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On spectral assignment and detectability of linear stochastic systems

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Abstract— **In this paper, we study the spectral assignment and detectability for linear stochastic time-invariant systems. It is shown that for deterministic systems and stochastic systems without a drift term, the spectrum of $\mathcal{L}_K$ cannot be assigned arbitrarily. By means of the spectral set of $\mathcal{L}_K$, the mean square stabilization with a given stabilizing degree is defined and discussed. Finally, stochastic detectability and its relation to exact observability have been investigated.**

I. INTRODUCTION

Stabilization and stability of Itô-type stochastic systems have been studied extensively by many authors, we refer the reader to [1,3-5] and [6] for the recent development. Different from most of the previous researchers, in [1], we presented a new approach to investigate the stabilization and exact observability of linear stochastic time-invariant systems, which can be called "spectrum method". One of the advantages of spectrum method may be that it helps us to understand how about the degree of stochastic stability and stabilization, second, it helps us to characterize exact observability and the other systemic concepts as Theorem 4 of [1]. Among the whole stochastic spectral theory, a closed-loop operator $\mathcal{L}_K$ plays an important role, by means of which, strong solutions of generalized algebraic Riccati equations (GAREs) was defined, and a stochastic Popov-Belevith-Hautus Criterion for exact observability also obtained [1]. At the end of [1], we put forward the spectral assignment problem of $\mathcal{L}_K$, similar to the pole placement of deterministic systems. This is an interesting and difficult issue, and will be answered partially in this present paper. We adopt the same notations as those of [1].

II. SPECTRAL ASSIGNMENT

In [1], to describe the mean square stabilizability of the following linear time-invariant stochastic system

$$\begin{cases} dx(t) = (Ax(t) + Bu(t))\,dt + (Cx + Du)\,dw(t) \\ y = D_1 x(t), x(0) = x_0 \in \mathcal{R}^n \end{cases} \tag{1}$$

we introduce a linear operator $\mathcal{L}_K$ associated with the closed-loop system

$$dx(t) = (A+BK)x(t)\,dt+(C+DK)x(t)\,dw(t), x(0) = x_0 \tag{2}$$

which is defined as

$$\begin{aligned}\mathcal{L}_K : X \in \mathcal{S}_n \longmapsto (A + BK)X + X(A + BK)' \\ +(C + DK)X(C + DK)'.\end{aligned} \tag{3}$$

In the above, $x(t) \in \mathcal{R}^n, u(t) \in \mathcal{R}^m$ are said to be the system state, and control input, respectively. $(A, B, C, D, K) \in \mathcal{R}^{n\times n} \times \mathcal{R}^{n\times m} \times \mathcal{R}^{n\times n} \times \mathcal{R}^{n\times m} \times \mathcal{R}^{m\times n}$ are real constant matrices, $w(\cdot)$ is one-dimensional, standard Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\{w(s), 0 \le s \le t\}$. The spectrum of $\mathcal{L}_K$ has a close connection to the mean square stabilizability of system (1), see Theorem 1 of [1]. For the reader's convenience, we adapt some materials of [1] as follows.

Definition 1. Stochastic system (1) is called stabilizable, if there exists a constant state feedback control $u = Kx$, such that for any $x_0 \in \mathcal{R}^n$, the closed-loop system

$$dx(t) = (A+BK)x(t)\,dt+(C+DK)x(t)\,dw(t), x(0) = x_0 \tag{4}$$

is asymptotically mean square stable.

Theorem 1. System (1) is stabilizable if and only if (iff) $\sigma(\mathcal{L}_K) \subset \mathcal{C}^-$ for some $K \in \mathcal{R}^{m\times n}$.

Different from the Lyapunov function-based approach, Theorem 1 gives a necessary and sufficient condition for the stabilizability of system (1) via the spectrum of $\mathcal{L}_K$, which has many applications, such as in defining strong solutions of GAREs [1]. The spectral set of $\mathcal{L}_K$ consists of $\frac{n(n+1)}{2}$ elements. Generally speaking, the spectrum of $\mathcal{L}_K$ cannot be assigned arbitrarily, please see the following example.

Definition 2. For system (1), if there exists $\{\lambda_1, \lambda_2, \cdots, \lambda_{n(n+1)/2}\} \subset \mathcal{C}$, such that for any $K \in \mathcal{R}^{m\times n}$, $\sigma(\mathcal{L}_K) \neq \{\lambda_1, \lambda_2, \cdots, \lambda_{n(n+1)/2}\}$, then we say that the spectrum of $\mathcal{L}_K$ cannot be assigned arbitrarily. Otherwise, we say the spectrum of $\mathcal{L}_K$ can be assigned arbitrarily.

The following theorem tells us that for more than one-dimensional deterministic system

$$\dot{x} = Ax + Bu, n > 1 \tag{5}$$

and stochastic system without a drift term

$$dx = (Cx + Du)dw \tag{6}$$

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the corresponding spectrum of $\mathcal{L}_K$ cannot be placed arbitrarily.

Theorem 2. For system (5) or (6), its corresponding spectrum of $\mathcal{L}_K$ cannot be assigned arbitrarily.

Proof. For the sake of page limitation, the proof is omitted.

Theorem 2 generates a slight more general result as follows.

At the present stage, we still don't know under what conditions does the spectrum of $\mathcal{L}_K$ associated with (1) can be assigned arbitrarily? By means of the spectrum of $\mathcal{L}_K$, we give the following definition, which guarantee the closed-loop system to be asymptotically mean square stable in a faster speed.

Definition 3. System (1) is said to be stabilizable with a stabilizing degree $\alpha > 0$, if there exists a constant matrix $K \in \mathcal{R}^{m\times n}$, such that $\sigma(\mathcal{L}_K) \subset \mathcal{C}^{-\alpha} := \{\lambda : Re\lambda < -\alpha\}$.

The following theorem gives a necessary and sufficient condition for (1) to be stabilizable with a stabilizing degree $\alpha > 0$.

Theorem 3. System (1) is stabilizable with a stabilizing degree $\alpha > 0$, iff the following linear matrix inequality (LMI)

$$\begin{bmatrix} (A+\frac{\alpha}{2}I)X + X(A+\frac{\alpha}{2}I)' + BY + Y'B' & CX+DY \\ XC' + Y'D' & -X \end{bmatrix} < 0$$

exist solutions $X > 0$ and Y. In this case, the corresponding feedback gain $K = YX^{-1}$.

Proof. By Theorem 1, system (1) is stabilizable with a stabilizing degree $\alpha > 0$ iff $\sigma(\mathcal{L}_K^\alpha) \subset \mathcal{C}^-$, where $\mathcal{L}_K^\alpha := \mathcal{L}_K + \frac{\alpha}{2}I$, which is also equivalent to

$$dx = [(A + \frac{\alpha}{2}I)x + Bu]\,dt + (Cx + Du)\,dw$$

being stabilizable. In addition, a direct application of Theorem 1 [5] immediately follows this theorem.

III. Stochastic detectability

On the basis of stabilizability, we can now define the stochastic detectability via duality.

Definition 4. We say $[A, C|D_1]$ is stochastically detectable, if $(A', D_1'; C', 0)$ is stabilizable.

Remark 1. In [8], in order to study a class of linearly perturbed Riccati equations, a new concept called "MS-detectability" was introduced. One can test that $[A, C|D_1]$ being stochastically detectable is equivalent to (D_1, A) being MS-detectable with $\Gamma(X(t)) = CX(t)C'$ in (3.1) of [8], where Γ denotes a bounded positive linear map from Hermitian matrices into itself.

Proposition 1.

(i) If $[A, C|D_1]$ is stochastically detectable, then there does not exist nonzero $X \in \mathcal{S}_n$, such that

$$XA' + AX + CXC' = \lambda X, D_1X = 0, Re(\lambda) \geq 0. \quad (7)$$

If there exists C_1, such that for any $X \in \mathcal{S}_n$, $CXC' = XC_1' + C_1X$, then (7) is also a necessary condition for stochastic detectability.

(ii) $[A, C|D_1]$ is stochastically detectable iff the following LMI

$$\begin{bmatrix} A'X + XA + D'Y + Y'D & C'X \\ XC & -X \end{bmatrix} < 0$$

admits solutions $X > 0$ and Y.

Proof. From Theorems 1 and 3 of [1], (i) is obvious. (ii) is a direct application of Theorem 1 [5].

Proposition 2. If $[A, C|D_1]$ is stochastically detectable, then (D_1, A) is detectable.

Proof. By Definition 4, if $[A, C|D_1]$ is stochastically detectable, then there exists a constant matrix H', such that

$$dx = (A' + D_1'H')x\,dt + C'x\,dw \quad (8)$$

is mean square stable, which implies [3]

$$dx = (A' + D_1'H')x\,dt \quad (9)$$

is asymptotically stable, so (D_1, A) is detectable.

Different from the deterministic case, it is easy to construct examples to show that there doesn't have any implication between exact observability [1] and stochastic detectability.

IV. Conclusions

This paper has discussed the spectral assignment, stabilization with a given stabilizing degree $\alpha > 0$ and detectability of linear stochastic systems based on the spectral theory. Obviously, there still remain many open problems to study. For instance, how do we treat with SLQR and stochastic H_∞ control with the constraint of given stability degree as done in [9] and [10]?

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