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Poli-criterial linearization technique in control problem of stochastic dynamic systems

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Abstract— **The problem of the determination of the response characteristics and quasi-optimal control for nonlinear stochastic dynamic systems by using multi-criteria linearization technique is presented in this paper. This idea was first introduced in previous author's paper [12] for a simple dynamic system. In this paper it is extended and the detailed analysis is given for a nonlinear oscillator with Gaussian external excitations and for a few criteria of statistical linearization. The obtained results are illustrated by a numerical example for Duffing oscillator.**

I. INTRODUCTION

Linearization methods are the most versatile methods for analysis of nonlinear systems and structures under stochastic excitations. The different criteria of linearization connected with well known linearization methods such as statistical linearization, equivalent linearization or exact linearization in three basic spaces were separately considered in the literature. They were discussed in the space, of moments of stochastic processes [3], of probability density functions of stochastic processes [11] and of spectral density functions of stochastic processes [1]

Since the number of different linearization criteria is large (more than 100). For details see, for instance a survey paper [3] or [14]. It is clear that the choice of a group of criteria of linearization depends on considered problem but in one group, for instance in the space of moments of stochastic processes the problem is open, the choice required an analysis. To study multicriteria problems special approaches called multicriteria optimization methods were developed in the literature. The objective of this paper is to show the relationship between these criteria using two approaches of poli-criterial optimization techniques, namely scalarization method and Pareto-optimal solution [9], [10] in application to the determination of the response characteristics and quasi-optimal control for nonlinear stochastic dynamic systems. This idea was first introduced in previous author's paper [12] for a simple dynamic system. In this paper it is extended and the detailed analysis is given for a nonlinear oscillator with Gaussian external excitations and for a few criteria of statistical linearization. The obtained results are illustrated by numerical example for Duffing oscillator.

Consider a nonlinear stochastic model of dynamic system described by the Ito vector differential equation

$$d\mathbf{x}(t) = \mathbf{\Phi}(\mathbf{x})dt + \sum_{k=1}^{M} \mathbf{G}_k d\xi_k(t) \tag{1}$$

where $\mathbf{x} = [x_1, ..., x_n]^T$ is the state vector, $\mathbf{\Phi} = [\Phi_1, ..., \Phi_n]^T$ is a nonlinear vector function such that $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$, $\mathbf{G}_k = [G_{k1}, ..., G_{kn}]^T$, are deterministic vectors, ξ_k are independent standard Wiener processes. We assume that the unique solution of equation (1) exists.

II. LINEARIZATION TECHNIQUES FOR STOCHASTIC SYSTEMS

A. Statistical and equivalent linearization

There are two basic groups of linearization methods for stochastic dynamic systems, namely statistical (or local) linearization and equivalent linearization. In the case of statistical linearization the objective is to find for nonlinear vector $bf\Phi = [\Phi_1, ..., \Phi_n]^T$ an equivalent one "in the sense of a linearization criterion", i.e., replacing

$$\mathbf{Y} = \mathbf{\Phi}(\mathbf{x}, t) \tag{2}$$

in equation (1) by a linearized form

$$\mathbf{Y} = \mathbf{\Phi}_0(\mathbf{m_x}, \mathbf{\Theta_x}, t) + \mathbf{K}(\mathbf{m_x}, \mathbf{\Theta_x}, t)\mathbf{x}^0 \tag{3}$$

where

$$\mathbf{m_x} = E[\mathbf{x}], \qquad \mathbf{\Theta_x} = [\Theta_{ij}] = E\left[x_i^0 x_j^0\right] \tag{4}$$

with $x_i^0 = x_i - m_{x_i}$ being the centralized stochastic process $\mathbf{\Phi}_0 = [\Phi_{01}, ..., \Phi_{0n}]^T$ is a nonlinear vector function of the moments of $\mathbf{x}$ and $\mathbf{K} = [k_{ij}]$ is $n \times n$ matrix of statistical linearization coefficients.

In the case of equivalent linearization the objective is to find for the nonlinear dynamic system (1) an equivalent one in the sense of a linearization criterion based on responses properties for nonlinear system (1) and for the following linearized system

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$$d\mathbf{x}(t) = [\mathbf{A}(t)\mathbf{x} + \mathbf{C}(t)]dt + \sum_{k=1}^{M} \mathbf{G}_k d\xi_k(t) \quad (5)$$

where

$\mathbf{A} = [a_{ij}], \quad i,j = 1,...,n, \; k = 1,...,M$ is a matrix and $\mathbf{C} = [C_1,...,C_n]^T$ is a vector of linearization coefficients.

B. Basic linearization criteria for stochastic dynamic systems

1a. Criterion of Equality of the First and Second Moments of Nonlinear and Linearized Variables (Kazakov [4])

$$E[Y_i] = \Phi_{i0} \quad (6)$$

$$E[(Y_i - E[Y_i])(Y_j - E[Y_j])] = \sum_{i=1}^{n}\sum_{j=1}^{n} k_i k_j \Theta_{ij} \quad (7)$$

1b. Criterion of Moment Error of Approximation

$$E\left[\Phi_i^{2p}(x,t) - \left(\Phi_{0i} + \sum_{j=1}^{n} k_{ij}x_j^0\right)^{2p}\right]^2, \quad i = 1,...,n, \; p = 1,2,... \quad (8)$$

where for $p = 1$ and $p = 2$ it is known in the literature as Criterion of Mean-square Error of Approximation (Kazakov [4]) and Energy Criterion (Elishakoff and Zhang [2])

Equivalence of response probability densities (Socha [11])

2a. Pseudomoment metric

$$I_{2a} = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \mathbf{x}^{[2r]} |g_N(\mathbf{x}) - g_L(\mathbf{x})| dx_1...dx_n \quad (9)$$

where $\mathbf{x} = [x_1,...,x_n]^T$, $\mathbf{x}^{[2r]}, = x_1^{2r_1}x_2^{2r_2}...x_n^{2r_n}$, $r_1,...r_n \in \mathbf{N}$, $\sum_{i=1}^{n} 2r_i = 2r$,

2b. Square probability metric

$$I_{2b} = \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} (g_N(\mathbf{x}) - g_L(\mathbf{x}))^2 dx_1...dx_n \quad (10)$$

where $g_N(\mathbf{x})$ and $g_L(\mathbf{x})$ are probability density functions of solutions of nonlinear (1) and linearized systems (5), respectively.

C. Poli-criteria optimization methods

In this section we quote some basic definitions and facts from multicriteria optimization theory [9] and [10].

A subset Θ of a linear space B is called a *convex cone* if and only if

$$\forall\alpha_1 \geq 0 \; \forall\alpha_2 \geq 0 \; \forall\mathbf{x}^1\mathbf{x}^2 \in \Theta \; (\alpha_1\mathbf{x}^1 + \alpha_2\mathbf{x}^2) \in \Theta \quad (11)$$

To every convex cone Θ there corresponds the ordering relation R in B defined by

$$\mathbf{x}^1 \leq_\Theta \mathbf{x}^2 <=> \mathbf{x}^2 - \mathbf{x}^1 \in \Theta \quad (12)$$

The relations of partial order induced by convex cones are generalization of the natural order in R^n defined as follows

$$\mathbf{x} \leq \mathbf{y} <=> \forall i = 1,...,n, \; x_i \leq y_i \quad (13)$$

where $\mathbf{x} = [x_1,...,x_n]^T$ and $\mathbf{y} = [y_1,...,y_n]^T$. This is equivalent to the relation $\mathbf{y} - \mathbf{x} \in R_+^n$. The positive orthant R_+^n satisfies all properties of convex cones.

The general problem of multicriteria optimization is

$$(F : U_d \rightarrow B) \rightarrow min(\Theta) \quad (14)$$

where the set of the admissible controls U_d is a subset of a linear space U, the goal space B is partially ordered Banach space with a closed convex cone Θ. Moreover it is assumed that the admissible set $F(U_d)$ is nonempty and closed.

D. Pareto-optimal approach

A control u_{opt} is said to be *nondominated or Pareto-optimal, or*

Θ *-optimal* if and only if

$$(F(u_{opt}) - \Theta) \cap F(U_d) = \{F(u_{opt})\} \quad (15)$$

The condition (15) means that no element of admissible set is better than u_{opt} in the sense of the partial order relation.

The relation (12) plays the fundamental role in the classical problems of multicriteria optimization which can be reduced to the simultaneous minimization of scalar functions

$$(F_1, F_2, ..., F_m) \rightarrow min \quad (16)$$

E. Scalarization methods

The most frequently used scalarization method for the problem

$$(F : U_d \rightarrow R^n) \rightarrow min(R_+^n) \quad (17)$$

is the positive convex combination of the criteria

$$F_w(u) = \sum_{i=1}^{N} w_i F_i(u) \quad (18)$$

where $u \in U_d$, $w_i > 0$ for $1 \leq i \leq N$ and $\sum_{i=1}^{N} w_i = 1$. The parameters $w_i > 0$, $1 \leq i \leq N$ are weight coefficients.

The scalarization by distance

$$F_d(u) = d(q, F(u)) \tag{19}$$

where d is a metric in the goal space, q is a fixed unattainable element of the goal space which dominates at least one point from $F(U_d)$, for instance

$$F_p(u) = \|q - F(u))\|_p^p \tag{20}$$

where $\|.\|_p^p$ is a p-th power of the norm in L_p space. For instance, as the scalarizing family for the finite-dimensional multicriteria optimization problem with respect to the natural partial order in R^n one can consider the following family of functionals

$$N_p(u, w) = \sum_{i=1}^{N} w_i (F_i(u) - q_i)^p, \; w \in R_+^n \backslash \{0\}, \; 1 \leq p \leq \infty \tag{21}$$

In particular case, when $n = 2$, $U_d = R^1 = \{k : -\infty < k < +\infty\}$ and $F_n = I_n$, $n = 1, 2$ an illustration of a dominated point q and relation (19) is given in Fig.1

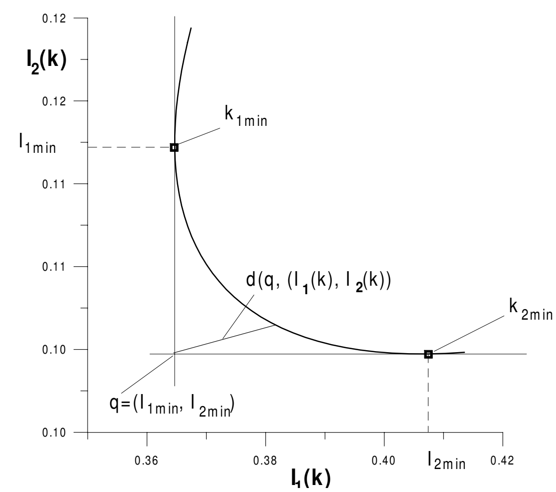

Fig. 1. A geometric illustration of condition 19

III. Applications for single degree of freedom systems

Consider the single degree of freedom system described by

$$\begin{aligned} dx_1 &= x_2 dt \\ dx_2 &= [-f(x_1) - 2hx_2]dt + \sigma d\xi(t) \end{aligned} \tag{22}$$

where h and σ are constant parameters f is a nonlinear function such that $f(0) = 0$. Then the mean value of the stationary solution is equal to zero i.e $E[x_1] = 0$.

An equivalent linearized system has the form

$$\begin{aligned} dx_1 &= x_2 dt \\ dx_2 &= [-kx_1 - 2hx_2]dt + \sigma d\xi(t) \end{aligned} \tag{23}$$

where k is a linearization coefficient.

The most frequently used scalarization method is the positive convex combination of considered criteria, i.e.

$$I_{opt}(k) = \sum_{i=1}^{N} \alpha_i I_i(k) \tag{24}$$

where I_{opt} and I_i, $i = 1, ..., N$ are multiobjective criterion of linearization and partial criteria of linearization, respectively, $\alpha_i > 0$, $i = 1, ..., N$ are weight coefficients such that $\sum_{i=1}^{N} \alpha_i = 1$.

The idea of the Pareto-optimal solution is to find a nondominated point $\mathbf{q}$ which coordinates are defined by minimal values of considered criteria i.e.

$$\mathbf{q}(\mathbf{k}) = \mathbf{q}(\mathbf{I_{i_{min}}}(\mathbf{k})) \quad i = 1, ..., N, \tag{25}$$

where

$$I_{i_{min}}(\mathbf{k})) = \min_{\mathbf{k}} I_i(\mathbf{k}) \quad i = 1, ..., N, \tag{26}$$

The scalarization distance d_w is defined, for instance by

$$d_w = \left(\sum_{i=1}^{N} \alpha_i (I_i(\mathbf{k}) - I_{i_{min}}(\mathbf{k}) \right)^{1/2} \tag{27}$$

where $\alpha_i > 0$, $i = 1, ..., N$ are weight coefficients such that $\sum_{i=1}^{N} \alpha_i = 1$.

To illustrate an application of Pareto-optimal approach and scalarization method in the determination of response characteristics we use two criteria of statistical linearization ($N = 2$).

The corresponding criteria and linearization coefficients have the following forms.

A. Statistical linearization criteria in state space

Mean-square Criterion

$$I_{MS} = \frac{E[f^2(x_1)]E[x_1^2] - (E[f(x_1)x_1])^2}{E[x_1^2]} \tag{28}$$

$$k_{MS} = \frac{E[f(x_1)x_1])}{E[x_1^2]} \tag{29}$$

Energy Criterion

$$I_E = \frac{E\left[\left(\int_0^{x_1} f(s)ds\right)^2\right] \frac{1}{4} E[x_1^4]}{\frac{1}{4}E[x_1^4]} - \frac{\left(E\left[\frac{x_1^2}{2}\int_0^{x_1} f(s)ds\right]\right)^2}{\frac{1}{4}E[x_1^4]} \tag{30}$$

$$k_E = \frac{E\left[\frac{x_1^2}{2}\int_0^{x_1} f(x_1)dx_1\right]}{\frac{1}{4}E[x_1^4]} \quad (31)$$

B. Probability density linearization criteria

Pseudomoment metric

$$I_{PMM} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} {x_1}^{2p}{x_2}^{2q}|g_N(\mathbf{x}) - g_L(\mathbf{x})|dx_1dx_2 \quad (32)$$

where $\mathbf{x} = (x_1, x_2)$, $p + q = r$, $p, q = 0, 1, ...,$

Square probability metric

$$I_{PSM} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (g_N(\mathbf{x}) - g_L(\mathbf{x}))^2 dx_1dx_2 \quad (33)$$

where $g_N(\mathbf{x})$ and $g_L(\mathbf{x})$ are probability density functions of solutions of nonlinear (22) and linearized systems (23), respectively. $g_N(x_1, x_2)$ is defined by the Gramm-Charlier expansion [8] and can be determined by stationary moments of nonlinear systems.

IV. Applications in control problems

The multicriteria analysis was also used to the determination of optimal control for linear stochastic systems by Piunovski [6] and by Radievski [8]. In this section we extend these results for a class of nonlinear systems combining multicriteria approach, statistical linearization with LQG technique.

Consider the following optimal control problem. The nonlinear stochastic model of dynamic system is described by

$$d\mathbf{x}(t) = [\mathbf{A}\mathbf{x}(t) + \mathbf{\Phi}(\mathbf{x}) + \mathbf{B}\mathbf{u}(t)]dt + \sum_{k=1}^{M} \mathbf{G}_k d\xi_k(t) \quad (34)$$

where $\mathbf{x} \in R^n$ and $\mathbf{u} \in R^m$ are the state vector and the control vector,respectively. $\mathbf{A}$ and $\mathbf{B}$ are time invariant matrices of appropriate dimensions, $\mathbf{\Phi} = [\Phi_1, ..., \Phi_n]^T$ is a nonlinear vector function, such that, $\mathbf{\Phi}(\mathbf{0}) = \mathbf{0}$, $\mathbf{G}_k$ are time invariant deterministic vectors, ξ_k are independent standard Wiener processes, for $k = 1, 2, ..., M$. We assume that the unique solution of equation (34) exists and the system is controllable.

The control strategy is designed to minimize the criterion,

$$I = E[\mathbf{x^T Q x} + \mathbf{u^T R u}], \quad (35)$$

where $\mathbf{Q}$ and $\mathbf{R}$ are time-invariant positive definite symmetric matrices.

A. Quasi-optimal control

We assume that the nonlinear vector $\mathbf{\Phi}(\mathbf{x})$ can be substituted by a linearized form

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{A_e x} \quad (36)$$

where $\mathbf{A_e}$ is a $n \times n$ matrix of linearization coefficients such that $(\mathbf{A} + \mathbf{A_e}, \mathbf{B})$ is stabilizable and detectable. Then the optimal control for the linearized system

$$d\mathbf{x}_L(t) = [(\mathbf{A} + \mathbf{A_e})\mathbf{x}_L(t) + \mathbf{B}\mathbf{u}(t)]dt + \sum_{k=1}^{M} \mathbf{G}_k d\xi_k(t) \quad (37)$$

can be found by a standard method [6] in the linear feedback form

$$\mathbf{u} = -\mathbf{K}\mathbf{x}_L, \quad \mathbf{K} = \mathbf{R^{-1}B}^T\mathbf{P} \quad (38)$$

where $\mathbf{K}$ is the gain matrix and $\mathbf{P}$ is a positive solution of the following algebraic Riccati equation:

$$\mathbf{P}(\mathbf{A} + \mathbf{A}_e) + (\mathbf{A} + \mathbf{A}_e)^T\mathbf{P} - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0}. \quad (39)$$

Substituting (38) into equation (37) yields:

$$d\mathbf{x}_L(t) = [(\mathbf{A} + \mathbf{A_e} - \mathbf{BK})\mathbf{x}_L(t)]dt + \sum_{k=1}^{M} \mathbf{G}_k d\xi_k(t). \quad (40)$$

The corresponding covariance equation and criterion have the form

$$(\mathbf{A} + \mathbf{A}_e - \mathbf{BK})\mathbf{V}_L + \mathbf{V}_L(\mathbf{A} + \mathbf{A}_e - \mathbf{BK})^T + \sum_{k=1}^{M} \mathbf{G}_k\mathbf{G}_k^T = \mathbf{0} \quad (41)$$

and

$$I_L = E[\mathbf{x}_L^T(\mathbf{Q} + \mathbf{K}^T\mathbf{RK})\mathbf{x}_L] = tr[(\mathbf{Q} + \mathbf{K}^T\mathbf{RK})\mathbf{V}_L] \quad (42)$$

where the subindex L corresponds to the linearized problem, "tr" denotes the trace of matrix.

$$\mathbf{V}_L = E[\mathbf{x}_L\mathbf{x}_L^T] \quad (43)$$

In the case of statistical linearization the elements of nonlinear vector $\mathbf{\Phi}(x)$ have to be replaced by corresponding equivalent elements "in the sense of a given criterion" in a linear form.

The following two moment criteria for scalar functions $\phi_i(x)$ are considered

Criterion 1 Mean-square error of displacements [4]

$$E[(c_{1i}x - \phi_i(x))^2] \to min \quad (44)$$

Criterion 2 Mean-square error of potential energies [2]

$$E\left[(\int_0^x (c_{2i}y - \phi_i(y))dy)^2\right] \to min \quad (45)$$

In the case of the application of linear feedback gain obtained for the linearized system to the nonlinear system, the state equation and the corresponding criterion have the form

$$d\mathbf{x}_N(t) = [(\mathbf{A}-\mathbf{BK})\mathbf{x}_N(t)+\mathbf{\Phi}(\mathbf{x}_N(t))]dt+\sum_{k=1}^{M}\mathbf{G}_k d\xi_k(t) \quad (46)$$

and

$$I_N = tr[(\mathbf{Q}+\mathbf{K}^T\mathbf{RK})\mathbf{V}_N], \quad \mathbf{V}_N = E[\mathbf{x}_N\mathbf{x}_N^T] \quad (47)$$

where the subindex N denotes the original nonlinear problem with

$$\mathbf{V}_N = E[\mathbf{x}_N\mathbf{x}_N^T] \quad (48)$$

In general, the covariance matrix $\mathbf{V}_N$ can be found approximately. To obtain the linearization matrix $\mathbf{A}_e$ and quasi-optimal control one of the four proposed criteria should be selected and used with an iterative procedure.

B. Iterative procedure for multicriteria nonlinear control problem

The following procedure is an extended version of a standard one given in [13]

Step 1. Choose a parameter of the nonlinear system (an element of eq. (34)) and criteria of linearization, then select $\mathbf{A}_e = 0$ in (36). Next for every criterion repeat Steps 2 - 11.

Step 2. Solve (39). The solution of (39) is $\mathbf{P}$.

Step 3. Substitute $\mathbf{P}$ obtained in step 2 into (38) and find the matrix $\mathbf{K}$. Next, substitute $\mathbf{K}$ and $\mathbf{A}_e = \mathbf{0}$ into equation (41) and solve the equation. The solution of equation (41) is $\mathbf{V}_L$.

Step 4. Substitute $\mathbf{P}$ obtained in step 2 into (42) and find I_L

Step 5. For each nonlinear element find the linearization coefficient which minimize the selected criterion, for instance, (44) or (45).

Step 6. Substitute the matrix of linearization coefficients $\mathbf{A}_e(\mathbf{V}_L)$ obtained in step 5 into equation (41) and then solve the equation.

Step 7. If the error of approximation of $\mathbf{V}_L$ is greater then a given parameter ε_1 then repeat steps 5 - 6 until $\mathbf{V}_L$ converges.

Step 8. Substitute the matrix of linearization coefficients $\mathbf{A}_e(\mathbf{V}_L)$ into Riccati equation (39) and then solve the equation.

Step 9. Substitute the matrix $\mathbf{P}$ obtained in step 8 into covariance equation (41) and then solve the equation.

Step 10. If the error of approximation of $\mathbf{V}_L$ and $\mathbf{P}$ is greater then a given ε_2 then repeat steps 5 - 9 until $\mathbf{V}_L$ and $\mathbf{P}$ converge.

Step 11. Calculate criteria I_L and I_N given by (42) and (47), respectively.

Step 12. Calculate a measure of multicriteria optimization problem based on criteria calculated in Step 11 for the different linearization criteria chosen in Step 1.

C. Example Duffing oscillator

Consider the Duffing oscillator described by

$$\begin{aligned} dx_1 &= x_2dt \\ dx_2 &= [-\omega_0^2x_1 - \varepsilon x_1^3 - 2hx_2 + bu]dt + \sigma d\xi(t) \end{aligned} \quad (49)$$

where $-\omega_0^2, \varepsilon, h, b$ and σ are constant parameters, u is a scalar control, $\xi(t)$ is the standard Wiener process and the mean-square criterion is

$$I = E[\mathbf{x}^T\mathbf{Q}\mathbf{x} + ru^2] \quad (50)$$

where $x = [x_1, x_2]^T$, $\mathbf{Q} = diag(Q_i)$, $i = 1, 2$; Q_i, r are positive constants. The linearized system has the following form

$$\begin{aligned} dx_1 &= x_2dt \\ dx_2 &= [-\omega_0^2x_1 - \varepsilon cx_1 - 2hx_2 + bu]dt + \sigma d\xi(t) \end{aligned} \quad (51)$$

where c is a linearization coefficient. The coordinates of the solutions of algebraic Riccati and covariance equations denoted by $\mathbf{P} = [p_{ij}]$ and $\mathbf{V}_L = [v_{L_{ij}}]$, respectively, for i,j=1 and 2, are the following

$$p_{12} = \frac{-c+\sqrt{c^2+Q_1\beta}}{\beta}, p_{22} = \frac{\sqrt{4h^2+\beta(Q_2+2p_{12})}}{\beta}$$
$$p_{11} = 2hp_{12} + cp_{22} + \beta p_{12}p_{22} \quad (52)$$

and

$$v_{L_{22}} = \frac{g^2}{2(2h+\beta p_{22})}, v_{L_{12}} = 0, \quad v_{L_{11}} = \frac{v_{22}}{\gamma+\beta p_{12}} \quad (53)$$

where $\beta = b^2/r$. The optimal value of the criterion for linearized system is

$$I_L = (Q_1+\beta p_{12}^2)v_{L_{11}} + (Q_2+\beta p_{22}^2)v_{L_{22}} \quad (54)$$

Applying the obtained linear feedback control to nonlinear system we obtain the state equation and the corresponding criterion

$$\begin{aligned} dx_1 &= x_2dt, \\ dx_2 &= [-2hx_2 - \omega_0^2 - \varepsilon x_1^3 - \beta(x_1p_{12} + x_2p_{22})]dt + \sigma d\xi \end{aligned} \quad (55)$$

and

$$I_{N_{opt}} = (Q_1+\beta p_{12}^2)v_{N_{11}} + (Q_2+\beta p_{22}^2)v_{N_{22}} \quad (56)$$

where the second order moments $v_{N_{11}}$ and $v_{N_{22}}$ can be found in analytical form, from

$$v_{N_{ii}} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_i^2 g_N(x_1,x_2)dx_1dx_2, i = 1, 2. \quad (57)$$

where

$$g_N(x_1,x_2) = \frac{1}{c_N} exp\{-\frac{2h+\beta p_{22}}{\sigma^2}(\omega_0^2+\beta p_{12})x_1^2+\alpha\frac{x_1^4}{2}+x_2^2\} \quad (58)$$

and c_N is a normalized constant.

One can show [2], [4] that linearization coefficients for two considered criteria have the form

$$c_1 = 3E[x_1^2], \quad c_2 = 2.5E[x_1^2] \quad (59)$$

To obtain the quasi-optimal controls and corresponding mean-square criteria I_1 and I_2 depending on the choice of linearization coefficients c_1 and c_2 one can use the iterative procedure proposed in previous section. To illustrate the obtained results a comparison of considered criteria is discussed. The set of dominating points for parameters $\omega_0^2 = 1$, $h = 0.05$, $b = 1$, $\varepsilon = 1$, $\sigma = 1$, $Q_1 = Q_2 = 1$, $r = 100$ is presented in Fig.2.

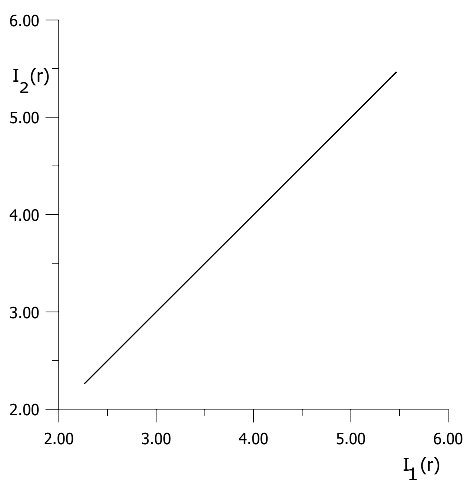

Fig. 2. A graphical illustration of the set of dominating points

Figure 2 shows that the considered mean-square criteria I_1 and I_2 are linearly dependent and there are not dominated points. It means the quasi-optimal controls obtained for both criteria are the same. This confirms an earlier observation presented in earlier author's paper [13].

V. Conclusions

Numerical studies show that in response analysis there are significant differences between the obtained linearization coefficients and corresponding criteria in contrast to control problems where for a given mean-square criterion of minimization (35) there are no significant differences between applied linearization methods. It means, the mean-square criteria corresponding to different linearization criteria are linearly dependent. This linear dependence also appears in application of linearization techniques with criteria in probability density functions space.

VI. Acknowledgement

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