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Nonlinear Stochastic H_2/H_∞ Control With State-Dependent Noise

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Abstract— **For a system governed by Itô-type nonlinear stochastic differential equation with state-dependent noise, the H_2/H_∞ control problem is considered, which combines the H_2 optimization with the robust H_∞ performance. A cross-coupled Hamilton-Jacobi equations associated with the nonlinear stochastic H_2/H_∞ control are obtained, based on which, sufficient conditions for designing the finite and infinite horizon nonlinear stochastic H_2/H_∞ controllers are derived. Some results on linear stochastic H_2/H_∞ control can be viewed as corollaries of this paper.**

I. INTRODUCTION

One of the most important robust control approaches is the so-called H_∞ control, which has made great progresses since the foundation work of [1]. H_∞ control demands that one design a controller to eliminate the external disturbance below a given level, obviously, there may be more than one controller to H_∞ control problem. In practice, we often need a control u^* not only to restrain the exogenous disturbance, but also to minimize a cost function when the worst case disturbance v is implemented, this is the so-called H_2/H_∞ control problem. Up to now, most of the results on H_∞ or mixed H_2/H_∞ control are concentrated on deterministic systems, we refer the reader to [2], [3], [9], [10], [14]-[16] and the references therein.

It is fair to say that stochastic H_∞ and mixed H_2/H_∞ control problems have become attractive research areas in the recent years, we can only mention the following work here. In [4], linear stochastic H_∞ control has been studied, and a stochastic bounded real lemma was also obtained. While [11] was on nonlinear stochastic H_∞ control problem, and an Hamilton-Jacobi equation (HJE) associated with nonlinear H_∞ was derived, which can be viewed as an extension of [14] in some sense. A recent paper [8] generalized the mixed H_2/H_∞ consequences of [2] to stochastic counterpart. [7] discussed the output feedback H_∞ control for stochastic uncertain systems. Now, in this present paper, we will continue the work of [8] to nonlinear case, basically follow the line of [9] for the treatment of deterministic nonlinear H_2/H_∞ control. Concretely speaking, the main contribution of this paper is as follows: A sufficient condition for finite /infinite horizon mixed H_2/H_∞ control is given via the coupled differential /algebraic HJEs, respectively, which are nonlinear second-order partial differential equations. Some further deserved study problems are also presented. This paper also extends the results of [9] to stochastic systems.

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For convenience, we adopt the following notations.

A' the transpose of the corresponding matrix A;

$A \geq 0 (A > 0)$ A is positive semidefinite (positive definite) real matrix;

I: the identity matrix;

$L_{f(x)}V(x) := \frac{\partial V'(x)}{\partial x} f(x)$;

$\mathcal{L}^2_{\mathcal{F}}(\mathcal{R}_+, \mathcal{R}^l)$ ($\mathcal{L}^2_{\mathcal{F}}([0,T], \mathcal{R}^l)$): the space of nonanticipative stochastic processes $y(t) \in \mathcal{R}^l$ with respect to an increasing σ-algebras $\mathcal{F}_t$ $(t \geq 0)$ satisfying $E\int_0^\infty \|y(t)\|^2 dt < \infty$ ($E\int_0^T \|y(t)\|^2 dt < \infty$).

II. FINITE HORIZON H_2/H_∞ CONTROL

Consider the following stochastic nonlinear system governed by Itô-type differential equation

$$\begin{cases} dx = (f(x) + g(x)u + h(x)v)\, dt + l(x)dW \\ f(0) = l(0) \equiv 0 \end{cases} \tag{1}$$

with controlled output

$$z = \begin{bmatrix} C(x) \\ u \end{bmatrix} \tag{2}$$

where $x(t) \in \mathcal{R}^n$ is called the system state, $z(t) \in \mathcal{R}^m$ is the penalty output, $u(t)$ and $v \in \mathcal{L}^2(\mathcal{R}_+, \mathcal{R}^{n_v})$ stand for the control and exogenous disturbance signal, respectively. f, g, h, l and C are smooth functions with suitable dimensions. $W(\cdot)$ is a standard one-dimensional Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ relative to an increasing family $(\mathcal{F}_t)_{t\in\mathcal{R}_+}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$. Now, we first define the finite horizon nonlinear stochastic H_2/H_∞ control as follows:

Definition 1 (Finite horizon nonlinear stochastic H_2/H_∞ control): Find, if possible, a state feedback control law $u = u^*(t,x)$ such that

(i) For any given $\gamma > 0$, $T > 0$, $v \in \mathcal{L}^2_{\mathcal{F}}([0,T], \mathcal{R}^{n_v})$, the trajectory of the closed-loop system (1) starting from $x(0) = x_0 = 0$ satisfies

$$E\int_0^T (\|C(x)\|^2 + \|u^*\|^2)\, dt \leq \gamma^2 E\int_0^T \|v\|^2\, dt. \tag{3}$$

(ii)When the worst case disturbance v^* is implemented in (1), u^* minimizes the quadratic performance

$$\begin{aligned} J_2^T(u^*, v^*) &= \min_{u\in\mathcal{L}^2_{\mathcal{F}}([0,T],\mathcal{R}^{n_u})} J_2^T(u, v^*) \\ &= \min_{u\in\mathcal{L}^2_{\mathcal{F}}([0,T],\mathcal{R}^{n_u})} E\int_0^T (\|C(x)\|^2 + \|u\|^2)\, dt. \end{aligned}$$

If we define

$$J_1^T(u,v) := E\int_0^T (\gamma^2\|v\|^2 - \|z\|^2)\,dt$$

and

$$J_2^T(u,v) := E\int_0^T \|z\|^2\,dt$$

then it can be seen that the mixed H_2/H_∞ control problem is equivalent to finding the Nash equilibria (u^*, v^*) defined as

$$J_1^T(u^*,v^*) \le J_1^T(u^*,v),\ \forall v \in \mathcal{L}^2_{\mathcal{F}}([0,T],\mathcal{R}^{n_v}) \tag{4}$$

$$J_2^T(u^*,v^*) \le J_2^T(u,v^*),\ \forall u \in \mathcal{L}^2_{\mathcal{F}}([0,T],\mathcal{R}^{n_u}). \tag{5}$$

The first Nash inequality is associated with the H_∞ performance, since $J_1^T(u^*,v^*) \ge 0$ implies (3), while the second one is related with the H_2 performance. Clearly, if the Nash equilibria (u^*,v^*) exist, u^* is our desired H_2/H_∞ controller, and v^* is the worst case disturbance. In this case, we also say that nonlinear stochastic H_2/H_∞ control admits a pair of solutions (u^*,v^*). The following theorem is a sufficient condition for the existence of a finite horizon H_2/H_∞ controller.

Theorem 1: Suppose there exist a non-positive definite function $V_1 \in C^{1,2}([0,T],\mathcal{R}^n)$, $V_1 : [0,T]\times\mathcal{R}^n \mapsto \mathcal{R}^-$ with $V_1(0,0)=0$, and a non-negative definite function $V_2 \in C^{1,2}([0,T],\mathcal{R}^n)$, $V_2 : [0,T]\times\mathcal{R}^n \mapsto \mathcal{R}^+$ with $V_2(0,0)=0$, such that they solve a pair of cross-coupled HJEs

$$\begin{array}{l}\mathcal{L}_{u=u^*,v=0}V_1(t,x) - \|C(x)\|^2 - \gamma^2\|v^*\|^2 - \|u^*\|^2 = 0,\\ V_1(T,x(T)) = 0\end{array} \tag{6}$$

$$\begin{array}{l}\mathcal{L}_{u=u^*,v=v^*}V_2(t,x) + \|C(x)\|^2 + \|u^*\|^2 = 0,\\ V_2(T,x(T)) = 0\end{array} \tag{7}$$

with

$$u^*(t,x) = -\frac{1}{2}L_{g(x)}V_2(t,x), \tag{8}$$

$$v^*(t,x) = -\frac{1}{2\gamma^2}L_{h(x)}V_1(t,x) \tag{9}$$

and $\mathcal{L}_{u,v}$ being the infinitesimal operator of (1), then the mixed H_2/H_∞ control problem admits a pair of solutions (u^*,v^*). Moreover,

$$J_2^T(u^*,v^*) = V_2(0,x_0).$$

Proof: We follow the line of [8] and [9]. By use of the completion technique of square argument and (6), it follows

$$\begin{aligned}
J_1^T(u,v) &= V_1(0,x_0) - EV_1(T,x(T))\\
&\quad + E\int_0^T [(\gamma^2\|v\|^2\\
&\quad -\|z\|^2)\,dt + dV_1(t,x(t))]\\
&= V_1(0,x_0) + E\int_0^T [(\gamma^2\|v\|^2\\
&\quad -\|z\|^2) + \mathcal{L}_{u,v}V_1(t,x(t))]\,dt\\
&= V_1(0,x_0) + E\int_0^T [(\gamma^2\|v\|^2 - \|z\|^2)\\
&\quad +\mathcal{L}_{u=u^*,v=0}V_1(t,x(t))\\
&\quad +L_{g(x)}V_1(t,x)(u-u^*)\\
&\quad +L_{h(x)}V_1(t,x)v]\,dt\\
&= V_1(0,x_0) + E\int_0^T [\gamma^2\|v\|^2 - \|u\|^2\\
&\quad +\|u^*\|^2 + \gamma^2\|v^*\|^2\\
&\quad +L_{g(x)}V_1(t,x)(u-u^*)\\
&\quad +L_{h(x)}V_1(t,x)v]\,dt
\end{aligned}$$

So we have

$$\begin{array}{l}J_1^T(u^*,v) = V_1(0,x_0) + E\int_0^T \gamma^2\|v-v^*\|^2\,dt\\ \ge J_1^T(u^*,v^*) = V_1(0,x_0)\end{array}$$

More specifically, for $x_0 = 0$, $J_1^T(u^*,v^*) = V_1(0,x_0) = 0$, which immediately concludes the H_∞ performance (3).

Additionally, to show the requirement (ii) of Definition 1, we only need to show (5). When the worst case v^* is implemented in system (1), considering (7), we have

$$\begin{aligned}
J_2^T(u,v^*) &= E\int_0^T (\|C(x)\|^2 + \|u\|^2)\,dt\\
&= V_2(0,x_0) - EV_2(T,x(T))\\
&\quad +E\int_0^T [(\|C(x)\|^2\\
&\quad +\|u\|^2)\,dt + dV_2(t,x(t))]\\
&= V_2(0,x_0) + E\int_0^T [(\|C(x)\|^2\\
&\quad +\|u\|^2 + \mathcal{L}_{u^*,v^*}V_2(t,x)\\
&\quad +L_{g(x)}V_2(t,x)(u-u^*))\,dt\\
&= V_2(0,x_0) + E\int_0^T \|u-u^*\|^2\,dt\\
&\ge J_2^T(u^*,v^*) = V_2(0,x_0)
\end{aligned}$$

which verifies the requirement (ii), the proof of Theorem 1 is complete.

For linear system

$$\left\{\begin{array}{l} dx(t) = (A(t)x(t) + B_2(t)u(t)\\ \qquad +B_1(t)v(t))\,dt + A_1(t)x(t)dw_1\\ x(0) = x_0\\ z(t) = \left[\begin{array}{c} C(t)x(t)\\ D(t)u(t)\end{array}\right], \quad D'(t)D(t) = I\end{array}\right. \tag{10}$$

if we take $V_1(t,x) = x'P_1(t)x, V_2(t,x) = x'P_2(t)x$ with $P_1 \leq 0, P_2 \geq 0$, then Theorem 1 yields the sufficiency part of Theorem 5 of [8]. In particular, the coupled HJEs (6) and (7) come down to a pair of coupled Riccati equations

$$\begin{aligned} -\dot{P}_1 &= A'P_1 + P_1A + A_1'P_1A_1 - C'C \\ &\quad -[P_1, P_2]\begin{bmatrix} \gamma^{-2}B_1B_1' & B_2B_2' \\ B_2B_2' & B_2B_2' \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \\ P_1(T) &= 0 \end{aligned} \tag{11}$$

$$\begin{aligned} -\dot{P}_2 &= A'P_2 + P_2A + A_1'P_2A_1 + C'C \\ &\quad -[P_1, P_2]\begin{bmatrix} 0 & \gamma^{-2}B_1B_1' \\ \gamma^{-2}B_1B_1' & B_2B_2' \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \\ P_2(T) &= 0. \end{aligned} \tag{12}$$

III. A UNIFIED TREATMENT FOR H_2, H_∞ AND MIXED H_2/H_∞ CONTROL

As done in [2] and [9], under the framework of a nonzero sum, two player Nash differential game, we can give a unified treatment for H_2, H_∞ and mixed H_2/H_∞ control problems. Consider system (1) with the penalty output (2), associated with the following two performance

$$J_1^T(u,v) = E\int_0^T (\gamma^2\|v\|^2 - \|z\|^2)\, dt$$

$$J_2^T(u,v) = E\int_0^T (\|z\|^2 - \rho\|v\|^2)\, dt$$

Similar to the discussion of Theorem 1, it can be shown that if the following cross-coupled HJEs

$$\begin{aligned} &\mathcal{L}_{u=u^*,v=0}\tilde{V}_1(t,x) - \|C(x)\|^2 - \gamma^2\|v^*\|^2 - \|u^*\|^2 = 0, \\ &\tilde{V}_1(T,x(T)) = 0 \end{aligned} \tag{13}$$

$$\begin{aligned} &\mathcal{L}_{u=u^*,v=v^*}\tilde{V}_2(t,x) + \|C(x)\|^2 + \|u^*\|^2 - \rho^2\|v^*\|^2 = 0, \\ &\tilde{V}_2(T,x(T)) = 0 \end{aligned} \tag{14}$$

admit solutions $\tilde{V}_1, \tilde{V}_2 \in C^{1,2}([0,T], \mathcal{R}^n)$, $\tilde{V}_1 : [0,T] \times \mathcal{R}^n \mapsto \mathcal{R}^-$, $\tilde{V}_2 : [0,T] \times \mathcal{R}^n \mapsto \mathcal{R}^+$, $\tilde{V}_1(0,0) = \tilde{V}_2(0,0) = 0$, then (u^*, v^*) is the so-called Nash equilibrium point, which satisfies (4) and (5), where

$$u^*(t,x) = -\frac{1}{2}L_{g(x)}\tilde{V}_2(t,x)$$

$$v^*(t,x) = -\frac{1}{2\gamma^2}L_{h(x)}\tilde{V}_1(t,x).$$

i) The nonlinear quadratic optimal control problem

$$\min_{u\in\mathcal{L}^2_{\mathcal{F}}([0,T],\mathcal{R}^{n_u})}\{J_2^T(u,0) = E\int_0^T \|z\|^2\, dt\}$$

subject to

$$\begin{cases} dx = (f(x) + g(x)u)\, dt + l(x)dW \\ x(0) = x_0 \in \mathcal{R}^n \end{cases}$$

can be solved by setting $\rho = 0, \gamma \to \infty$. It can be shown that the solutions $\tilde{V}_1$ and $\tilde{V}_2$ of the coupled HJEs (13) and (14) are $\tilde{V}_1(t,x) \to -\tilde{V}(t,x)$ and $\tilde{V}_2(t,x) \to \tilde{V}(t,x)$ respectively, where $\tilde{V}(t,x)$ solves the following HJE

$$\mathcal{L}_{u=u^*,v=0}\tilde{V}(t,x) + \|C(x)\|^2 + \|u^*\|^2 = 0,\ \tilde{V}(T,x(T)) = 0$$

with $u^*(t,x) = -\frac{1}{2}L_{g(x)}\tilde{V}(t,x)$, or equivalently,

$$\begin{cases} \frac{\partial \tilde{V}}{\partial t} + \frac{\partial \tilde{V}'}{\partial x}f(x) + \frac{1}{2}l'(x)\frac{\partial \tilde{V}^2}{\partial x^2}l(x) + \|C(x)\|^2 \\ \quad + \frac{1}{4}\frac{\partial \tilde{V}'}{\partial x}g(x)g'(x)\frac{\partial \tilde{V}}{\partial x} = 0 \\ \tilde{V}(T,x(T)) = 0. \end{cases} \tag{15}$$

Moreover,

$$\min_{u\in\mathcal{L}^2_{\mathcal{F}}([0,T],\mathcal{R}^{n_u})} J_2^T(u,0) = J_2^T(u^*,0) = \tilde{V}(0,x_0).$$

ii) If we set $\rho = \gamma$, then $\tilde{V}_\infty(t,x) = \tilde{V}_2(t,x) = -\tilde{V}_1(t,x)$, where $\tilde{V}_\infty$ is a solution to HJE

$$\begin{cases} \mathcal{L}_{u=u^*,v=0}\tilde{V}_\infty(t,x) + \|C(x)\|^2 + \|u^*\|^2 + \gamma^2\|v^*\|^2 = 0, \\ \tilde{V}_\infty(T,x(T)) = 0 \end{cases} \tag{16}$$

or

$$\begin{cases} \frac{\partial \tilde{V}_\infty}{\partial t} + \frac{\partial \tilde{V}_\infty'}{\partial x}f(x) + \frac{1}{2}l'(x)\frac{\partial \tilde{V}_\infty^2}{\partial x^2}l(x) \\ \quad -\frac{1}{4}\frac{\partial \tilde{V}_\infty'}{\partial x}g(x)g'(x)\frac{\partial \tilde{V}_\infty}{\partial x} \\ \quad + \frac{1}{4\gamma^2}\frac{\partial \tilde{V}_\infty'}{\partial x}h(x)h'(x)\frac{\partial \tilde{V}_\infty}{\partial x} + \|C(x)\|^2 = 0 \\ \tilde{V}_\infty(T,x(T)) = 0 \end{cases} \tag{17}$$

with

$$u^*(t,x) = -\frac{1}{2}L_{g(x)}\tilde{V}_\infty(t,x),\ \ v^*(t,x) = \frac{1}{2\gamma^2}L_{h(x)}\tilde{V}_\infty(t,x)$$

More specifically, the above $u^*(t,x)$ is our desired H_∞ control law, which makes

$$E\int_0^T \|z(t)\|^2\, dt \leq \gamma^2 E\int_0^T \|v(t)\|^2\, dt$$

hold for any nonzero $v \in \mathcal{L}^2_{\mathcal{F}}([0,T], \mathcal{R}^{n_v})$.

iii) By taking $\rho = 0$, the mixed H_2/H_∞ control is retrieved. In this case, $V_1 = \tilde{V}_1, V_2 = \tilde{V}_2$.

Remark 1: It can be seen that all results obtained in this section still hold for the time-varying stochastic system

$$\begin{cases} dx = (f(t,x) + g(t,x)u + h(t,x)v)\, dt + l(t,x)dW \\ f(0,t) = l(0,t) \equiv 0,\ \forall t \geq 0 \end{cases}$$

with penalty output

$$z = \begin{bmatrix} C(t,x) \\ u \end{bmatrix}.$$

Remark 2: A more general HJE than (15) was derived in [12], while (17) is also a special case of the corresponding one of [11].

IV. INFINITE HORIZON H_2/H_∞ CONTROL

To discuss the infinite horizon nonlinear stochastic H_2/H_∞ control problem, the internal stability requirement is needed, so we should introduce the following definition on stochastic stability.

Definition 2 [5]: Consider the following uncontrolled stochastic system

$$dx = f(x)\,dt + l(x)\,dW,\ x(0) = x_0,\ f(0) = l(0) = 0. \tag{18}$$

1) $x \equiv 0$ of (18) is said to be stable in probability if for any $\epsilon > 0$

$$\lim_{x_0 \to 0} P(\sup_{t \geq 0} |x| > \epsilon) = 0. \tag{19}$$

2) $x \equiv 0$ of (18) is said to be locally asymptotically stable in probability if (19) holds and there exists a neighborhood U_0 of the origion, such that

$$P(\lim_{t \to \infty} |x(t)| = 0, \forall x_0 \in U_0) = 1 \tag{20}$$

Remark 3: In the previous references, we can find another definition form on locally asymptotic stability (e.g. [17]), which said that $x \equiv 0$ of (18) is locally asymptotically stable in probability if (19) holds and

$$\lim_{x_0 \to 0} P(\lim_{t \to \infty} |x(t)| = 0) = 1$$

Here, we adopt Definition 2 in order to be consistent with the deterministic one [9]. The following lemma is well known for stability in probability.

Lemma 1: If there exists a neighborhood U_0 of 0, a Lyapunov function $V(x) \in C^2(U)$, $V(x) > 0$ for $x \neq 0$ in the domain U_0, such that

$$\mathcal{L}_{u=0,v=0} V(x) = \frac{\partial V'(x)}{\partial x} f(x) + \frac{1}{2} l'(x) \frac{\partial^2 V(x)}{\partial x^2} l(x) \leq 0 \tag{21}$$

for $x \neq 0$, then $x \equiv 0$ of system (18) is stable in probability.

Below, we state the infinite horizon nonlinear stochastic H_2/H_∞ control as follows:

Definition 3 (Infinite horizon nonlinear stochastic H_2/H_∞ control): Find, if possible, a static state feedback control law $u = u^*(x) \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{R}^+, \mathcal{R}^{n_u})$ such that

(i) For any given $\gamma > 0$ and any nonzero $v \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{R}^+, \mathcal{R}^{n_v})$, the trajectory

$$dx = (f(x) + g(x)u^*(x) + h(x)v)\,dt + l(x)\,dW \tag{22}$$

starting from $x_0 = 0$ satisfies

$$E\int_0^\infty (\|C(x)\|^2 + \|u^*(x)\|^2)\,dt \leq \gamma^2 E\int_0^\infty \|v\|^2\,dt \tag{23}$$

(ii) When the worst case disturbance v^* is implemented in (1), u^* minimizes the quadratic performance

$$\begin{aligned} J_2^\infty(u^*, v^*) &= \min_{u \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{R}^+, \mathcal{R}^{n_u}) \cap \mathcal{U}^\infty_{ad}} J_2^\infty(u, v^*) \\ &= \min_{u \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{R}^+, \mathcal{R}^{n_u}) \cap \mathcal{U}^\infty_{ad}} E\int_0^\infty (\|C(x)\|^2 + \|u(x)\|^2)\,dt \end{aligned}$$

where $\mathcal{U}^\infty_{ad}$ consists of all measurable, adaptive process $u(x)$ (with respect to $\mathcal{F}_t$)), which makes the following trajectory

$$dx = (f(x) + g(x)u + h(x)v^*)\,dt + l(x)\,dW \tag{24}$$

to be locally asymptotically stable in probability.

(iii) The system

$$dx = (f(x) + g(x)u^*(x))\,dt + l(x)\,dW \tag{25}$$

is locally asymptotically stable in probability.

If we define

$$J_1^\infty(u, v) := \int_0^\infty (\gamma^2\|v\|^2 - \|z\|^2)\,dt$$

and

$$J_2^\infty(u, v) := \int_0^\infty \|z\|^2\,dt$$

then the nonlinear stochastic H_2/H_∞ control problem can be converted into solving the following two persons, nonzero sum Nash game associated with the H_∞ and H_2 performance:

$$J_1^\infty(u^*, v^*) \leq J_1^\infty(u^*, v),\ \forall v \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{R}^+, \mathcal{R}^{n_v})$$

$$J_2^\infty(u^*, v^*) \leq J_2^\infty(u, v^*),\ \forall u \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{R}^+, \mathcal{R}^{n_u})$$

The following definition generalizes the zero-state detectability to the stochastic system

$$\begin{cases} dx = f(x)\,dt + l(x)\,dW, \\ y = C(x). \end{cases} \tag{26}$$

Definition 4: System (26) is said to be locally zero-state detectable, if there exists a neighborhood U_0 of 0, that for all $x_0 \in U_0$, we have

$$y(t) \equiv 0, \quad \forall t \geq 0 \Rightarrow P\{\lim_{t \to \infty} x(t) = 0, x(0) = x_0\} = 1. \tag{27}$$

If $U_0 = \mathcal{R}^n$, then (26) is called zero-state detectable. In the sequel, when (26) is locally zero-state detectable (zero-state detectable), we also call $[f, l|C]$ locally zero-state detectable (zero-state detectable).

Theorem 2: Suppose the following assumptions hold.

1) $[f, l|C]$ is locally zero-state detectable.

2) there exists a locally negative definite function $V_1 : \Omega_0 \mapsto \mathcal{R}^-$, defined on a neighborhood Ω_0 of the origion, and a locally positive definite function $V_2 : \Omega_0 \mapsto \mathcal{R}^+$, such that they satisfy a pair of cross-coupled HJEs as follows:

$$\mathcal{L}_{u=u^*,v=0} V_1(x) - \|C(x)\|^2 - \gamma^2\|v^*\|^2 - \|u^*\|^2 = 0 \tag{28}$$

$$\mathcal{L}_{u=u^*,v=v^*} V_2(x) + \|C(x)\|^2 + \|u^*\|^2 = 0 \tag{29}$$

where u^* and v^* take the same form as in (8) and (9), respectively.

3) the pair $[f(x)+h(x)v^*, l(x)|C(x)]$ is locally zero-state detectable.

Then the state feedback control law (8) and (9) solve the infinite time horizon H_2/H_∞ control problem.

Proof: We first show (iii) of Definition 3 holds. For system (25) with u^* given by (8), we apply Itô's formula to Lyapunov function $-V_1(x)$, and consider equation (28), it concludes

$$\mathcal{L}_{u=u^*,v=0}(-V_1(x)) = -\|C(x)\|^2 - \gamma^2\|v^*\|^2 - \|u^*\|^2 \leq 0$$

therefore, system (25) is stable in probability from Lemma 1 . Moreover, $\mathcal{L}_{u=u^*,v=0}(-V_1(x)) = 0$ if and only if $C(x) = 0, v^* = 0, u^* = 0$. By stochastic LaSalle's invariance principle [13] and condition 1), the locally asymptotic stability in probability is immediately obtained. By the same way, condition 3) and (29) yields the following closed-loop system

$$dx = (f(x) + g(x)u^* + h(x)v^*)\,dt + l(x)\,dW \qquad (30)$$

being locally asymptotically stable in probability.

Second, by means of the completion of square argument technique, together with considering equation (28), we have

$$\begin{aligned} J_1^T(u,v) &= E\int_0^T (\gamma^2\|v\|^2 - \|z\|^2)\,dt + E\int_0^T dV_1 \\ &\quad +V_1(x_0) - EV_1(x(T)) \\ &= E\int_0^T (\gamma^2\|v\|^2 - \|z\|^2)\,dt \\ &\quad +E\int_0^T \mathcal{L}_{u,v}V_1(x)\,dt \\ &\quad +E\int_0^T L_{l(x)}V_1(x)\,dW \\ &\quad +V_1(x_0) - EV_1(x(T)) \\ &= E\int_0^T (\gamma^2\|v\|^2 - \|z\|^2)\,dt \\ &\quad +E\int_0^T [\mathcal{L}_{u=u^*,v=0}V_1(x) \\ &\quad +\frac{1}{2}l'(x)\frac{\partial^2 V_1(x)}{\partial x^2}l(x)]\,dt \\ &\quad +E\int_0^T [L_{g(x)}V_1(x)(u-u^*) \\ &\quad +L_{h(x)}(x)V_1(x)v]\,dt + V_1(x_0) \\ &\quad -EV_1(x(T)) \\ &= E\int_0^T [(\gamma^2\|v\|^2 - \|u\|^2 + \|u^*\|^2 \\ &\quad +\gamma^2\|v^*\|^2 + L_{h(x)}V_1(x)v \\ &\quad +L_{g(x)}V_1(x)(u-u^*)]\,dt + V_1(x_0) \\ &\quad -EV_1(x(T)) \end{aligned}$$

which concludes

$$\begin{aligned} J_1^T(u^*,v) &= V_1(x_0) - EV_1(x(T)) \\ &\quad +E\int_0^T \gamma^2\|v-v^*\|^2\,dt \\ &\geq V_1(x_0) - EV_1(x(T)) \\ &= J_1^T(u^*,v^*). \end{aligned} \qquad (31)$$

Especially, for $x_0 = 0$, $J_1^T(u^*,v) = -EV_1(x(T)) \geq 0$, which follows

$$E\int_0^\infty \|z\|^2\,dt \leq \gamma^2 E\int_0^\infty \|v\|^2, \quad x_0 = 0, \forall v \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{R}^+, \mathcal{R}^{n_v}).$$

Hence, the requirement (i) of Definition 3 is achieved. As to the requirement (ii), it is an immediate corollary of [12], which asserted that $J_2^\infty(u^*,v^*) = V_2(x_0)$.

Remark 4: From $J_2^\infty(u^*,v^*) = V_2(x_0)$, we know $u^*(x) \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{R}^+, \mathcal{R}^{n_u})$, but can we assert $v^*(x) \in \mathcal{L}^2_{\mathcal{F}}(\mathcal{R}^+, \mathcal{R}^{n_v})$? Additionally, is it right for $J_1^\infty(u^*,v^*) = V_1(x_0)$ or $\lim_{t\to\infty} EV(x(t)) = 0$? where $x(\cdot)$ is the trajectory of the closed-loop system (30). It is known that the above two conjectures are valid for deterministic systems [9].

Remark 5: In [12], local zero-state observability (zero-state observability) was defined, which is in fact by replacing (27) with

$$y(t) \equiv 0, \quad \forall t \geq 0 \Rightarrow x_0 = 0$$

It can be shown that if we substitute detectability of Theorem 2 with observability, then V_1 and V_2 are strictly negative and positive, respectively. In particular, for linear stochastic systems, zero-state observability was introduced by [6].

Obviously, in the case of linear stochastic system (10), the coupled HJEs (28) and (29) come down to the following coupled algebraic Riccati equations

$$\begin{aligned} &A'P_1 + P_1A + A_1'P_1A_1 - C'C \\ &-\begin{bmatrix} P_1 & P_2 \end{bmatrix}\begin{bmatrix} \gamma^{-2}B_1B_1' & B_2B_2' \\ B_2B_2' & B_2B_2' \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 0 \end{aligned} \qquad (32)$$

and

$$\begin{aligned} &A'P_2 + P_2A + A_1'P_2A_1 + C'C \\ &-\begin{bmatrix} P_1 & P_2 \end{bmatrix}\begin{bmatrix} 0 & \gamma^{-2}B_1B_1' \\ \gamma^{-2}B_1B_1' & B_2B_2' \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 0 \end{aligned} \qquad (33)$$

respectively, which were derived in [8].

V. CONCLUDING REMARKS

In the above sections, we have discussed the finite and infinite horizon stochastic H_2/H_∞ control problems, as well as the associated two classes of cross-coupled HJEs, respectively. Since it is too difficult to solve the cross-coupled HJEs, searching for the numerical solutions of the coupled HJEs is a valuable topic; In additions, even for the linear stochastic systems, when the control enters into the diffusion term, the H_2/H_∞ control problem remains an open problem. Finally, in comparison with the results on linear stochastic H_2/H_∞ control [8], there is a definite gap, because Theorems 1 and 2 are only sufficient, not necessary conditions for finite/infinite horizon H_2/H_∞ problems.

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