidefinite, rather than positive definite, bilinear form.

Section II describes the model of vehicles under consideration. Section III briefly describes the framework of degenerate mechanical systems to deal with the idea of allowing nonzero velocities to have zero kinetic energy. A hurdle that arises in the application to roll-over prevention is the fact that the system is underactuated; since the control inputs arise solely from tire forces, the roll mode is not directly actuated. Section IV lays out some propositions concerning which closed loop systems are possible to implement, accomodating both the fact that the "metric" may be degenerate and the possibility that the system is underactuated. Section V describes the details of the control law, using results from Section IV. Finally, an example is presented in Section VI to illustrate this control strategy.

II. Vehicle Model

As a starting point, this section briefly presents a model of a road vehicle which captures position, heading, and roll. The vehicle is modeled as two rigid bodies, B_1 and B_2, as illustrated in figures 1 and 2. Body B_1 represents the unsprung mass, and is limited to planar motion; it has mass m_1 and moment of inertia I_{1zz}. (x, y) describes the position of the center of mass of B_1 in the plane, and ψ is the heading angle. Body B_2 represents the sprung mass, and it has one degree of freedom for motions relative to B_1, parametrized by ρ. The mass of B_2 is m_2, and its inertia matrix is I_2. The position of the center of mass of B_1 relative to that of B_2 in the vertical/lateral plane is represented by $h(\rho)$, $\phi(\rho)$, as shown in figure 1, and the longitudinal distance between the two centers of mass is $d(\rho)$, as figure 2 depicts. The orientation of B_2 relative to B_1 is given by the rotation matrix $R^{12}(\rho)$.

As an example, if the sprung mass is assumed to rigidly rotate about the longitudinal axis of the unsprung mass, d would be how far the center of mass of B_1 is ahead (longitudinally) of the center of mass of B_2, h would be the distance from the center of mass of B_2 to the roll axis, and $\phi(\rho)$ would be the roll angle; d and h would be constant and

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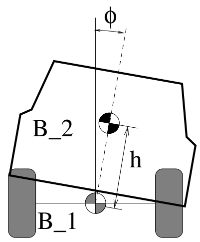

Fig. 1. Vehicle model

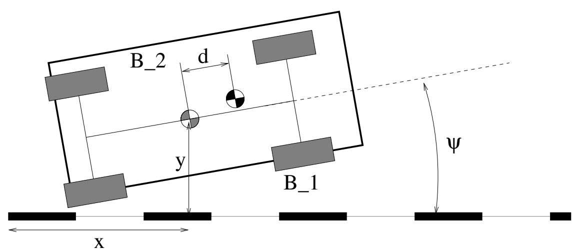

Fig. 2. Vehicle model

$\phi(\rho)$ would simply be equal to ρ. In general, the suspension geometry of the vehicle determines $d(\rho)$, $h(\rho)$, $\phi(\rho)$ and $R^{12}(\rho)$.

The system can be parametrized by $q = (x, y, \psi, \rho)$. In this coordinate system, the mass matrix $M(q)$ is given by

$$M(q) = \begin{bmatrix} (m_1 + m_2)I_{2\times 2} & m_2 R(\psi)D(\rho) \\ m_2 D(\rho)^T R(\psi)^T & E(\rho) \end{bmatrix}, \quad (1)$$

where

$$R(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}, \quad (2)$$

$$D(\rho) = \begin{bmatrix} h(\rho)\sin\phi(\rho) & d'(\rho) \\ d(\rho) & -h'(\rho)\sin\phi(\rho) - h(\rho)\phi'(\rho)\cos\phi(\rho) \end{bmatrix}, \quad (3)$$

$$\begin{aligned} E(\rho) = &\begin{bmatrix} I_{1zz} & 0 \\ 0 & m_2\eta(\rho) \end{bmatrix} + m_2 D(\rho)^T D(\rho) \\ &+ J_b(\rho)^T I_2 J_b(\rho), \end{aligned} \quad (4)$$

$J_b(\rho)$ is the body angular Jacobian of B_2, and $\eta = (h'\cos\phi - h\phi'\sin\phi)^2$.

The trajectories of the system evolve according to

$$\frac{\nabla\dot{q}}{dt} + M^{-1}dV = M^{-1}F_d + M^{-1}\tau \quad (5)$$

where ∇ is the Levi-Civita connection defined by M (for details, a text on Riemannian geometry like [4] can be consulted), $V(\rho)$ is the potential energy, F_d is the damping force from the motion of B_2 relative to B_1, and τ is the resultant generalized force from the tires. For this paper, it is assumed that $V''(0) > 0$, and the damping force F_d is assumed to take the form $F_d(\dot{q}) = -K_d\dot{\rho}d\rho$, where K_d is a positve constant, that is, linear damping for relative motion of the two bodies.

Since τ is generated by the tires, $\tau = \tau_1 dx + \tau_2 dy + \tau_3 d\psi$. For this paper, any such $\tau \in \mathrm{span}\{dx, dy, d\psi\}$ is assumed to be available via control of the throttle, steering, and differential braking; of course, in reality, there are limitations on the magnitude of τ that must be considered.

In coordinates, (5) can be expressed as

$$\begin{aligned} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\psi} \\ \ddot{\rho} \end{bmatrix} + C(\dot{x}, \dot{y}, \dot{\psi}, \dot{\rho}) + \frac{\partial V}{\partial \rho} M^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} &= \\ - K_d\dot{\rho} M^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} &+ M^{-1} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ 0 \end{bmatrix} \end{aligned} \quad (6)$$

where C represents accelerations due to centrifugal and Coriolis forces.

Assuming the center of the lane is the x-axis, the object of the controller is to keep the values of y, ψ, and ρ sufficiently close to zero. Thus, the energy of the error should be zero if and only if $y = \psi = \rho = 0$ and $\dot{y} = \dot{\psi} = \dot{\rho} = 0$; this would require the kinetic energy to be based on a positive semidefinite bilinear form because it shouldn't be affected by the value of $\dot{x}$.

If the lateral displacement, heading, and roll can be controlled independently, then any second order behavior for (y, ψ, ρ) can be effected; this is not the case, however, because actuation of the roll mode is coupled to actuation of the lateral displacement. In other words, the system is underactuated because the span of $M^{-1}dx$, $M^{-1}dy$, and $M^{-1}d\psi$ does not contain the span of $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial \psi}$, and $\frac{\partial}{\partial \rho}$, so some care must be taken to ensure that the desired closed-loop dynamics are implementable.

The next two sections describe a more general framework for working with semidefinite kinetic energy forms for underactuated systems.

III. Degenerate Mechanical Systems

To accomodate the possibility of non-zero generalized velocities for which the kinetic energy is zero, the notion of a degenerate mechanical system, which also appears in [3], is described here. The basic structures do not differ much from the usual, nondegenerate case. The account here and in the following sections relies on notation from differential geometry, which can be found in textbooks like [4]. For example, $\Gamma(E)$ is a section of the vector bundle E.

Definition 1: A *degenerate metric* of rank r on an n-dimensional configuration manifold Q is a symmetric positive semidefinite $G \in \Gamma(T^*Q \otimes T^*Q)$ with an $n - r$ dimensional null space. As a matter of notation, $G(X, Y)$ will be written $\langle X, Y\rangle_G$. Since G is of constant rank, its null space is a subbundle of TQ, which will be denoted $\mathcal{N}_G$, or just $\mathcal{N}$ when there is no danger of confusion.

In coordinates, G can be represented as an $n \times n$ matrix valued function of Q which is symmetric, positive semidefinite, and of rank r at each point of Q. A degenerate metric of rank n is also known as a Riemannian metric.

The concept of metric compatibility for affine connections is defined in much the same way as in the nondegenerate case.

Definition 2: An affine connection D is *compatible* with a degenerate metric G if $X\langle Y, Y\rangle_G = 2\langle D_X Y, Y\rangle_G$ for all $X, Y \in \Gamma(TQ)$, or, equivalently, $X\langle Y, Z\rangle_G = \langle D_X Y, Z\rangle_G + \langle Y, D_X Z\rangle_G$ for all $X, Y, Z \in \Gamma(TQ)$.

For a Riemannian metric, the classic example of a compatible connection is the Levi-Civita connection.

A notion of compatibility can also be defined for functions.

Definition 3: A function $V \in C^\infty(Q)$ is *compatible* with a degenerate metric G if $\langle dV, W\rangle = 0$ for all W such that $\langle W, W\rangle_G = 0$, or equivalently, there exists grad $V \in \Gamma(TQ)$ such that Ggrad $V = dV$.

This definition is vacuous in the nondegenerate case, because every smooth function is compatible with any Riemannian metric. In the degenerate case, it creates a class of functions suitable for use as the "potential energy" part of a total energy dissipated by a degenerate mechanical system, as defined below.

Definition 4: A *degenerate mechanical system* is a system whose trajectories obey

$$\frac{D\dot{q}}{dt} + \text{grad } V = F(\dot{q}), \tag{7}$$

where D is an affine connection compatible with a degenerate metric G, V is a function compatible with G, and $\langle \dot{q}, F(\dot{q})\rangle_G$ is always nonpositive.

As with usual dissipative mechanical systems, the total energy $\frac{1}{2}\langle \dot{q}, \dot{q}\rangle_G + V$ of a degenerate mechanical system is always nonincreasing [3]. Taking $\frac{1}{2}\langle \dot{q}, \dot{q}\rangle_G + V$ as the energy of error, the fact that it is always nonincreasing guarantees that the error is bounded. It makes sense, then, to use control inputs to make the closed-loop system a degenerate mechanical system whose energy represents error energy. This is always possible when the space of available control inputs and the null space of the desired degenerate metric add up to all of TQ, but otherwise, as in the case of the application to road departure and roll-over prevention, care needs to be taken to find implementable degenerate mechanical systems.

The next section gives some conditions the desired closed-loop degenerate metric must satisfy.

IV. Kinetic Energy Shaping

Given an original system of the form

$$\frac{D_1\dot{q}}{dt} + G_1^{-1}dV_1 = F_1(\dot{q}) + u, \tag{8}$$

where the control input u must lie in a subbundle $\mathcal{C}$ of TQ, (7) is feasible if $\frac{D\dot{q}}{dt} - \frac{D_1\dot{q}}{dt} \in \mathcal{C}$, grad $V - G_1^{-1}dV_1 \in \mathcal{C}$, and $F(\dot{q}) - F_1(\dot{q}) \in \mathcal{C}$. In other words, in order for (7) to describe the closed loop dynamics, the corrections necessary must lie in the space of available control inputs $\mathcal{C}$.

This section puts forward some propositions to determine when, given a desired G, there exists an affine connection D such that $\frac{D\dot{q}}{dt} - \frac{D_1\dot{q}}{dt} \in \mathcal{C}$. The general idea is to study the behavior of D_1 on the null space of G and the orthogonal complement of $\mathcal{C}$ under G. These propositions are used in Section V to demonstrate the feasibility of controlling lane departure and roll by making the closed loop behave as a degenerate mechanical system

TQ can be decomposed as the direct sum of four subbundles, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{D}_1$, $\mathcal{D}_2$, where $\mathcal{C}_2 = \mathcal{N} \cap \mathcal{C}$, $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$, $\mathcal{N} = \mathcal{C}_2 \oplus \mathcal{D}_2$, and $\langle \mathcal{C}, \mathcal{D}_1\rangle_G = 0$. Given this decomposition, let $\bar{\sigma}_1$, $\bar{\sigma}_2$, σ_1, and σ_2 be the projections onto $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{D}_1$, $\mathcal{D}_2$, respectively, and let $\bar{\sigma} = \bar{\sigma}_1 + \bar{\sigma}_2$ and $\sigma = \sigma_1 + \sigma_2$.

The following lemma lies at the heart of the results of this paper.

Lemma 1: An affine connection D is compatible with G if and only if

1) $X\langle \bar{\sigma}Y, \bar{\sigma}Y\rangle_G = 2\langle D_X\bar{\sigma}Y, \bar{\sigma}Y\rangle_G$,
2) $X\langle \sigma Y, \sigma Y\rangle_G = 2\langle D_X\sigma Y, \sigma Y\rangle_G$, and
3) $\langle \bar{\sigma}D_X\sigma Y, \bar{\sigma}Y\rangle_G + \langle \sigma D_X\bar{\sigma}Y, \sigma Y\rangle_G = 0$

for all vector fields $X, Y \in \Gamma(TQ)$.

This lemma is straightforward to prove, and a proof is omitted here.

From Lemma 1 follows a criterion for when corrections in $\mathcal{C}$ can make a connection compatible with G.

Proposition 1: Given an affine connection $\tilde{D}$, there exists $\Delta \in \Gamma(T^*Q \otimes T^*Q \otimes \mathcal{C})$ such that $\tilde{D} + \Delta$ is compatible with G if and only if

1) $X\langle \sigma Y, \sigma Y\rangle_G = 2\langle \tilde{D}_X\sigma Y, \sigma Y\rangle_G$ and
2) $\sigma\tilde{D}_X W \in \Gamma(\mathcal{N})$,

for all $X, Y \in \Gamma(TQ)$ and $W \in \Gamma(\mathcal{N})$.

Proof: Suppose $\tilde{D}$ satisfies both conditions 1 and 2. Define $A \in \Gamma(T^*Q \otimes T^*Q \otimes \mathcal{C})$ via

$$\langle A_XY, Z\rangle_G = -\langle \bar{\sigma}\tilde{D}_X\sigma Y, \bar{\sigma}Z\rangle_G - \langle \sigma\tilde{D}_X\bar{\sigma}Z, \sigma Y\rangle_G. \tag{9}$$

This is possible because condition 2 implies that the right hand side is zero for all $Z \in \Gamma(\sigma(TQ) + \mathcal{N})$, and the right hand side is $C^\infty(Q)$-linear in X, Y, and Z. Similarly, it is possible to define $B \in \Gamma(T^*Q \otimes T^*Q \otimes \mathcal{C})$ via

$$\begin{aligned}2\langle B_XY, Z\rangle_G = X\langle \bar{\sigma}Y, \bar{\sigma}Z\rangle_G - \langle \tilde{D}_X\bar{\sigma}Y, \bar{\sigma}Z\rangle_G \\ - \langle \bar{\sigma}Y, \tilde{D}_X\bar{\sigma}Z\rangle_G + \langle \bar{\sigma}Y, \tilde{D}_X\bar{\sigma}_2 Z\rangle_G \\ - \langle \tilde{D}_X\bar{\sigma}_2 Y, \bar{\sigma}Z\rangle_G.\end{aligned} \tag{10}$$

Set $\Delta_XY = A_X\sigma Y + B_X\bar{\sigma}Y$, and let $D_XY = \tilde{D}_XY + \Delta_XY$.

It is easy to check that D satisfies all three conditions of Lemma 1, and thus establish the "if" part of the proposition. Since it is unnecessary for the rest of the paper, the proof of the converse is omitted. ■

Since D only matters in the form $\frac{D\dot{q}}{dt}$, any anti-symmetric correction is also possible. The following proposition builds on Proposition 1 for the special case when $\dim TQ = \dim(\mathcal{C} + \mathcal{N}) + 1$.

Proposition 2: Suppose $\mathcal{D}_1 = \text{span}(Z)$, for some nonzero $Z \in \Gamma(TM)$.

Let $G_1 \in \Gamma(T^*M \otimes T^*M)$ be a symmetric, positive semidefinite, constant rank tensor whose null space is contained in $\mathcal{N}$, and let D_1 be an affine connection compatible with G_1.

There exists an affine connection D compatible with G such that $D_XX - D_{1X}X \in \Gamma(\mathcal{C})$ for all $X \in \Gamma(TM)$ if and only if

1) $\sigma_1 D_{1W} W = 0$,
2) $\frac{1}{2}\langle Z, Z\rangle_{G_1} Z\lambda + \langle D_{1Z}Z, Z\rangle_{G_1} = \langle \sigma_1 D_{1Z} Z, Z\rangle_{G_1}$, and
3) $\frac{1}{2}\langle Z, Z\rangle_{G_1} W\lambda + \langle D_{1W}Z, Z\rangle_{G_1} = \langle \sigma_1(D_{1Z}W + D_{1W}Z), Z\rangle_{G_1}$,

for all $W \in \Gamma(\mathcal{N})$, where $\alpha = \frac{\langle Z,Z\rangle_G}{\langle Z,Z\rangle_{G_1}}$ and $\lambda = \log\alpha$.

Proof: Suppose conditions 1, 2, and 3 hold.

Define $A, B_1, B_2 \in \Gamma(T^*M \otimes T^*M)$ via

$$B_{1X}Y = \langle \tfrac{1}{2}(X\log\alpha)Z + (\bar\sigma + \sigma_2)D_{1X}Z, Y\rangle_{G_1} \quad (11)$$

$$B_{2X}Y = \tfrac{1}{2}\langle \sigma_1 D_{1\nu X}\nu Y, Z\rangle_{G_1} - \langle \sigma_1 D_{1X}\nu Y, Z\rangle_{G_1} \quad (12)$$

$$A_X Y = \frac{B_{1\bar\nu X}\sigma_1 Y - B_{1\bar\nu Y}\sigma_1 X}{\langle Z, Z\rangle_{G_1}} + \frac{B_{2X}Y - B_{2Y}X}{\langle Z, Z\rangle_{G_1}} \quad (13)$$

where $\nu = \sigma_2 + \bar\sigma_2$ and $\bar\nu = \sigma_1 + \bar\sigma_1 = \mathrm{id} - \nu$, and let $\tilde D$ be the affine connection defined by $\tilde D_X Y = D_{1X}Y + (A_X Y)Z$.

Straightforward computations yield $\sigma_1 \tilde D_X W = 0$ and $X\langle Z, Z\rangle_G = 2\langle \tilde D_X Z, Z\rangle_G$ from which it can be deduced that $X\langle \sigma Y, \sigma Y\rangle_G = 2\langle \tilde D_X \sigma Y, \sigma Y\rangle_G$. The "if" direction of the proposition then follows from Proposition 1.Again, because the converse is unnecessary for the rest of the paper, its proof is omitted. ■

Thus, if $TQ = \mathrm{span}(Z) \oplus \mathcal{N} \oplus \mathcal{C}_1$ satisfies the three conditions in Proposition 2 for some function λ, there exists an implementable affine connection D such that D is compatible with the degenerate metric G defined by $\langle X, Y\rangle_G = e^\lambda\langle \sigma_1 X, \sigma_1 Y\rangle_{G_1} + \langle \bar\sigma_1 X, \bar\sigma_1 Y\rangle_g$ for any nondegenerate metric g.

V. Lane Departure and Roll-over Prevention

Returning to the object of preventing lane departure and roll-over, recall that the original system, as described in Section II, obeys

$$\frac{\nabla \dot q}{dt} + M^{-1}dV = M^{-1}F_d + u, \quad (14)$$

where u must lie in the span of $M^{-1}dx$, $M^{-1}dy$, and $M^{-1}d\psi$. The goal is to choose u such that the closed-loop system obeys

$$\frac{D\dot q}{dt} + \mathrm{grad}\ U = F(\dot q), \quad (15)$$

where the affine connection D and the function U are compatible with a degenerate metric G, and $\langle \dot q, F(\dot q)\rangle_G$ is always nonpositive.

The strategy is to first choose what the null space and the orthogonal complement of $\mathcal{C}$ should be under G, then construct G and D using results from the previous section.

The degenerate metric G should be designed so that $\langle \dot q, \dot q\rangle_G = 0$ if and only if $\dot y = \dot\psi = \dot\rho = 0$, and the function U should be designed so that the smaller U is, the closer y, ψ, and ρ are to zero.

This implies the null space of G should be the span of $\frac{\partial}{\partial x}$. It can be shown that $\frac{\partial}{\partial x}$ is in the span of $M^{-1}dx$ and $M^{-1}d\psi$ when $d'\cos\psi = (h'\sin\phi + h\phi'\cos\phi)\sin\psi$, so the space of available control inputs and the null space cannot span all of TQ. However, $\frac{\partial}{\partial x}$, $M^{-1}dy$, and $M^{-1}d\psi$ are always independent, so $\dim TQ = \dim(\mathcal{N} + \mathcal{C}) + 1$, where $\mathcal{N} = \mathrm{span}(\frac{\partial}{\partial x})$ and $\mathcal{C} = \mathrm{span}\{M^{-1}dy, M^{-1}d\psi\}$. Thus, the results of Section IV can be used to find a feasible G.

Proposition 3: Choose any symmetric positive definite $g \in \Gamma(T^*Q \otimes T^*Q)$, any positive $\alpha_0 \in \mathbb{R}$, and any $Z \in \Gamma(TQ)$ such that $[\frac{\partial}{\partial x}, Z] = 0$, and $\langle v, Z\rangle_M \neq 0$, where v is any vector orthogonal to $\mathcal{N} + \mathcal{C}$ with respect to M. Let σ_1, σ_2, $\bar\sigma$ be the projections to $\mathrm{span}(Z)$, $\mathcal{N}$, $\mathcal{C}$, respectively, according to the decomposition $TQ = \mathrm{span}(Z) \oplus \mathcal{N} \oplus \mathcal{C}$.

There exists, near $\{y = \psi = \rho = 0\}$, a degenerate metric G, an affine connection D compatible with G, a function U compatible with G, and a tensor $K \in \Gamma(T^*Q \otimes TQ)$ such that

1) $\frac{D\dot q}{dt} - \frac{\nabla \dot q}{dt}$ is in $\mathcal{C}$
2) grad $U - M^{-1}dV$ is in $\mathcal{C}$
3) The Hessian of $U(y, \psi, \rho)$ (as a function of y, ψ, and ρ) is positive definite at $y = \psi = \rho = 0$
4) $K\dot q - M^{-1}F_d$ is in $\mathcal{C}$
5) $\langle \dot q, K\dot q\rangle_G \leq 0$
6) $\langle \bar\sigma X, \bar\sigma Y\rangle_G = \langle \bar\sigma X, \bar\sigma Y\rangle_g$
7) $\langle Z, Z\rangle_G = \alpha_0\langle Z, Z\rangle_M$ on $\{y = \psi = \rho = 0\}$

Proof: Let $v = -\frac{M_{14}}{m_1+m_2}\frac{\partial}{\partial x} + \frac{\partial}{\partial\rho}$. Observe that $\langle v, \mathcal{N} + \mathcal{C}\rangle_M = 0$. Let $w \in \Gamma(TM)$ be a vector field such that $\langle w, \mathrm{span}(Z) + \mathcal{C}\rangle_M = 0$. Then $\sigma_1 X = \frac{\langle v,X\rangle_M}{\langle v,Z\rangle_M}Z$, $\sigma_2 X = \frac{\langle w,X\rangle_M}{\langle v,\frac{\partial}{\partial x}\rangle_M}\frac{\partial}{\partial x}$, and $\bar\sigma X = X - \sigma_1 X - \sigma_2 X$.

Define G via $\langle X, Y\rangle_G = \alpha\langle \sigma_1 X, \sigma_1 Y\rangle_M + \langle \bar\sigma X, \bar\sigma Y\rangle_g$.

From straightforward calculations, $\nabla_X\frac{\partial}{\partial x} = 0$ for all X, and $\nabla_{\frac{\partial}{\partial x}}Z = 0$. Thus, the conditions of Proposition 2 reduce to $\frac{\partial}{\partial x}\lambda = 0$,

$$Z\lambda = \frac{2\langle(\sigma_1 - \mathrm{id})\nabla_Z Z, Z\rangle_M}{\langle Z, Z\rangle_M}, \quad (16)$$

and $\lambda = \log\alpha$. It can be shown that the right hand side of (16) is a function only of y, ψ, and ρ, and that $Z = Z^1(y,\psi,\rho)\frac{\partial}{\partial x} + Z^2(y,\psi,\rho)\frac{\partial}{\partial y} + Z^3(y,\psi,\rho)\frac{\partial}{\partial\psi} + Z^4(y,\psi,\rho)\frac{\partial}{\partial\rho}$, so there is a solution $\lambda(y,\psi,\rho)$ to the linear PDE above (via, for example, the method of characteristics) with $\lambda(0,0,0) = \log\alpha_0$. Conditions 1, 6, and 7 then follow from Proposition 2.

Since $[\frac{\partial}{\partial x}, Z] = 0$, there exists a coordinate system (u^1, u^2, u^3, u^4) such that $u^1 = u^2 = u^3 = u^4 = 0$ corresponds to $x = y = \psi = \rho = 0$, $\frac{\partial}{\partial u^1} = \frac{\partial}{\partial x}$, and $\frac{\partial}{\partial u^4} = Z$. Then $\langle \mathrm{span}(Z) + \mathcal{N}, \mathcal{C}\rangle_G = 0$ implies that $G\mathcal{C} = \mathrm{span}\{du^2, du^3\}$.

Let $\zeta = \langle M^{-1}dV, Z\rangle_G = \frac{\alpha\langle v, M^{-1}dV\rangle_M\langle Z,Z\rangle_M}{\langle v,Z\rangle_M} = \alpha V'\frac{\langle Z,Z\rangle_M}{\langle v,Z\rangle_M}$, and let $f(u^1,u^2,u^3,u^4) = \int_0^{u^4}\zeta(u^1,u^2,u^3,t)dt$. It can be shown that $\frac{\partial\zeta}{\partial x} = \frac{\partial\zeta}{\partial u^1} = 0$, so $df = \int\frac{\partial\zeta}{\partial u^2}dtdu^2 + \int\frac{\partial\zeta}{\partial u^3}dtdu^3 + \zeta du^4$. Since $GM^{-1}dV = \langle M^{-1}dV, \frac{\partial}{\partial u^2}\rangle_G du^2 + \langle M^{-1}dV, \frac{\partial}{\partial u^3}\rangle_G du^3 + \langle M^{-1}dV, \frac{\partial}{\partial u^4}\rangle_G du^4$, $df - GM^{-1}dV$ is in the span of du^2 and du^3, so there exists $Y \in \Gamma(\mathcal{C})$ such that

$G(M^{-1}dV + Y) = df$. Let $\bar{f} = k((u^2)^2 + (u^3)^2)$. Since $d\bar{f}$ is in the span of du^2 and du^3, there exists $\bar{Y} \in \Gamma(\mathcal{C})$ such that $G(M^{-1}dV + Y + \bar{Y}) = d(f + \bar{f})$. When $y = \psi = \rho = 0$,

$$\frac{\partial^2(f+\bar{f})}{\partial(x^4)^2} = Z\langle M^{-1}dV + Y + \bar{Y}, Z\rangle_G \tag{17}$$

$$= Z(\alpha\langle v, M^{-1}dV\rangle_M \frac{\langle Z, Z\rangle_M}{\langle v, Z\rangle_M}) \tag{18}$$

$$= \alpha\frac{\langle Z, Z\rangle_M}{\langle v, Z\rangle_M} Z\langle v, M^{-1}dV\rangle_M \tag{19}$$

$$= \alpha\frac{\langle Z, Z\rangle_M}{\langle v, Z\rangle_M} ZV' = \alpha\frac{\langle Z, Z\rangle_M}{\langle v, Z\rangle_M} V''(0), \tag{20}$$

and $\frac{\partial^2(f+\bar{f})}{\partial(x^i)^2} = k + \frac{\partial^2 f}{\partial(x^i)^2}$ for $i = 2, 3$. Thus, if $V''(0) > 0$, the Hessian of $f + \bar{f}$, as a function of u^2, u^3, u^4, is positive definite for large enough k. Since $\frac{\partial\zeta}{\partial x} = 0$, $\frac{\partial f+\bar{f}}{\partial x} = 0$, so $f + \bar{f}$ is compatible with G.

This covers conditions 2 and 3.

Let $\bar{F}_d = -\bar{\sigma}(\dot{q} + M^{-1}F_d)$, and let $K\dot{q} = M^{-1}F_d + \bar{F}_d$. Then

$$\langle \dot{q}, K\dot{q}\rangle_G = \langle \dot{q}, \sigma_Z M^{-1}F_d\rangle_G - \langle \dot{q}, \bar{\sigma}\dot{q}\rangle_G \tag{21}$$

$$= -K_d\alpha\frac{\langle Z, Z\rangle_M}{\langle v, Z\rangle_M}\dot{\rho}^2 - \langle \bar{\sigma}\dot{q}, \bar{\sigma}\dot{q}\rangle_G, \tag{22}$$

so $\langle \dot{q}, K\dot{q}\rangle_G$ is always non-positive, and conditions 4 and 5 are covered. ∎

Let the "error energy" be the quantity $H = \frac{1}{2}\langle \dot{q}, \dot{q}\rangle_G + U(q)$. Since the object is to keep U below some value c and H is non-increasing when (15) is enforced, interference with the driver commands can be minimized by turning the controls on, so to speak, only when the error energy gets too high. That is, τ is set to τ_{cl} when $H \geq c$ and τ is set to the driver command when $H < c$, where τ_{cl} enforces (15). The harshness of the transition can be reduced by letting $\tau = \mu(H)\tau_{cl} + (1 - \mu(H))\tau_d$, where μ is a continuous function such that $\mu(H) = 0$ for $H \leq c_0$ and $\mu(H) = 1$ for $H \geq c$, for some $c_0 < c$.

The next section describes a specific example applying the more general result of Proposition 3.

VI. Example

The specific vehicle model used is the simple case when the center of mass of B_2 sits directly above the center of mass of B_1 when $\rho = 0$ and B_2 simply rotates about the longitudinal axis of B_1 which goes through the center of mass of B_1. Then $d(\rho) = 0$, $h(\rho) = 0.6m$, $\phi(\rho) = \rho$, and $J_b(\rho) = \begin{bmatrix} 0 & 1 \\ \sin\phi & 0 \\ \cos\phi & 0 \end{bmatrix}$. The inertia matrix of B_2 is $I_2 = \begin{bmatrix} I_{2x} & 0 & 0 \\ 0 & I_{2y} & 0 \\ 0 & 0 & I_{2z} \end{bmatrix}$. The potential energy is $V(\rho) = \frac{1}{2}K_r\phi^2$.

In this example, Z is chosen to be

$$Z = -\frac{M_{14}}{m_1+m_2}\frac{\partial}{\partial x} + \frac{\partial}{\partial\rho} = -\frac{h\sin\psi\cos\phi}{m_1+m_2}\frac{\partial}{\partial x} + \frac{\partial}{\partial\rho}. \tag{23}$$

This reduces (16) to $\frac{\partial\lambda}{\partial\rho} = 0$. Choosing $\lambda = 1$ and $g = M$, the resulting G is given by $\langle X, Y\rangle_G = e\langle\sigma_1 X, \sigma_1 Y\rangle_M + \langle\bar{\sigma}X, \bar{\sigma}Y\rangle_M$, and can be represented by the matrix

$$M - \gamma M \begin{bmatrix} \frac{m_2^2 h^2\sin^2\psi\cos^2\phi}{m_1+m_2} & 0 & 0 & -\frac{m_2 h\sin\psi\cos\phi}{m_1+m_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{m_2 h\sin\psi\cos\phi}{m_1+m_2} & 0 & 0 & 1 \end{bmatrix} M, \tag{24}$$

where $\gamma = \frac{(m_1+m_2)(e-1)}{(m_1+m_2)(I_{2x}+m_2h^2) - m_2^2h^2\sin^2\psi\cos^2\phi}$.

The closed loop potential energy is $U(y, \psi, \rho) = e^\lambda V(\rho) + \frac{1}{2}k_y y^2 + \frac{1}{2}k_\psi\psi^2$. The constants k_y and k_ψ are tuned so that $U \leq c$ implies $-0.25m \leq y \leq 0.25m$, $-\pi/2 \leq \psi \leq \pi/2$, and $-0.1\text{rad} \leq \rho \leq 0.1\text{rad}$.

The function μ is chosen to be $\mu(H) = 0$ for $H \leq 0.7c$, and $\mu(H) = \min\{\frac{H-0.7c}{0.3c}, 1\}$.

Figures 3-7 are results of a simulation representing what might happen if a driver tries to make fast changes in direction, perhaps in an attempt to dodge an obstacle. The driver is crudely simulated as a PD controller tracking the vehicle's displacement from lane center which is subjected to an input veering away from the lane center and back.

Figures 3 and 4 show the trajectory of the vehicle and the vehicle roll with and without the controller. The controller ensures that the vehicle stays with $0.25m$ of the lane center ($y = 0$) and that the roll angle never exceeds 0.1 radians.

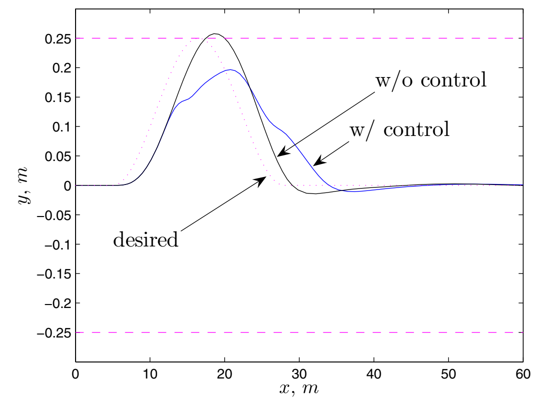

Fig. 3. Trajectory

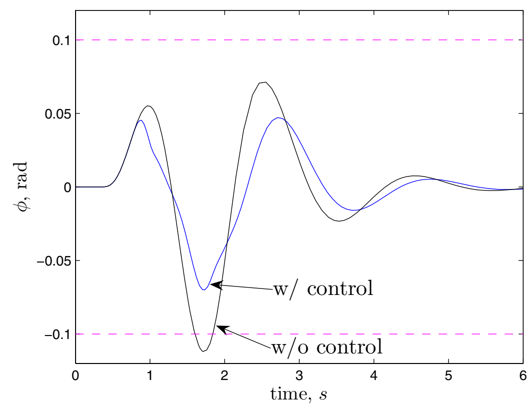

Fig. 4. Roll angle

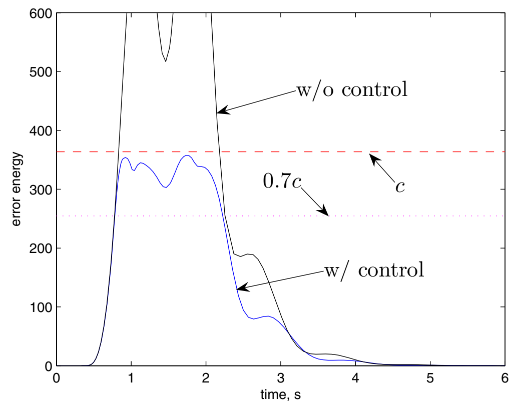

Fig. 5. Error energy

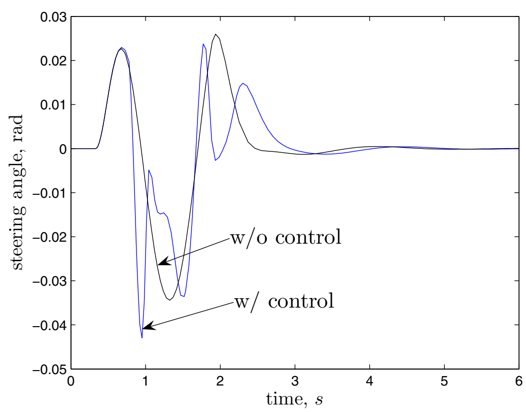

Fig. 6. Steering angle

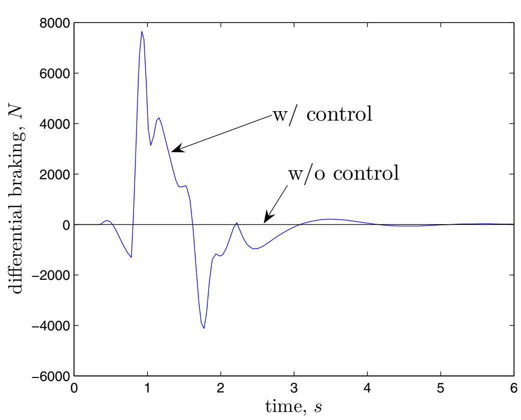

Fig. 7. Differential braking force

Figure 5 shows the "error energy" $H = \frac{1}{2}\langle \dot{q}, \dot{q}\rangle_G + U(q)$ over time. The region considered safe is the one where $U(y, \psi, \rho) \leq c$. The controller starts to come in when H reaches $0.7c$.

Figures 6 and 7 show the actuator commands with and without the controller from Section V. In this case, there is a tendency to produce a lateral force with steering while counteracting the resulting moment with differential braking. It also appears that rather large forces are commanded from the differential braking.

By using the degenerate mechanical system framework of Section III, a quantity representing error energy can be used to keep the vehicle within a safe operating region without interfering with the driver commands when there is no danger. For implementation on a real system, some more investigation would be needed into how to ensure the controller does not saturate the actuators. It is possible that this issue can be addressed by tuning the design choices such as Z, λ, and g.

VII. Conclusion

In this paper, the differential geometric structures arising from Riemannian metrics and affine connections were used to create a control scheme that guarantees that a vehicle will stay within an acceptable distance to the road center with an acceptable amount of body roll. To allow for the possibility of nonzero velocities with zero kinetic energy, the framework of degenerate mechanical systems was used to define the closed loop dynamics of part of the system. Since the system is not fully actuated, some care had to be taken to make sure that the desired closed loop system is feasible.

An example illustrates how this might be used to prevent drivers from accidentally venturing out of the safe region. Because of the flexibility of Proposition 3, there are more possibilities to explore in terms of choosing Z and g to alter the behavior of the system. Different choices may lead to more desirable characteristics in terms of actuator saturation and deviation from the intended trajectory.

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