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A Multiobjective Cost Cumulant Control Problem: The Nash Game Solution

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Abstract— **Recently cost cumulant control has been effectively applied to vibration control problems. Similarly so have multiobjective control paradigms. In this paper the cost cumulant control idea is used to solve the case when the system has some structured uncertainty, with two players, the control and disturbance, having their own performance indices. The development at first is done for a class of nonlinear systems, non-quadratic cost and then is later applied to the linear, quadratic special case. The two performance indices will be linear combinations of cumulants, in which one shall be for the control and the other will constrain the input-output relationship of the disturbance to a output. The control solution is then applied to the First Generation Benchmark for a 3-story building under seismic excitation. The results will be compared with those of several different controllers.**

I. Introduction

In recent times the k cost cumulant (kCC) and the minimum cost variance (MCV) control problems have gained attention [7]-[12], [13]. These control methods generalize the approach of minimizing the mean of a cost function that is so prevalent in the area of control. They let the control minimize a linear combination of the cumulants. In the MCV problem this means minimizing a linear combination of the mean and the variance of a cost function, whereas the kCC goes beyond these two most well known cumulants. These methods have been applied successfully to vibration control problems, in particular the control of structures excited by winds and seismic disturbances. Also, there has been the application of multiobjective methods [5] to this same problem. The approach taken in this paper is to combine these two problems. The idea presented is to allow the control to minimize a linear combination of the first two cumulants while satisfying some constraint on the systems induced norm. In doing this a Nash game approach shall be taken in a manner similar to that of [6], [3]. The development will be carried out for a class of nonlinear systems with non-quadratic costs. It shall then be applied to the case when the system is linear and costs are quadratic. The Nash game shall involve two players, a control and a disturbance. Later the disturbance will be given as the result of some "structured" uncertainty inherent in the system. Lastly the control will be applied to the First Generation Structural Benchmark for buildings under seismic excitation.

II. Preliminaries

Consider the following stochastic differential equation

$$dx(t) = f(t,x(t),u(t),w(t))dt + \sigma(t,x(t))d\xi(t) \qquad (1)$$

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where $x(t_0) = x_0$ is a random variable independent of ξ, ξ is d-dimensional Brownian motion on the probability space $(\Omega, \mathscr{F}, P)$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathscr{U}$ is the control, $w(t) \in \mathscr{W}$ is the disturbance, and $t \in T = [t_0, t_f]$. Let $Q_0 = (t_0, t_f) \times \mathbb{R}^n$ and $\bar{Q}_0$ be its closure, that is $\bar{Q}_0 = T \times \mathbb{R}^n$. Assume the functions f and σ to be Borel measurable and of class $C^1(\bar{Q}_0 \times \mathscr{U} \times \mathscr{W})$ and $C^1(\bar{Q}_0)$ respectively. This means that the functions f and σ have continuous partial derivatives of first order. Furthermore assume that f and σ satisfy the following conditions.

(i) There exists a constant C such that

$$\|f(t,x,u,w)\| \le C(1 + \|x\| + \|u\| + \|w\|)$$
$$\|\sigma(t,x)\| \le C(1 + \|x\|)$$

for all $(t,x,u,w) \in \bar{Q}_0 \times \mathscr{U} \times \mathscr{W}$, $(t,x) \in \bar{Q}_0$, and $\|\cdot\|$ is the Euclidean norm.

(ii) There is a constant K so that

$$\begin{aligned}\|f(t,\tilde{x},\tilde{u},\tilde{w}) - f(t,x,u,w)\| \le& K(\|\tilde{x}-x\| + \|\tilde{u}-u\| \\ &+ \|\tilde{w}-w\|)\\ \|\sigma(t,\tilde{x}) - \sigma(t,x)\| \le& K\|\tilde{x}-x\|\end{aligned}$$

for all $t \in T$; $x, \tilde{x} \in \mathbb{R}^n$; $u, \tilde{u} \in \mathscr{U}$; $w, and \tilde{w} \in \mathscr{W}$.

Now we shall assume some conditions on the strategies of the control and disturbance. First assume that the strategies are of the form $u(t) = \mu(t,x(t))$ and $w(t) = \nu(t,x(t))$. Furthermore the functions $\mu : \bar{Q}_0 \to \mathscr{U}$ and $\nu : \bar{Q}_0 \to \mathscr{W}$ are assumed to be Borel measurable and to satisfy

(i) for some constant $\tilde{C}$

$$\|\mu(t,x)\| \le \tilde{C}(1 + \|x\|) \quad \text{and} \quad \|\nu(t,x)\| \le \tilde{C}(1 + \|x\|)$$

(ii) there exists a constant $\tilde{K}$ such that

$$\|\mu(t,\tilde{x}) - \mu(t,x)\| \le \tilde{K}(\|\tilde{x}-x\|)$$
$$\|\nu(t,\tilde{x}) - \nu(t,x)\| \le \tilde{K}(\|\tilde{x}-x\|)$$

where $t \in T$ and $x, \tilde{x} \in \mathbb{R}^n$. Often we will suppress the dependence on t and x and refer to the strategies as μ and ν. If the strategies μ and ν satisfy these conditions, then they are admissible strategies. We can rewrite the stochastic differential equation as

$$dx(t) = \tilde{f}(t,x(t))dt + \sigma(t,x(t))d\xi(t) \qquad x(t_0) = x_0 \qquad (2)$$

where the strategy (μ, ν) has been substituted into f and now called $\tilde{f}$. The conditions of Theorem V4.1 of [4] are now satisfied and we see that if $E\|x(t_0)\|^2 < \infty$, then a solution of (1) exists. Furthermore the solution $x(t)$ is unique in the sense that if there exists another solution $\tilde{x}(t)$ with $\tilde{x}(t_0) = x_0$, then the two solutions have the same sample paths with probability 1. The resulting process is a Markov diffusion process ([4] pg. 123) and the moments of $x(t)$ are bounded.

Let $C^{1,2}(\bar{Q}_0)$ be the class of functions Φ that have continuous first partial derivatives with respect to t and continuous second partial derivatives with respect to x: $\Phi_t, \Phi_{x_i}, \Phi_{x_i x_j}$ for $i,j = 1,2,\cdots,n$. Now let $C_p^{1,2}(\bar{Q}_0)$ be the class of functions $\Phi(t,x)$ that are of class $C^{1,2}(\bar{Q}_0)$ but where $\Phi, \Phi_t, \Phi_{x_i}, \Phi_{x_i,x_j}$ satisfy a polynomial growth condition. A polynomial growth condition for a function Φ is such that there exist constants c_1 and c_2 so that $\|\Phi(t,x)\| \leq c_1(1+\|x\|^{c_2})$ for all $(t,x) \in \bar{Q}_0$. This yields the Dynkin formula

$$\begin{aligned}\Phi(t,x) =& E\left\{\int_t^{t_f} -\mathscr{O}^{\mu,\nu}\Phi(s,x(s))ds|x(t)=x\right\} \\ &+ E\left\{\Phi(t_f,x(t_f))|x(t)=x\right\}\end{aligned} \tag{3}$$

where $\mathscr{O}^{\mu,\nu}$ is the backward evolution operator given by

$$\begin{aligned}\mathscr{O}^{\mu,\nu} =& \frac{\partial}{\partial t} + f'(t,x,u,w)\frac{\partial}{\partial x} \\ &+ \frac{1}{2}tr\left(\sigma(t,x)W(t)\sigma'(t,x)\frac{\partial^2}{\partial x^2}\right)\end{aligned} \tag{4}$$

with $E\{d\xi(t)d\xi'(t)\} = W(t)$, superscript $'$ denotes transpose, and tr refers to the trace operator. The expectation in (3) will now be referred to as E_{tx}.

III. Problem Definition

The game described by (1) has with it two cost functions. The first cost function, J_1, is to be associated with the control u and the second, J_2, is for the disturbance w. Both players wish to minimize their respective cost functions. When the arguments of the state, control, or disturbance w are missing, it should be assumed that they are just suppressed. The players' cost functions are given by

$$J_1(t,x,u,w) = \int_t^{t_f} L_1(\tau,x,u,w)d\tau + \psi_1(x_f) \tag{5}$$

$$J_2(t,x,u,w) = \int_t^{t_f} L_2(\tau,x,u,w)d\tau + \psi_2(x_f) \tag{6}$$

where L_1, L_2 are the running cost functions, ψ_1, ψ_2 are the terminal cost functions for each player respectively, and $x(t_f) = x_f$. Assume the running cost L_i satisfies a polynomial growth condition $\|L_i(t,x,u,w)\| \leq k(1+\|x\|^c + \|u\|^c + \|w\|^c)$ and the terminal cost ψ_i satisfies a polynomial growth condition $\|\psi_i(t,x)\| \leq k(1+\|x\|^c)$, where k,c are some constants and for $i=1,2$. The game to be considered here is one in which the first player, the control u, wishes to minimize a performance index consisting of a linear combination of cumulants given by

$$\phi_1(t,x,u,w) = E_{tx}\{J_1(t,x,u,w)\} + \gamma Var_{tx}\{J_1(t,x,u,w)\} \tag{7}$$

where γ is some positive constant and Var_{tx} is the normal definition of variance only using the condition expectation. On the other hand, the second player, the disturbance w, wishes to minimize the mean of its cost function. That is the disturbance has

$$\phi_2(t,x,u,w) = E_{tx}\{J_2(t,x,u,w)\} \tag{8}$$

as its own performance index.

Since both players will be assumed to have feedback information available to them, $\mathscr{U}_F$ will be the information pattern for the control and $\mathscr{W}_F$ will be the information pattern for the disturbance. Thus, $\mathscr{U}_F$ is the class of all feedback strategies μ already described, and similarly for $\mathscr{W}_F$. Now we define what is meant by a Nash equilibrium solution to the game.

Definition 1: The pair (μ^*, ν^*) is a Nash equilibrium solution if it satisfies the inequalities

$$\phi_1(0,x,\mu^*,\nu^*) \leq \phi_1(0,x,\mu,\nu^*)$$

$$\phi_2(0,x,\mu^*,\nu^*) \leq \phi_2(0,x,\mu^*,\nu)$$

$\forall \mu \in \mathscr{U}_F$ and $\forall \nu \in \mathscr{W}_F$.

Now let $V_1(t,x;\mu,\nu) = E_{tx}\{J_1(t,x,u,w)\}$ and $V_2(t,x;\mu,\nu) = E_{tx}\{J_1^2(t,x,u,w)\}$ be the first and second moments of the cost function $J_1(t,x,u,w)$.

Definition 2: A function $M : \bar{Q}_0 \to \mathbb{R}^+$ is an admissible mean cost function if there exists an admissible strategy μ such that $M(t,x) = V_1(t,x;\mu,\nu^*)$ for $t \in T, x \in \mathbb{R}^n$.

From now on we shall assume that M is an admissible mean cost function.

Definition 3: M defines a class of admissible strategies $\mathscr{U}_M$ such that $\mu \in \mathscr{U}_M$ if and only if the strategy μ is admissible and satisfies Definition 2.

Definition 4: An MCV control strategy $\mu^* \in \mathscr{U}_M$ is one that minimizes the second moment, i.e. $V_2(t,x,\mu^*,\nu^*) = V_2(t,x) \leq V_2(t,x,\mu,\nu^*)$ for $t \in T, x \in \mathbb{R}^n, \nu^* \in \mathscr{W}_F$, where $\mu \in \mathscr{U}_M$. Furthermore the variance is found through $V(t,x) = V_2(t,x) - M^2(t,x)$.

IV. Nonlinear Nash Solution

We shall begin this section by giving several lemmas that will be used in the proof of the control's Nash equilibrium strategy. The first lemma will help by providing a necessary condition for the mean of the cost function.

Lemma 1: Let $M \in C_p^{1,2}(\bar{Q}_0)$ be an admissible mean cost function and μ be an admissible control strategy such that it satisfies Definition 2. Under these assumptions the admissible mean cost function M satisfies

$$\mathscr{O}^{\mu,\nu^*}M(t,x) + L_1(t,x,\mu,\nu^*) = 0 \tag{9}$$

where $M(t_f,x_f) = \psi_1(x_f)$.

Now we have the following Verification Lemma for the mean of the cost function. It provides sufficient conditions for the mean value function. Here the set Q is to be an open subset of Q_0.

Lemma 2 (Verification Lemma): Let $M \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be a solution to

$$\mathscr{O}^{\mu,\nu^*}M(t,x) + L_1(t,x,\mu,\nu^*) = 0 \tag{10}$$

with boundary condition $M(t_f,x_f) = \psi_1(x_f)$. Then $M(t,x) = V_1(t,x;\mu,\nu^*)$ for all $\mu \in \mathscr{U}_M$.

Now that we have the results for the mean of the cost, we have the following Verification Lemma for the second moment of the cost.

Lemma 3 (Verification Lemma): Let $V_2 \in C_p^{1,2}(Q) \cap C(\bar{Q})$ be a nonnegative solution to the partial differential equation

$$\min_{\mu \in \mathscr{U}_M} \left\{ \mathscr{O}^{\mu,\nu^*} V_2(t,x) + 2M(t,x)L_1(t,x,\mu,\nu^*) \right\} = 0 \quad (11)$$

with boundary condition $V_2(t_f,x_f) = \psi_1^2(x_f)$. Then $V_2(t,x) \leq V_2(t,x;\mu,\nu^*)$ for every $\mu \in \mathscr{U}_M$, and $(t,x) \in \bar{Q}_0$. If μ also satisfies

$$\begin{aligned} \min_{\tilde{\mu} \in \mathscr{U}_M} &\left\{ \mathscr{O}^{\tilde{\mu},\nu^*} V_2(t,x) + 2M(t,x)L_1(t,x,\tilde{\mu},\nu^*) \right\} \\ &= \mathscr{O}^{\mu,\nu^*} V_2(t,x) + 2M(t,x)L_1(t,x,\mu,\nu^*) \end{aligned} \quad (12)$$

for all $(t,x) \in \bar{Q}_0$, then $V_2(t,x) = V_2(t,x;\mu,\nu^*)$.

Proof. The proof follows closely that of Theorem 4.2 of [13]. □

From these lemmas, we can begin to discuss the Nash equilibrium solution. The following theorem provides sufficient conditions for the Nash equilibrium solution.

Theorem 1: Consider the two player game described by (1), (7), and (8). Let M be an admissible mean cost function, $M \in C_p^{1,2}(Q) \cap C(\bar{Q})$, with an associated $\mathscr{U}_M$. Also consider the function $V \in C_p^{1,2}(Q) \cap C(\bar{Q})$ that is a solution to

$$\begin{aligned} \min_{\mu \in \mathscr{U}_M} &\left\{ \frac{\partial V}{\partial t}(t,x) + f'(t,x,\mu,\nu^*) \frac{\partial V}{\partial x}(t,x) \right. \\ &+ \frac{1}{2} tr \left(\sigma(t,x) W(t) \sigma'(t,x) \frac{\partial^2 V}{\partial x^2}(t,x) \right) \\ &\left. + \left| \frac{\partial M}{\partial x}(t,x) \right|^2_{\sigma(t,x)W(t)\sigma'(t,x)} \right\} = 0 \end{aligned} \quad (13)$$

with $V(t_f,x_f) = 0$ and the function $P \in C_p^{1,2}(Q) \cap C(\bar{Q})$ that satisfies

$$\begin{aligned} \min_{\nu \in \mathscr{W}_F} &\left\{ \frac{\partial P}{\partial t}(t,x) + f'(t,x,\mu^*,\nu) \frac{\partial P}{\partial x}(t,x) \right. \\ &+ \frac{1}{2} tr \left(\sigma(t,x) W(t) \sigma'(t,x) \frac{\partial^2 P}{\partial x^2}(t,x) \right) \\ &\left. + L_2(t,x,\mu^*,\nu) \right\} = 0 \end{aligned} \quad (14)$$

with $P(t_f,x_f) = \psi_2(x_f)$. If the strategies μ^* and ν^* are the minimizing arguments of (13) and (14), then the pair (μ^*,ν^*) constitutes a Nash equilibrium solution.

Proof. To start out the proof, assume that the control's Nash equilibrium solution has been played and $P \in C_p^{1,2}(Q) \cap C(\bar{Q})$. Then we have a minimal mean of the cost problem for the disturbance w. Assume the disturbance plays the strategy $\nu(t,x(t))$, which may or may not minimize (14). This yields

$$\mathscr{O}^{\mu^*,\nu} P(t,x) + L_2(t,x,\mu^*,\nu) \geq 0. \quad (15)$$

But by the Dynkin formula and (15) we have

$$\begin{aligned} P(t,x) &= E_{tx} \left\{ \int_t^{t_f} -\mathscr{O}^{\mu^*,\nu} P(s,x) ds + \psi_2(x_f) \right\} \\ &\leq E_{tx} \{ J_2(t,x,\mu^*,\nu) \} \end{aligned} \quad (16)$$

where E_{tx} is as previously defined. Notice that if the disturbance plays a strategy ν^* that minimizes (14), we have $P(t,x) = E_{tx}\{J_2(t,x,\mu^*,\nu^*)\}$, and thus if μ^* is the control's Nash equilibrium solution, then Definition 1 is satisfied and ν^* is the disturbance's Nash equilibrium strategy.

For the second part of the proof, let the disturbance play its Nash equilibrium strategy ν^*. Now assume M is an admissible mean cost function such that $M^2 \in C_p^{1,2}(Q) \cap C(\bar{Q})$ and assume $V \in C_p^{1,2}(Q) \cap C(\bar{Q})$ satisfies (13). Now let $\mu \in \mathscr{U}_M$ be an admissible control strategy which may or may not be the minimal strategy in (13). Recall that if the control strategy μ is in the class of admissible mean strategies $\mathscr{U}_M$, then it is such that

$$\mathscr{O}^{\mu,\nu^*} M(t,x) + L_1(t,x,\mu,\nu^*) = 0 \quad (17)$$

where $M(t_f,x_f) = \psi_1(x_f)$. Since μ may or may not be optimal we have

$$\mathscr{O}^{\mu,\nu^*} V(t,x) + \left| \frac{\partial M}{\partial x}(t,x) \right|^2_{\sigma(t,x)W(t)\sigma'(t,x)} \geq 0 \quad (18)$$

where $V(t_f,x_f) = 0$. Manipulating the above equation yields

$$\begin{aligned} V(t,x) &= E_{tx} \left\{ \int_t^{t_f} -\mathscr{O}^{\mu,\nu^*} V(s,x) ds \right\} \\ &\leq E_{tx} \left\{ \int_t^{t_f} \left| \frac{\partial M}{\partial x}(s,x) \right|^2_{\sigma(s,x)W(s)\sigma'(s,x)} ds \right\} \end{aligned} \quad (19)$$

where once again the Dynkin formula is used. Recall that in order for $V(t,x)$ to be a value function for the variance, it must be such that $V(t,x) = V_2(t,x) - M^2(t,x)$. But since $M^2 \in C_p^{1,2}(Q) \cap C(\bar{Q})$, then $V_2 \in C_p^{1,2}(Q) \cap C(\bar{Q})$. Using the definition of variance we have

$$V_2(t,x) - M^2(t,x) \leq E_{tx} \left\{ \int_t^{t_f} \left| \frac{\partial M}{\partial x}(s,x) \right|^2_{\sigma W \sigma'} ds \right\} \quad (20)$$

which yields

$$\begin{aligned} V_2(t,x) \leq & E_{tx} \left\{ \int_t^{t_f} \left| \frac{\partial M}{\partial x}(s,x) \right|^2_{\sigma W \sigma'} ds \right\} \\ & - E_{tx} \left\{ \int_t^{t_f} \mathscr{O}^{\mu,\nu^*} M^2(s,x) ds \right\} + E_{tx}\{\psi_1^2(x_f)\} \end{aligned} \quad (21)$$

by the application of the Dynkin formula to the function $M^2(t,x)$. From Lemma 3 we also have

$$\begin{aligned} V_2(t,x) \leq & E_{tx} \left\{ \int_t^{t_f} 2M(s,x)L_1(s,x,\mu,\nu^*) ds \right\} \\ & + E_{tx}\{\psi_1^2(x_f)\} \end{aligned} \quad (22)$$

with another usage of the Dynkin formula. If we can show that the right members of (21) and (22) are equal, then we can employ the techniques of Lemma 3 to obtain the desired result. To do so, we examine the equality

$$\begin{aligned} \mathscr{O}^{\mu,\nu^*} M^2(t,x) &+ 2M(t,x)L_1(t,x,\mu,\nu^*) \\ &= \left| \frac{\partial M}{\partial x}(t,x) \right|^2_{\sigma W \sigma'} \end{aligned} \quad (23)$$

for $(t,x) \in Q$. The first step in this pursuit is to let $\mathscr{O}^{\mu,\nu} = \mathscr{O}_1^{\mu,\nu} + \mathscr{O}_2$ where

$$\mathscr{O}_1^{\mu,\nu} = \frac{\partial}{\partial t} + f'(t,x,\mu(t,x),\nu(t,x))\frac{\partial}{\partial x}$$
$$\mathscr{O}_2 = \frac{1}{2} tr\left(\sigma(t,x)W(t)\sigma'(t,x)\frac{\partial^2}{\partial x^2}\right)$$

and with this definition we have for $\mathscr{O}^{\mu,\nu^*}M^2(t,x)$

$$\mathscr{O}^{\mu,\nu^*}M^2(t,x) = 2M(t,x)\mathscr{O}_1^{\mu,\nu^*}M(t,x) + \mathscr{O}_2 M^2(t,x).$$

Recall that M is an admissible mean cost function, therefore $-\mathscr{O}^{\mu,\nu^*}M(t,x) = L_1(t,x,\mu,\nu^*)$. With these two observations, (23) reduces to

$$\mathscr{O}_2 M^2(t,x) - 2M(t,x)\mathscr{O}_2 M(t,x) = \left|\frac{\partial M}{\partial x}(t,x)\right|^2_{\sigma W \sigma'}. \quad (24)$$

We wish to show that (24) holds. Recall that

$$\frac{\partial^2 M^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial M^2}{\partial x}\right)'$$

so that the left members of (24) becomes

$$\begin{aligned}&\frac{1}{2} tr\left(\sigma W \sigma'\left[\frac{\partial^2 M^2}{\partial x^2} - 2M(t,x)\frac{\partial^2 M}{\partial x^2}\right]\right)\\ &= tr\left(\sigma W \sigma'\left(\frac{\partial M}{\partial x}\right)\left(\frac{\partial M}{\partial x}\right)'\right)\end{aligned} \quad (25)$$

where the arguments are suppressed. But notice that the right members in (25) equals the right members of (24). Therefore the equality (23) is established. This results in (21) and (22) being equivalent, which in turn says that $V_2(t,x) \le V_2(t,x;\mu,\nu^*)$ for all $\mu \in \mathscr{U}_M$ and $(t,x) \in Q$. With another application of Definition 4 we see that $V(t,x) \le V(t,x;\mu,\nu^*)$ for all $\mu \in \mathscr{U}_M$ and $(t,x) \in Q$ and therefore we have reached the desired result. Note that if indeed $\mu = \mu^*$, the inequalities in the second part of the proof become equalities and therefore $V(t,x) = V(t,x;\mu^*,\nu^*)$. □

V. Application to the Linear Quadratic Special Case

Now we consider the case when the system given is linear. The system will be described by

$$\begin{aligned} dx(t) &= (A(t)x(t) + B(t)u(t) + D(t)w(t))dt + E(t)d\xi(t)\\ z_1(t) &= H_1(t)x(t) + G_1(t)u(t)\\ z_2(t) &= H_2(t)x(t) + G_2(t)u(t)\end{aligned}$$

where $x(t_0) = x_0$ and z_1, z_2 are the regulated outputs of the system. It also will be assumed that $H_i'H_i = Q_i$, $G_i'H_i = 0$, and $G_i'G_i = R_i$ for $i = 1,2$, where Q_i is positive semidefinite and R_i is positive definite. Furthermore the costs will be assumed to be quadratic;

$$\begin{aligned} J_1 &= \int_{t_0}^{t_f} z_1'(t)z_1(t)dt\\ J_2 &= \int_{t_0}^{t_f} (\delta^2 w'(t)w(t) - z_2'(t)z_2(t))dt\end{aligned}$$

where $Q_f^1 = Q_f^2 = 0$.

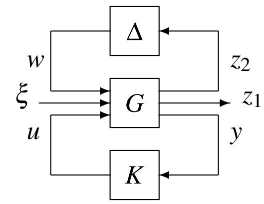

Fig. 1. Block Diagram

Notice that minimizing the performance index of the disturbance will then be imposing a constraint on the input output properties of the disturbance w to the regulated output z_2. To see this consider that for the performance index $E\{J_2\} \ge 0$ we have

$$E\left\{\int_{t_0}^{t_f} (\delta^2 w'(t)w(t) - z_2'(t)z_2(t))dt\right\} \ge 0$$

but this is the same as

$$\int_{t_0}^{t_f} E\left\{||z_2(t)||^2\right\}dt \le \delta^2 \int_{t_0}^{t_f} E\left\{||w(t)||^2\right\}dt$$

which says that δ is a constraint on the H_∞ norm of the system. The problem can be viewed by Fig. 1 where G is the plant transfer function and Δ is a structured plant uncertainty. That is the interesting part of this problem. The cost cumulant control problem involves the first two cumulants, but also has the ability to incorporate some uncertainty into our control designing equations. Notice that if we let $\gamma = 0$, then we have the H_2/H_∞ control problem. This suggests that the multiobjective cumulant control is a generalization of H_2/H_∞ control. We now apply the previous results to the linear quadratic cost cumulant problem.

Let us assume that the costs are quadratic. That is $M(t,x) = x'\mathscr{M}(t)x + m(t)$ and similarly with $V(t,x)$ and $P(t,x)$ where $\mathscr{M}, \mathscr{V}, \mathscr{P}$ are matrix functions of time and m, v, p are scalar functions of time. Now consider the HJB equation for the disturbance. We obtain

$$\begin{aligned}\min_{\nu \in \mathscr{W}_F}\Big\{&x'\dot{\mathscr{P}}x + \dot{p} + 2(Ax + B\mu^* + D\nu)'\mathscr{P}x\\ &+ tr(EWE'\mathscr{P}) + \delta^2\nu'\nu - x'Q_2x - \mu^{*\prime}R_2\mu^*\Big\} = 0\end{aligned}$$

and now performing the minimization

$$w^*(t) = \nu^*(t,x(t)) = -\frac{1}{\delta^2}D'(t)\mathscr{P}(t)x(t) \quad (26)$$

which is the form of the disturbance's Nash equilibrium solution. Now for the control. The control's performance index was a linear combination of the mean and variance. This yields

$$\begin{aligned}\min_{\mu \in \mathscr{U}_M}\Big\{&x'\dot{\mathscr{M}}x + \dot{m} + 2(Ax + B\mu + D\nu^*)'\mathscr{M}x\\ &+ tr(EWE'(\mathscr{M} + \gamma\mathscr{V})) + x'Q_1x + \mu'R_1\mu + \gamma[x'\dot{\mathscr{V}}x + \dot{v}\\ &+ 2(Ax + B\mu + D\nu^*)'\mathscr{V}x + 4\mathscr{M}EWE'\mathscr{M}]\Big\} = 0\end{aligned}$$

and minimizing this gives

$$u^*(t) = \mu^*(t,x(t)) = -R_1^{-1}B'(t)(\mathscr{M}(t)+\gamma\mathscr{V}(t))x(t) \quad (27)$$

which is the form of the controller's Nash equilibrium solution. Using this Nash equilibrium solution (μ^*, ν^*), we can determine three Riccati equations by substitution. First consider the mean of the control's cost function

$$\begin{aligned}&\dot{\mathscr{M}} + A'\mathscr{M} + \mathscr{M}A + Q_1 - \mathscr{M}BR_1^{-1}B'\mathscr{M}\\ &-\frac{1}{\delta^2}\mathscr{P}DD'\mathscr{M} - \frac{1}{\delta^2}\mathscr{M}DD'\mathscr{P}\\ &+\gamma^2\mathscr{V}BR_1^{-1}B'\mathscr{V} = 0\end{aligned} \quad (28)$$

where $\mathscr{M}(t_f) = Q_f^1$. Next we derive an expression for the variance. In a similar way use (μ^*, ν^*) to give the equation

$$\begin{aligned}&\dot{\mathscr{V}} + A'\mathscr{V} + \mathscr{V}A - \gamma\mathscr{M}BR_1^{-1}B'\mathscr{V} - \gamma\mathscr{V}BR_1^{-1}B'\mathscr{M}\\ &-\frac{1}{\delta^2}\mathscr{P}DD'\mathscr{V} - \frac{1}{\delta^2}\mathscr{V}DD'\mathscr{P} - 2\gamma\mathscr{V}BR_1^{-1}B'\mathscr{V}\\ &+4\mathscr{M}EWE'\mathscr{M} = 0\end{aligned} \quad (29)$$

with $\mathscr{V}(t_f) = 0$. Finally an expression for the mean of the disturbance's cost is given by

$$\begin{aligned}&\dot{\mathscr{P}} + A'\mathscr{P} + \mathscr{P}A - (\mathscr{M}+\gamma\mathscr{V})BR_1^{-1}B'\mathscr{P}\\ &-\mathscr{P}BR_1^{-1}B'(\mathscr{M}+\gamma\mathscr{V}) - \frac{1}{\delta^2}\mathscr{P}DD'\mathscr{P}\\ &-Q_2 - \mathscr{M}BR_1^{-1}R_2R_1^{-1}B'\mathscr{M}\\ &-\gamma\mathscr{M}BR_1^{-1}R_2R_1^{-1}B'\mathscr{V} - \gamma\mathscr{V}BR_1^{-1}R_2R_1^{-1}B'\mathscr{M}\\ &-\gamma^2\mathscr{V}BR_1^{-1}R_2R_1^{-1}B'\mathscr{V} = 0\end{aligned} \quad (30)$$

where $\mathscr{P}(t_f) = Q_f^2$. Notice that if these Riccati equations are satisfied then we know the strategy (μ^*, ν^*) given in (26) and (27). This leads to the following theorem.

Theorem 2: Consider the stochastic game in which the system is linear and the costs are quadratic. Suppose the $\mathscr{M}(t), \mathscr{V}(t), \mathscr{P}(t)$ are unique solutions to the coupled Riccati equations (28), (29), (30), then the Nash equilibrium solution $(\mu^*(t,x), \nu^*(t,x))$ is given by (27) and (26). $M(t,x)$, $V(t,x)$, and $P(t,x)$ are then constructed with the aid of

$$\begin{aligned}\dot{m}(t) &= -tr(E(t)W(t)E'(t)\mathscr{M}(t))\\ \dot{v}(t) &= -tr(E(t)W(t)E'(t)\mathscr{V}(t))\\ \dot{p}(t) &= -tr(E(t)W(t)E'(t)\mathscr{P}(t))\end{aligned}$$

where $m(t_f) = 0, v(t_f) = 0, p(t_f) = 0$.

Now that we have the linear quadratic problem solved, the solution will be used to determine a controller for the First Generation Benchmark for buildings under earthquake excitation.

VI. Structural Control Benchmark

The benchmark problem of controlling a three-story building with an active mass damper is described in [14]. For the evaluation of this benchmark there is a 28 state model given by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + E\ddot{x}_g(t)\\ y(t) &= C_yx(t) + D_yu(t) + F_y\ddot{x}_g(t) + v\\ z(t) &= C_zx(t) + D_zu(t) + F_z\ddot{x}_g(t)\end{aligned} \quad (31)$$

Fig. 2. Test Structure

where $\ddot{x}_g$ is the earthquake ground acceleration, v is the sensor noise, $y = [x_m, \ddot{x}_{a1}, \ddot{x}_{a2}, \ddot{x}_{a3}, \ddot{x}_{am}, \ddot{x}_g]'$ is the output, and $z = [x_1, x_2, x_3, x_m, \dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_m, \ddot{x}_{a1}, \ddot{x}_{a2}, \ddot{x}_{a3}, \ddot{x}_{am}]'$ is the regulated output. The test structure for this benchmark is shown in Fig. 2.

Along with this model there are ten different evaluation criteria $J_1 - J_{10}$ that are given in [14]. The first five evaluation criteria are based on the rms response of the building excited by a random process $\ddot{x}_g$ with specral density characterized by the Kanai-Tajimi spectrum. The first two performance criteria help describe the effect of the controller on the vibration of the building, while the next three help account for the performance of the actuator itself. The last five performance criteria are characterized in terms of the peak response of the building. In this case the excitation $\ddot{x}_g$ takes the form of two historical earthquakes, the 1940 El Centro earthquake or the 1968 Hachinohe earthquake. The first two of these are again used to account for the performance of the building whereas the last three help to evaluate the control resources being used.

For the control design a reduced order model is to be used. This is given by

$$\begin{aligned}\dot{x}_r(t) &= A_rx_r(t) + B_ru(t) + E_r\ddot{x}_g(t)\\ y_r(t) &= C_{yr}x(t) + D_{yr}u(t) + F_{yr}\ddot{x}_g(t) + v_r\end{aligned} \quad (32)$$

where x_r is a 10-dimensional state and $y_r = [\ddot{x}_{a1}, \ddot{x}_{a2}, \ddot{x}_{a3}, \ddot{x}_{am}]'$. From the LQG design in [14], we obtain $H_1 = (C_{yr}'Q, 0')'$ and $G_1 = (D_{yr}'Q, \tilde{G}')'$ where $R = 50$, $Q = diag(1,1,1,0)$, and $\tilde{G}'\tilde{G} = R$.

To account for some unmodeled dynamics in the system a weighting function shall be used. In particular from [5] we see that the weighting function

$$W_{z_2}(s) = \frac{3s^2 + 90s + 600}{s^2 + 300s + 30000} \quad (33)$$

can be used to accommodate a multiplicative uncertainty. This can be seen in more detail in Fig. 3.

Along with the ten performance criteria J_1-J_{10} there are control and actuator constraints. These constraints are $\max_t |u(t)| \le 3$ V, $\sigma_u \le 1$ V, $\max_t |x_m(t)| \le 9$ cm, $\sigma_{x_m} \le 3$ cm, $\max_t |\ddot{x}_{am}(t)| \le 6$ g's, and $\sigma_{\ddot{x}_{am}} \le 2$ g's.

The results from the simulation for the LQG, MCV, and the Multiobjective controllers may be found in Table I.

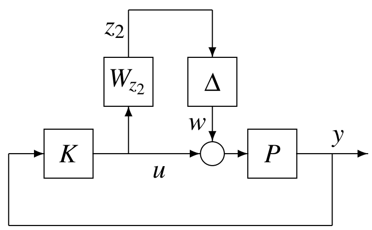

Fig. 3. Block Diagram with Multiplicative Uncertainty

For the MCV controller the parameter is set to $\gamma = 10^{-5}$. Similarly for the Multiobjective Cost Cumulant controller the parameter for the weight on the variance is $\gamma = 10^{-5}$ and also the parameter for the disturbance is set to $\delta = 3.0$. As one might expect there is a significant reduction from the LQG results in J_1, J_2, J_6, and J_7 in the Multiobjective case. Since these quantities correspond to the vibration of the building this makes sense. Likewise the rest of the performance criteria are worse for the Multiobjective Cost Cumulant controller. This too make sense since these criteria correspond to how much action is required from the actuator. Also from these simulation results one can notice that the switch to the two cost cumulant case results in a large reduction in the first several performance criteria from the LQG case. Also note that the reduction in J_1 and J_2 for the multiobjective cost cumulant controller is roughly 36% less than that from the LQG controller. Similarly it is roughly 18% less than the peak response J_6 for the LQG case.

VII. Conclusion

In this paper a Nash Game approach was taken to solve the case when there are two players, a control and a disturbance, which wish to minimize their own performance index, namely two different linear combinations of cost cumulants. The problem was developed for a class of nonlinear systems and non-quadratic costs. Later the case of a linear system and quadratic cost was examined, and in particular the case when the system has some structured uncertainty present. Also a connection with multiobjective control was established, in particular a connection with the H_2/H_∞ control method.

Once the linear case was established, the theory was applied to the First Generation Benchmark for seismic excited buildings. With a multiplicative uncertainty the controller was designed and the response was simulated. The results of the simulation were examined and found that the multiobjective cost cumulant control performed well compared to other different controller types.

	LQG	2CC	*MCC*
J_1	0.2898	0.2073	0.1846
J_2	0.4439	0.3127	0.2814
J_3	0.4843	0.7263	0.8236
J_4	0.4856	0.7199	0.8130
J_5	0.5976	0.7452	0.8223
J_6	0.4559	0.3847	0.3742
J_7	0.7096	0.6683	0.6448
J_8	0.6695	1.3295	1.6989
J_9	0.7807	1.3252	1.6525
J_{10}	1.3142	1.5146	1.6947
σ_u	0.1441	0.2345	0.2722
σ_{x_m}	0.6341	0.9508	1.0782
$\sigma_{\ddot{x}_{am}}$	1.0696	1.3339	1.4720
$\max_t \|u\|$	0.5259	1.0063	1.1924
$\max_t \|x_m\|$	2.0060	3.6012	4.2715
$\max_t \|\ddot{x}_{am}\|$	4.7454	5.5398	5.9663

TABLE I
Benchmark Results

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