Discrete Stochastic Approximation via Simultaneous Difference Approximations

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***Abstract*—A stochastic approximation method for optimizing a class of discrete functions is considered. The procedure is a version of the Simultaneous Perturbation Stochastic Approximation (SPSA) method that has been modified to obtain a stochastic optimization method for cost functions defined on a discrete set of points. We discuss the algorithm and examine its convergence and also the rate of convergence.**

I. Introduction

This paper discusses a discrete stochastic optimization algorithm based on the simultaneous perturbation stochastic approximation algorithm (SPSA) for continuous parameter optimization ([1]). The SPSA algorithm is essentially a randomized version of the Kiefer-Wolfowitz method and is a computationally efficient algorithm, especially in problems of high dimension. It is the efficiency of the method that we wish to exploit for discrete parameter optimization.

The problem in the discrete setting, in broad terms, is to minimize a cost function that is defined on the grid of points in R^p with integer coordinates. As with the continuous parameter problem, it is assumed that the form of the cost function is unknown, however noisy measurements of it are available.

The motivation for the discrete algorithm is a class of discrete resource allocation problems ([2], [3], [4]), where the problem is to distribute a finite number of resources, in discrete amounts, to finitely many users in such a way that the allocation optimizes some performance measure. A common feature of these problems, which makes them difficult to solve, is the cardinality of the search space, which is large even in small dimensions.

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II. Notation and Problem Formulation

Let Z denote the set of integers and consider the grid Z^p of points in R^p with integer coordinates. Suppose that a real-valued cost function $L: Z^p \to R$ is given. The analytic form of the function is unknown; however noisy measurements y_n of it are available, where

$$y_n(\theta) = L(\theta) + \varepsilon_n(\theta) \tag{1}$$

and $\{\varepsilon_n(\theta)\}$ is a zero-mean stochastic process. The ε_n's are not necessarily independent; however, sufficient conditions are imposed to ensure that the $y_n(\theta)$'s are integrable. We assume also that L is bounded below. The problem is to minimize L using only the measurements y_n.

Similar to [2], we restrict our attention to a class of cost functions that arise in resource allocation. Specifically, the cost functions are assumed to satisfy the following integer convexity condition ([5]). For each $x \in R^p$, let $[x_i]$ denote the integer part of its *i*-th component, $1 \le i \le p$, and let $N(x) = \{\theta \in Z^p : [x_i] \le \theta_i < [x_i]+1, 1 \le i \le p\}$, which is the smallest hypercube in Z^p about *x*. The cost function L is integer convex if for each scalar λ, such that $0 \le \lambda \le 1$, and points $\theta', \theta'' \in Z^p$, the following inequality holds:

$$\min_{\theta \in N(\lambda\theta' + (1-\lambda)\theta'')} L(\theta) \le \lambda L(\theta') + (1-\lambda) L(\theta''). \tag{2}$$

For $p = 1$, integer convexity reduces to the inequality

$$L(\theta+1) - L(\theta) \ge L(\theta) - L(\theta-1) \tag{3}$$

or, equivalently,

$$2L(\theta) \le L(\theta+1) + L(\theta-1) \tag{4}$$

for each $\theta \in Z$. The latter inequality is the discrete analogue of mid-convexity. If strict inequality holds in (2) then L is said to be strictly convex.

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Analogous to the continuous case, the problem of minimizing L then reduces to the problem of finding its stationary values, i.e., any point $\theta \in Z$ for which

$$L(\theta+1)-L(\theta) \geq 0 \geq L(\theta)-L(\theta-1). \qquad (5)$$

If the function is strictly convex, then the stationary point is unique.

III. Finite-Difference Based Algorithm

The discrete stochastic approximation method here is motivated by the SPSA method ([1]). In the discrete case, difference approximations replace gradients. To estimate the differences of $L(\theta)$ we use simultaneous differences, analogous to the simultaneous perturbation gradient approximation in [1]. At each iteration k of the algorithm, take a random perturbation vector $\Delta_k = (\Delta_{k1},\ldots,\Delta_{kp})^T$, where the Δ_{ki}'s form an i.i.d. sequence of Bernoulli random variables taking the values ± 1. The perturbations are assumed to be independent of the measurement noise process. The difference estimate at iteration k is obtained by evaluating $y_k(\cdot)$ at two values:

$$y_k^+(\theta) = L(\theta+\Delta_k)+\varepsilon_{2k-1}(\theta+\Delta_k),$$

$$y_k^-(\theta) = L(\theta-\Delta_k)+\varepsilon_{2k}(\theta-\Delta_k).$$

The difference estimate $\hat{g}$ has i-th component, $1 \leq i \leq p$, given by

$$\hat{g}_i(k,\theta) = \left(y_k^+(\theta)-y_k^-(\theta)\right)/2\Delta_{ki}. \qquad (6)$$

The discrete stochastic approximation algorithm is

$$\hat{\theta}_{k+1} = \hat{\theta}_k - a_k\,\hat{g}\left(k+1,\left[\hat{\theta}_k\right]\right) \qquad (7)$$

with initial estimate $\hat{\theta}_1 \in Z^p$, where $[x]$ denotes the vector with i-th component equal to $[x_i]$. The sequence $\{a_n\}$ satisfies the standard conditions in stochastic approximation: $a_k > 0$, for all $k \geq 1$, $\sum a_k^2 < \infty$, and $\sum a_k = \infty$.

The above algorithm is a version of an algorithm presented in [6] that allowed variable step-sizes to estimate the differences in (6). The proof of convergence in [6] required a continuous extension of the loss function to all of R^p, which exists by Proposition 3.3 of [5], and used an SPSA estimate of a subgradient of the continuous extension ([7]). The algorithm in (7) is also similar to a version introduced in [8], which used a different method to project iterates onto to the set Z^p.

Proposition 1: Assume that L is strictly integer convex on Z^p and that L is separable, i.e. $L(\theta)=\sum_{j=1}^{p} L_j(\theta_j)$. Assume also that L is bounded below and that $\left(L_i(\theta_i)-L_i(\theta_i-1)\right)^2 + E\left(\varepsilon^2(\theta)\right) \leq \mathrm{O}\left(1+\|\theta\|^2\right)$, $1 \leq i \leq p$. Then the algorithm in (7) converges almost surely to a global minimum θ^* of L.

Proof: Each term $L_j(\cdot)$ of the cost function is an integer convex function on Z. Observe also that the difference approximations in (6) are (unbiased) estimates of the vector-valued function $g(\theta)$, the i-th component of which equals $g_i(\theta) = \left(L_i(\theta_i+1)-L_i(\theta_i-1)\right)/2$. Thus, the result follows as a consequence of Theorem 4 of [9] applied to $g(\theta)$.

A direct application of Theorem 3 of [9] to the components of $g(\theta)$ yields a rate-of-convergence for (7), i.e., Corollary 1 of [9]:

Proposition 2: Assume the conditions of Proposition 1 and take $a_k = 1/k$, $k \geq 1$, in (7). Assume also that for K sufficiently large $\left|L_i(z_i)-L_i(z_i-1)\right| \leq K\left|z_i-\theta_i^*\right|$, $1 \leq i \leq p$. Let $\varepsilon > 0$ be given. Then there exists a positive constant C such that $\mathrm{P}\left(\left\|\hat{\theta}_k-\theta^*\right\| \geq \varepsilon\right) \leq \exp(-Ck)$.

References

[1] J. C. Spall. "Multivariate stochastic approximation using a simultaneous perturbation gradient approximation," IEEE Trans. on Automat. Contr., vol. 37, pp. 332--341, 1992.

[2] C. G. Cassandras, L. Dai, and C. G. Panayiotou. "Ordinal optimization for a class of deterministic and stochastic discrete resource allocation problems," IEEE Trans. Auto. Contr., vol. 43(7): pp. 881--900, 1998.

[3] T. Ibaraki and N. Katoh. Resource Allocation Problems: Algorithmic Approaches. MIT Press, 1988.

[4] B. L. Miller, "On Minimizing Nonseparable Functions Defined on the Integers with an Inventory Application, SIAM Journal on Applied Mathematics, vol. 21, pp. 166-185.

[5] P. Favati and F. Tardella, "Convexity in nonlinear integer programming," Ricerca Operativa, vol. 53, pp. 3-34, 1990.

[6] S. D. Hill, L. Gerencsér and Z. Vágó, Stochastic Approximation on Discrete Sets Using Simultaneous Difference Approximations, pp. 2795- 2798, Proc. of the 2004 American Control Conference.

[7] Y. He, M. C. Fu, and S. I. Marcus, "Convergence of Simultaneous Perturbation Stochastic Approximation for Nondifferentiable Optimization," IEEE Trans. on Auto. Contr., vol. 48, pp. 1459-1463.

[8] L. Gerencsér, S. D. Hill, and Z. Vágó. "Optimization over discrete sets via SPSA," Proceedings of the Conference on Decision and Control, CDC 38, 1999.

[9] V. Dupac and U. Herkenrath, "Stochastic approximation on a discrete set and the multi-armed bandit problem," Communications in Statistics--Sequential Analysis, vol 1, pp. 1025, 1982.