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On Zero Semimodules of Systems over Semirings with Applications to Queueing Systems

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***Abstract*— In this paper, a zero semimodule for a linear system over a semiring is introduced. Unlike a linear system over a field, an (A, B)-controlled invariant sub-semimodule V is not equivalent to an (A, B)-controlled invariant sub-semimodule of feedback type V_{FB}. The solvability condition for the disturbance decoupling problem (DDP) cannot be established by knowing the maximal (A, B)-controlled invariant sub-semimodule. In this paper, an extended zero semimodule is used to find a tight upper bound for the maximal (A, B)-controlled invariant sub-semimodule of feedback type V^*_{FB}, if one exists. Thus a connection is established between the geometric control method and the frequency domain method. This connection implies a necessary condition for the solvability of DDP. For example systems over some special semirings, this tight upper bound is equal to V^*_{BF}; then we have a necessary and sufficient condition for the solvability of DDP. A queueing system, described by (Max,+)-algebra, is studied to illustrate the main results.**

Keywords: **Semirings, (A, B)-controlled invariance, disturbance decoupling.**

I. INTRODUCTION

This paper studies linear dynamical systems over a semiring. One example of this type of system is a discrete event system described by the $(\mathrm{Max}, +)$-algebra. The control problem in this paper is the disturbance decoupling problem (DDP), which is a well known entry-point problem for geometric control theory. The main contribution of this paper is to propose a new computational method for the DDP of linear dynamical systems over a semiring, which has potential use in various fields, such as scheduling, public transportation and queueing systems.

For a linear system over a field, the (A, B)-controlled invariant subspace is the same as the (A, B)-controlled invariant subspace of feedback type, which may be denoted as an $(A+BF)$-invariant subspace. The DDP is solvable by analyzing the maximal (A, B)-controlled invariant subspace (MCIS), which can be obtained from a recursive algorithm. The maximal (A, B)-controlled invariant sub-semimodule (MCISM) V^* and the maximal (A, B)-controlled invariant sub-semimodule of feedback type (MCISMF) V^*_{FB}, if one exists, need not be the same for linear dynamical systems over a semiring. Moreover, the recursive algorithm to compute maximal (A, B)-controlled invariant subspaces for linear systems over a field does not extend directly to linear systems over a semiring. The research for systems over a ring has been studied in [2], where a class of injective systems are studied for which the (A, B)-controlled invariant sub-module is the same as the (A, B)-controlled invariant sub-module of feedback type. In this research, we study the existence and properties of mappings which relate zero semimodules to MCISMFs. We also establish a computational method of MCISMs for linear systems over a semiring. The purpose of the study is to evaluate constructions on zero semimodules to make the types of calculations needed to solve the DDP. This paper establishes the connection between the semimodule of extended zeros to the MCISMF. In this way, a connection is established between the geometric control method [5] and the frequency domain method [6]. This connection implies a necessary condition for the solvability of DDP. For example systems over some special semirings, this tight upper bound is equal to V^*_{BF}; then we have a necessary and sufficient condition for the solvability of DDP. A queueing system is modelled as a timed Petri net to illustrate the main results.

The remainder of this paper is organized as follows. Section II introduces some mathematical preliminaries. Section III introduces linear dynamical systems over a semiring. Section IV presents pole and zero semimodules for a given transfer function and the extension of Kalman's realization diagram to linear dynamical systems over a semiring. Section V presents a tight upper bound for V^*_{FB}. Section VI states a necessary and sufficient condition for the solvability of DDP and a computational method for MCISM. Section VII studies a queueing system modelled as a timed Petri net to illustrate the main results. Section VII is the conclusion.

II. MATHEMATICAL PRELIMINARIES

This section introduces some preliminary definitions about semirings and semimodules [3] and some preliminary results, which will be used in the sequel.

A *semigroup* $(S, \Box)$ is a set S together with a binary operation $\Box : S \times S \to S$ which is associative. A *monoid* $(M, \Box, e_M)$ is a semigroup $(M, \Box)$ with the unit element e_M for $\Box$, i.e. $e_M \Box x = x \Box e_M = x$ for all $x \in M$. For generality, we use symbols to represent the operators, instead of the traditional [1] addition $(+)$ and multiplication $(\cdot)$. A *semiring* $R = (R, \Box, e_R, \circ, 1_R)$ is a set R with two binary operations, box $\Box$ and circle $\circ$, such that:

1) $(R, \Box, e_R)$ is a monoid where $\Box$ is commutative;

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[1] the reader is cautioned that, in new applications of system theory, operators may be counter-intuitive.

2) $(R, \circ, 1_R)$ is a monoid under $\circ$;
3) $r \circ e_R = e_R = e_R \circ r$ for all $r \in R$;
4) circle $\circ$ is distributive on both sides over Box $\Box$.

A *commutative* semiring is a semiring in which the operator $\circ$ is commutative. Let $(R, \Box, e_R, \circ, 1_R)$ be a semiring, and $(M, \triangle_M, e_M)$ a commutative monoid. M is called a *left R-semimodule* if there exists a map $\mu : R \times M \to M$, denoted as $\mu(r, m) = rm$ for all $r \in R$, and $m \in M$, such that for any $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$, we have $r(m_1 \triangle_M m_2) = rm_1 \triangle_M rm_2$; $re_M = e_M = e_R m$; $(r_1 \Box r_2)m = r_1 m \triangle_M r_2 m$; $r_1(r_2 m) = (r_1 \circ r_2)m$; $1_R m = m$. A *sub-semimodule* of M is a subset N of M such that N is a submonoid of M and, for all $r \in R$ and $n \in N$, $rn \in N$. Let $N = (N, \triangle_N, e_N)$ also be an R-semimodule. An R-semimodule *morphism* from M to N is a map $f : M \to N$ such that, for all $m, m_1, m_2 \in M$, and $r \in R$,

1) $f(m_1 \triangle_M m_2) = f(m_1) \triangle_N f(m_2)$;
2) $f(rm) = rf(m)$.

If $f : M \to N$ is an R-semimodule morphism, then the kernel of f is defined as $\ker f = \{m \in M : f(m) = e_N\}$. The kernel of f is a sub-semimodule of M.

An equivalence relation ρ defined on a left R-semimodule M is an R-congruence relation if $m\rho m'$ and $n\rho n'$, imply $(m \triangle_M n)\rho(m' \triangle_M n')$ and $(rm)\rho(rm')$. If $f : M \to N$ is an R-morphism of left R-semimodules, then f defines an R-congruence relation $\equiv_f$ on M by $m \equiv_f m'$ if $f(m) = f(m')$. If K is a sub-semimodule of M, then K induces an R-congruence relation $\equiv_K$ on M, called the *Bourne relation*, defined by setting $m \equiv_K m'$ if and only if there exist elements k and k' in K such that $m \triangle_M k = m' \triangle_M k'$. So if m and m' are elements in M satisfying $m \equiv_{\ker f} m'$ then surely $m \equiv_f m'$, but the converse does not necessarily hold. If $\equiv_{\ker f}$ and $\equiv_f$ coincide, then f is called *steady*. A steady R-morphism f is monic if and only if $\ker f = \{e_M\}$.

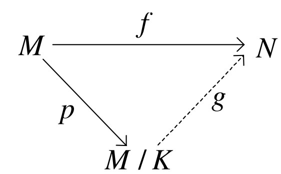
Theorem 1: [3, p.167] Let R be a semiring and $f : M \to N$ be an R-morphism between left R-semimodules. Let $p : M \to P$ be a surjective steady R-morphism between left R-semimodules satisfying $\ker p \subset \ker f$. Then there exists a unique R-morphism $g : P \to N$ satisfying $f = g\, p$; if f is monic so is g; $\ker g = p \ker f$ and $g(P) = f(M)$.

Lemma 1: Suppose that p is the surjective R-morphism of $M \to M/K$, where K is a sub-semimodule of semimodule M and $p(m) = m/K = \{x | x \triangle_M k_1 = m \triangle_M k_2, \exists k_1, k_2 \in K\}$; then p is steady.

Proof: To prove that p is steady we need to show that $p(x) = p(y)$ is equivalent to $x \equiv_{\ker \mathrm{p}} y$. "$\Longleftarrow$": This direction is obvious. "$\Longrightarrow$": If, for $x, y \in M$, we have $p(x) = p(y)$, this implies that $x/K = y/K$. This means $x \triangle_M k_1 = y \triangle_M k_2$ for some $k_1, k_2 \in K$. Then $x \equiv_K y$, which means [2] $x \equiv_{e_M/K} y$ since $K \subset e_M/K$. $\ker\, p = e_M/K$, so $x \equiv_{\ker \mathrm{p}} y$. $\diamondsuit$

[2]The notation e_M/K, in semiring literature, is used two ways: first, to represent a sub-semimodule of M, and second, to represent an element of M/K. The context usually makes this clear.

Theorem 2: Let $M = (M, \triangle_M, e_M)$ and $N = (N, \triangle_N, e_N)$ be left R-semimodules, and K be a sub-semimodule of M and $p : M \to M/K$ be the natural semimodule morphism. Let $f : M \to N$ be any R-semimodule morphism whose kernel contains $\ker\, p = e_M/K$. There exists a unique R-semimodule morphism $g : M/K \to N$ making the diagram commute and $\ker\, g = \ker\, f/K$. If



$\ker\, f = \ker\, p = e_M/K$, then g is a left R-semimodule monomorphism if and only if f is steady.

Proof: For $x, y \in M$ such that $p(x) = p(y)$, we have $x/K = y/K$, i.e. $x \triangle_M k_1 = y \triangle_M k_2$, for some $k_1, k_2 \in K$. Since $K \subset e_M/K \subset \ker\, f$, $k_1, k_2 \in \ker\, f$. Therefore, $f(x) \triangle_N f(k_1) = f(y) \triangle_N f(k_2) \Longrightarrow f(x) = f(y)$. Therefore, there exists a map $g : M/K \to N$ such that the diagram commutes.

We will prove that g is an R-semimodule morphism. For elements $x/K, y/K \in M/K$ and $r \in R$, $g(x/K \triangle_{M/K} y/K) = g(p(x) \triangle_{M/K} (p(y)) = gp(x \triangle_M y) = f(x) \triangle_N f(y) = gp(x) \triangle_N gp(y) = g(x/K) \triangle_N g(y/K)$; $g(rx/K) = g(r(p(x)) = g(p(rx)) = f(rx) = r(f(x)) = rg(x/K)$. Therefore, g is an R-semimodule morphism from M/K to N.

Next, we will prove $\ker\, g = p(\ker\, f)$. "$\subset$": For any $x/K \in \ker\, g \Longrightarrow g(x/K) = e_N \Longrightarrow gp(x) = f(x) = e_N$. Therefore, $x \in \ker\, f$ and $p(x) \in p(\ker\, f)$. "$\supset$": For all $x/K \in p(\ker\, f) \Longrightarrow x \in \ker\, f \Longrightarrow f(x) = g(p(x)) = e_N$. Therefore, $x/K \in \ker\, g$.

If f is steady and $\ker\, f = \ker\, p = e_M/K$, then for $x/K, y/K \in M/K$ such that $g(x/K) = g(y/K)$, we have $g(p(x)) = g(p(y))$, i.e. $f(x) = f(y)$; by steadiness of f, we have $x \equiv_{\ker f} y$, i.e. $x \triangle_M k_1 = y \triangle_M k_2$, for $k_1, k_2 \in \ker\, f$. Since $\ker\, f = \ker\, p = e_M/K$, $p(x \triangle_M k_1) = p(y \triangle_M k_2) \Longrightarrow p(x) = p(y)$. Therefore, $x/K = y/K$ and g is an R-semimodule monomorphism. Conversely, if we know that g is an R-semimodule monomorphism, then $f(x) = f(y)$ implies that $gp(x) = gp(y)$, i.e. $g(x/K) = g(y/K)$. Since g is monic, we have $x/K = y/K$, i.e. $x \triangle_M k_1 = y \triangle_M k_2$, for some k_1, k_2 in K. Therefore, $x \equiv_K y \Rightarrow x \equiv_{e_M/K} y \Leftrightarrow x \equiv_{\ker f} y$, i.e. f is steady. $\diamondsuit$

III. Linear Dynamical Systems over a Semiring

In this paper, we consider a linear dynamical system over a semiring R, written here in the discrete time case

$$\begin{aligned} x(k+1) &= Ax(k) \Box Bu(k), \\ y(k) &= Cx(k) \star Du(k), \end{aligned} \tag{1}$$

where the three finitely genarated R-semimodules $(X, \Box, e_X)$, $(U, \triangle, e_U)$ and $(Y, \star, e_Y)$ are the state semimodule, the input semimodule and the output semimodule, respectively, and $A : X \to X$, $B : U \to X$, $C : X \to Y$ and $D : U \to Y$ are four semimodule R-morphisms. When X, U and Y are vector spaces, there is the corresponding "external" or "input-output" description $G(z) = C(zI - A)^{-1}B + D : U(z) \to Y(z)$, where $Y(z) = \sum_{i=i_y}^{\infty} y(i)z^{-i}$, and $U(z) = \sum_{i=i_u}^{\infty} u(i)z^{-i}$. For linear dynamical systems over a semiring, we have a corresponding δ-transformation, with $X(\delta) = \square_{i=i_x}^{\infty} x(i)\delta^{-i}$, $U(\delta) = \triangle_{i=i_u}^{\infty} u(i)\delta^{-i}$ and $Y(\delta) = \bigstar_{i=i_y}^{\infty} y(i)\delta^{-i}$, where $\infty < i_x, i_u, i_y \le 0$. The indeterminate δ is viewed as a time-marker and δ^{-k} means time $t = k$. If i_x, i_y and i_u are different, for instance, $i_x < i_y < i_u$, we always can fill in zero terms in $U(\delta)$ and $Y(\delta)$ for $i_x < i < i_u, i_y$ to make them start at the same time instant. Without loss of generality, we assume they are the same, i.e. $i_x = i_u = i_y \stackrel{\text{def}}{=} i_0$. If we define another sequence of states as $\{\bar{x}(k), k = 0, 1, \cdots\}$, where $\bar{x}(k) = x(k+i_0)$, then its corresponding δ-transformation is $\bar{X}(\delta) = \square_{i=0}^{\infty}\bar{x}(i)\delta^{-i}$, which is equal to $X(\delta)\delta^{i_0}$. Similarly, we have $\bar{U}(\delta) = U(\delta)\delta^{i_0}$ and $\bar{Y}(\delta) = Y(\delta)\delta^{i_0}$. We know that, if there exists a transfer function $G(\delta) : U(\delta) \to Y(\delta)$ such that $G(\delta)U(\delta) = Y(\delta)$, then $G(\delta)$ is also the transfer function from $\bar{U}(\delta)$ to $\bar{Y}(\delta)$. In the remaining section, we will establish the transfer function representation with $i_0 = 0$, because the value of the starting time instant will not affect the formula of the transfer function. The solution of the equation (1) is given in [5, pp. 155]:

$$x(k+1) = A^{k+1}x_0\square(\mathop{\square}_{j=0}^{k} A^{k-j}Bu(j)).$$

Multiply δ^{-k-1} on both sides of this solution to obtain

$$\delta^{-k-1}x(k+1) = \delta^{-k-1}A^{k+1}x_0\square(\mathop{\square}_{j=0}^{k} \delta^{-k-1}A^{k-j}Bu(j)).$$

These equations are "summed" with respect to $k = 0, 1, \cdots$

$$\mathop{\square}_{k=0}^{\infty} \delta^{-k-1}x(k+1) =$$

$$(\mathop{\square}_{k=0}^{\infty} \delta^{-k-1}A^{k+1}x_0)\square(\mathop{\square}_{k=0}^{\infty}\mathop{\square}_{j=0}^{k} \delta^{-k-1}A^{k-j}Bu(j)) =$$

$$(\mathop{\square}_{k=0}^{\infty} \delta^{-k-1}A^{k+1}x_0)\square\delta^{-1}(\mathop{\square}_{k=0}^{\infty}\mathop{\square}_{j=0}^{k} \delta^{-k}A^{k-j}Bu(j)). \quad (2)$$

Lemma 2: $(\mathop{\square}_{m=0}^{\infty} \delta^{-m}A^{m})(\mathop{\square}_{n=0}^{\infty} \delta^{-n}Bu(n)) =$

$$(\mathop{\square}_{m=0}^{\infty} \delta^{-m}A^{m})BU(\delta) = \mathop{\square}_{k=0}^{\infty} \alpha_k\delta^{-k},$$

where $\alpha_k = \mathop{\square}_{i=0}^{k} A^{k-i}Bu(i)$.

Therefore, the right hand side of Eq.(2) is written as

$$\mathop{\square}_{k=0}^{\infty} \delta^{-k-1}x(k+1) = (\mathop{\square}_{k=0}^{\infty} \delta^{-k-1}A^{k+1}x_0)\square \delta^{-1}(\mathop{\square}_{k=0}^{\infty} \delta^{-k}A^{k})BU(\delta).$$

Boxing x_0 on both sides of the equation gives

$$\mathop{\square}_{k=0}^{\infty} \delta^{-k}x(k) = (\mathop{\square}_{k=0}^{\infty} \delta^{-k}A^{k}x_0)\square\delta^{-1}(\mathop{\square}_{k=0}^{\infty} \delta^{-k}A^{k})BU(\delta),$$

which is equivalent to

$$X(\delta) = (\delta^{-1}A)^*(\delta^{-1}BU(\delta)\square x_0),$$

where

$$(\delta^{-1}A)^* = I \,\square\, \delta^{-1}A \,\square\, \cdots \,\square\, (\delta^{-1}A)^n \,\square\, \cdots .$$

Thus we obtain the corresponding input-output description or *transfer function* of system (1) over semiring R without using inverses: $Y(\delta) = G(\delta)U(\delta)$, where

$$G(\delta) = C(\delta^{-1}A)^*B\delta^{-1}\star D = CB\delta^{-1}\star CAB\delta^{-2}\cdots\star D.$$

The linear system $(X, U, Y; A, B, C, D)$ is a realization of $G(\delta)$ when A, B, C, D can produce the transfer function $G(\delta)$. The state semimodule X can have $R[\delta]$-semimodule structure if, for any polynomial $r(\delta) \in R[\delta]$, the scalar multiplication is defined by the $R[\delta]$-semimodule isomorphism $r(\delta)x = r(A)x$. This $R[\delta]$-semimodule is called the pole semimodule of the system (A, B, C, D).

IV. Pole and Zero Semimodules

This section introduces zero and pole semimodules of a given transfer function. The definitions of these semimodules are based on the extension of Kalman's realization diagram to linear dynamical systems over a semiring. The zero semimodule is an extension to the zero module defined in [6]. Without loss of generality, here we assume $D = 0$.

Consider a transfer function $G(\delta) : U(\delta) \to Y(\delta)$, where

$$G(\delta) = C(\delta^{-1}A)^*B\delta^{-1} = CB\delta^{-1}\star CAB\delta^{-2}\star\cdots .$$

Denote the $R[\delta]$-polynomial semimodules $U[\delta]$ and $Y[\delta]$ by ΩU and ΩY, respectively. Observe that $Y(\delta)$ is an $R[\delta]$-semimodule, and that ΩY is one of its sub-semimodules. Define the $R[\delta]$-semimodule $\Gamma Y = Y(\delta)/\Omega Y$. Considering the $R[\delta]$-semimodule X and the R-linear map $B : U \to X$, we define a unique $R[\delta]$-linear map $\widetilde{B} : \Omega U \to X$ which restricts to B on $U \subset \Omega U$, by $\widetilde{B}(u\delta^i) = \delta^i Bu$. A similar construction can be given to an R-linear map $\phi : \Gamma Y \to Y$. For any element $y(\delta) \in Y(\delta)$, we write

$$y(\delta) = y_{-n}\delta^n \star\cdots\star y_{-1}\delta\star y_0\star y_1\delta^{-1}\star\cdots$$

as a formal Laurent series, and $\phi(y(\delta)) = y_1$. Let $C : X \to Y$ be an R-linear map with domain the $R[\delta]$-semimodule X. There exists a unique $R[\delta]$-linear map $\widetilde{C} : X \to \Gamma Y$ such that $\phi\circ\widetilde{C} = C$, given by

$$\widetilde{C}x = Cx\delta^{-1}\star C(\delta x)\delta^{-2}\star\cdots .$$

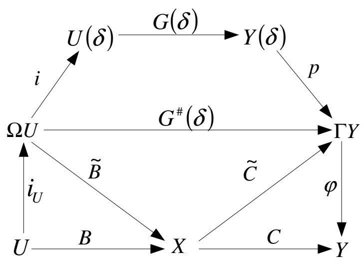


Fig. 1. A Kalman realization diagram of $G(\delta)$ [7].

An extension of Kalman's realization diagram [7] based on Theorem 2 in Section II is given in Fig. 1. A realization of $G^{\#}(\delta)$ or $G(\delta)$ is a commutative diagram of $R[\delta]$-semimodules as shown in the figure. We shall say that the realization is *reachable* if $\widetilde{B}$ is epic and *observable* if $\widetilde{C}$ is monic. Theorem 2 tells us that we can construct a $\widetilde{C}$, with $\ker \widetilde{C} = e_X$, by selecting $K = \ker G^{\#}(\delta)$. However, in the case of semimodules, this does not guarantee that $\widetilde{C}$ is monic. If, in addition, $G^{\#}(\delta)$ is steady, then $\widetilde{C}$ is monic. When the realization of $G(\delta)$ is both reachable and observable, we shall call it *canonical* or *minimal*. Steady transfer functions are not hard to find; for example, invertible transfer functions are steady. If $G(\delta) : U(\delta) \to Y(\delta)$ is invertible, then there exists a $G^{'}(\delta) : Y(\delta) \to U(\delta)$ such that $G^{'}(\delta)G(\delta) = 1_{U(\delta)}$. If we are given $G(\delta)u_1 = G(\delta)u_2$, then $G^{'}(\delta)G(\delta)u_1 = G^{'}(\delta)G(\delta)u_2$ implies $u_1 = u_2$, i.e. $u_1 \equiv_{\ker G} u_2$. The *pole semimodule* $X(G)$ of a steady $G(\delta)$ is defined as

$$X \cong \frac{\Omega U}{\ker G^{\#}} = \frac{\Omega U}{G^{-1}(\Omega Y) \cap \Omega U}.$$

The *zero semimodule* $Z(G)$ of a steady $G(\delta)$ is defined as

$$Z(G) \cong \frac{G^{-1}(\Omega Y) \triangle \Omega U}{\ker G \triangle \Omega U},$$

where the ΩU in the numerator of the quotient is provided so that the denominator is contained in the numerator.

V. $(A \square BF)$-Invariant Semimodules

Consider a linear system $(X, Y, Z; A, B, C)$ as in Eq. (1). A sub-semimodule V of a sub-semimodule K of X is called an (A, B)-controlled invariant semimodule (CISM) if for each $x_0 \in V$ there exists an input sequence, $\underline{u} = (u_0, u_1, u_2, \cdots)$, such that every component in the state sequence, $\underline{x} = (x_0, x_1, \cdots)$, produced by the input sequence $\underline{u}$, is in V. The MCISM of K, which always exists, is denoted as V^*. A sub-semimodule $V_{FB} \subset K$ is called a CISMF or $(A \square BF)$-invariant sub-semimodule of K if there exists a state feedback $F : X \to U$ such that $(A \square BF)V_{FB} \subset V_{FB}$. The MCISMF in V, if it exists, is denoted by V^*_{FB}. For linear dynamical systems over a field, a CISM is equivalent to a CISMF; however, this is not the case for linear systems over a semiring.

Define the extended zero $R[\delta]$-semimodule,

$$Z_1 = \frac{G^{-1}(\Omega Y)}{G^{-1}(\Omega Y) \cap \Omega U}.$$

Define the R-linear map $p' : G^{-1}(\Omega Y) \to \Omega U$ as $p'(u(\delta)) = u_{poly}$, where $u(\delta) = u_{poly} \triangle u_{sp}$. Then, this map induces an R-linear map $p_1 : Z_1 \to X(G)$. The following proposition establishes that $p_1(Z_1)$ is an upper bound of V^*_{FB} within ker C, if it exists.

Proposition 1: For a linear dynamical system (1) over a semiring R, $p_1(Z_1)$ is an (A, B)-controlled invariant sub-semimodule of $\ker C$. If the MCISMF V^*_{FB} exists within $\ker C$, then it is contained in $p_1(Z_1)$, i.e. $V^*_{FB} \subset p_1(Z_1) \subset V^*$, where V^* is the MCISM of $\ker C$.

Proof: $p_1(Z_1) \subset \ker C$: Let $\zeta \in Z_1$ and $\zeta = u(\delta) \mod (G^{-1}(\Omega Y) \cap \Omega U)$, where $u(\delta) \in U(\delta)$ and $u(\delta) = u_{poly} \triangle u_{sp}$. Then we have

$$G(\delta)u(\delta) = G(\delta)u_{poly} \star G(\delta)u_{sp}.$$

We know that

$$G(\delta)u_{poly} = y_{poly} \star (C\widetilde{B}u_{poly}) \cdot \delta^{-1} \star (C\widetilde{B}\delta u_{poly}) \cdot \delta^{-1} \cdots,$$

where $y_{poly} \in \Omega Y$. Since $G(\delta)u(\delta) \in \Omega Y$ and $G(\delta)u_{sp}$ has no δ^{-1} terms, $G(\delta)u_{poly}$ has no δ^{-1} terms either, i.e. $C\widetilde{B}u_{poly} = Cp_1(\zeta) = e_{Y(\delta)} \Rightarrow p_1(\zeta) \in \ker C$.

$p_1(Z_1)$ is CISM, i.e., $p_1(Z_1) \subset V^*$: Suppose x in $p_1(Z_1)$, i.e, $x = p_1(\zeta) = \widetilde{B}u_{poly}$ with

$$u_{poly} \triangle u_1 \delta^{-1} \triangle u_2 \delta^{-2} \triangle \cdots \in G^{-1}(\Omega Y).$$

Then $Ax = \widetilde{B}\delta u_{poly}$. We know $\widetilde{B}(\delta u_{poly} \triangle u_1) \in p_1(Z_1)$ because

$$\delta u_{poly} \triangle u_1 \triangle u_2 \delta^{-1} \triangle \cdots \in G^{-1}(\Omega Y).$$

Therefore, $p_1(Z_1) \subset V^*$.

$V^*_{BF} \subset p_1(Z_1)$: For a state $x \in V^*_{BF} \subset X$, there exists a state feedback $F : X \to U$ such that $(A \square BF)x \in V^*_{BF}$. Then $(A \square BF)\widetilde{B}u_{poly} = A\widetilde{B}u_{poly} \square BF\widetilde{B}u_{poly} = \widetilde{B}\{(\delta \triangle F\widetilde{B})u_{poly}\}$.

Lemma 3: [7, p. 630] Suppose $x = \widetilde{B}u_{poly}$ in V^*_{FB}. For $i \geq 1$, define u_i in U by $u_i = F\widetilde{B}\{(\delta \triangle F\widetilde{B})^{i-1}u_{poly}\}$, then $u_{poly} \triangle u_1 \delta^{-1} \triangle u_2 \delta^{-2} \triangle \cdots \in G^{-1}(\Omega Y)$.

This lemma shows that $V^*_{FB} \subset p_1(Z_1)$, because if $x = \widetilde{B}u_{poly} \in V^*_{FB}$, then there exists an element $\zeta \in Z_1$ with $p_1(\zeta) = x$. $\diamondsuit$

The image of p_1 contains V^*_{FB} and it is tight because, for example systems over some special semirings, these two are the same. For instance, we consider a linear dynamical system over a semiring $(\mathbb{R}^+, +, 0, \cdot, 1)$ as follows

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k, \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and $C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, The transfer function $G(\delta) = CB\delta^{-1} + CAB\delta^{-2} + \cdots = \delta^{-1}$. Therefore, $G^{-1}(\Omega Y) = u_{poly} = (u_1\delta + u_2\delta^2 + \cdots)$ for any $u_i \in U$. The $\widetilde{B}u_{poly} = p_1(Z_1) = (ABu_1 + A^2Bu_2 + \cdots) = \text{Im} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We know $\mathcal{K} = \ker C = \text{Im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, so $p_1(Z_1)$ is contained in $\mathcal{K}$. We can find a feedback $F : X \to U$ such that $(A+BF)p_1(Z_1) \subset p_1(Z_1)$, for instance, $F = [f_1\ f_2\ 0]$ for any $f_1, f_2 \in \mathbb{R}^+$. Therefore $p_1(Z_1)$ is of feedback type, i.e. $V^*_{FB} = p_1(Z_1)$.

VI. DDP for Systems over a Semiring

If we consider a linear dynamical system over a semiring R with some disturbance signal $d_k \in D$ as follows,

$$\begin{aligned} x_{k+1} &= Ax_k \,\square\, Bu_k \,\square\, Sd_k, \\ y_k &= Cx_k,\ u_k = Fx_k, \end{aligned} \tag{3}$$

where $x_k \in X$, $u_k \in U$ and $d_k \in D$, then we say the system (3) is disturbance decoupled if $\langle A\square BF | \text{Im}\ S\rangle \subset \ker\ C$, which is a trivial extension to Lemma 4.1 in [5]. Let $\mathcal{K} = \ker\ C$ and $\mathcal{T} = \text{Im}\ S$. The DDP is to find (if possible) a state feedback $F : X \to U$ such that $\langle A\square BF|\mathcal{T}\rangle \subset \mathcal{K}$. For linear systems over a field, the (A, B)-controlled invariant subspaces are equivalent to the (A, B)-controlled invariant subspaces of feedback type. The solvability of DDP can be obtained by knowing the MCIS.

Theorem 3: [5] The DDP is solvable for linear systems over fields if and only if the MCIS V^* in $\mathcal{K}$ contains $\mathcal{T}$.

Proposition 2: The DDP is solvable for linear systems over semirings, if and only if there exists a CISMF $V_{FB} \subset \mathcal{K}$ which contains $\mathcal{T}$.

Proof: (If): If there exists a $V_{FB} \subset \mathcal{K}$, such that $V_{FB} \supset \mathcal{T}$, then $\langle A\square BF|\mathcal{T}\rangle \subset \langle A\square BF|V_{FB}\rangle = V_{FB} \subset \mathcal{K}$. Then the DDP is solvable using Lemma 4.1 in [5].

(Only if) If the DDP is solvable for linear systems over semirings, the sub-semimodule $\mathcal{T}$ is a subset of $\langle A\square BF|\mathcal{T}\rangle \subset V_{FB}$. Therefore, $V_{FB} \supset \mathcal{T}$. $\diamondsuit$

Proposition 3: If the DDP is solvable for linear systems over semirings, then the MCISM V^* in $\mathcal{K}$ and $p_1(Z_1)$ both contain $\mathcal{T}$.

Proof: If the DDP is solvable for linear systems over semirings, then $\mathcal{T} \subset V_{FB} \subset \mathcal{K}$ using Proposition 2. Since $V_{FB} \subset p_1(Z_1) \subset V^*$, we have $p_1(Z_1) \supset \mathcal{T}$ and $V^* \supset \mathcal{T}$.

The preceding proposition shows that, for linear dynamical systems over semirings, the necessary and sufficient condition becomes only necessary. A computational method for controlled invariant sub-semimodules can be obtained by the following proposition.

Proposition 4: V is an (A, B)-controlled invariant sub-semimodule if and only if $AV \subset V \hat{\square}_X \mathcal{B}$, where $V\hat{\square}_X\mathcal{B} = \{x \in X | \exists b \in \mathcal{B},\ s.t.\ x\square b \in V\}$, with $\mathcal{B} = \text{Im}\ B$.

Proof: "$\Longleftarrow$", if $AV \subset V\hat{\square}_X\mathcal{B}$, then for any $x_0 \in V$, there exists a $b_0 \in \mathcal{B}$, such that $Ax_0\square b_0 = x_1 \in V$. Let $Bu_0 = b_0$, then $Ax\square Bu_0 = x_1 \in V$. Repeating this step we obtain a sequence of control inputs $\underline{u} = (u_0, u_1, \cdots)$ such that $x_k(x_0; \underline{u}) \in V$ for all $k = 0, 1, \cdots$. Therefore, V is a controlled invariant sub-semimodule.

"$\Longrightarrow$", If V is a CISM, then for any $x_0 \in V$ there exists a control signal u_0 such that $Ax_0\square Bu_0 \in V$. So $Ax_0 \in V\hat{\square}_X\mathcal{B}$. $\diamondsuit$

The MCISM V^* of a sub-semimodule $\mathcal{K}$ in X can be computed by the following algorithm:

$$\begin{aligned} V_0 &= \mathcal{K} \\ V_{l+1} &= \mathcal{K} \cap A^{-1}(V_l \hat{\square}_X\ \mathcal{B}), \end{aligned}$$

where A^{-1} is the set inverse map of $A : X \to X$. The proof of this algorithm is similar to Theorem 4.3 in [5].

The commutative diagram in the preceding section presents a computational method for V^*_{FB}, using the map $p_1 : Z_1 \to V^*_{FB}$. Then $p_1(\zeta) = \widetilde{B}u_{poly} \in V^*_{FB}$, for any $\zeta \in Z_1$.

VII. A Queueing System as a Timed Petri Net

In this section, the DDP for a queueing system is studied. The queueing system is modelled as a timed Petri net.

A. Petri Nets and Timed Petri Nets

This subsection introduces the definition of Petri nets [1]. A *Petri net* is a four-tuple (P, T, A, w) where P is a finite set of *places*; T is a finite set of *transitions*; A is a set of *arcs*, a subset of the set $(P \times T) \cap (T \times P)$ and w is a *weight function*, $w : A \to \{1, 2, 3, \cdots\}$. We use $I(t_j)$ to represent the set of input places to transition t_j and $O(t_j)$ to represent the set of output places from transition t_j and

$$I(t_j) = \{p_i : (p_i, t_j) \in A\} \text{ and } I(t_j) = \{p_i : (t_j, p_i) \in A\}.$$

A marking x of a Petri net is a function $x : P \to \{0, 1, 2, \cdots\}$. The number represents how many tokens in a place. A marked Petri net is a five-tuple (P, T, A, w, x_0) where (P, T, A, w) is a Petri net and x_0 is the initial marking. For a timed Petri net, when the transition t_j is enabled for the k-th time, it does not fire immediately, but it has a firing delay, $v_{j,k}$, during the tokens are kept in the input places of t_j. The clock structure associated with the set of timed transitions, $T_D \subseteq T$, of a marked Petri net (P, T, A, w, x) is a set $\mathbf{V} = \{\mathbf{v}_j : t_j \in T_D\}$ of lifetime sequences $\mathbf{v}_j = \{v_{j,1}, v_{j,2}, \cdots\}, t_j \in T_D, v_{j,k} \in \mathbb{R}^+, k = 1, 2, \cdots$. A *timed Petri net* is a six-tuple $(P, T, A, w, x, \mathbf{V})$ where (P, T, A, w, x) is a marked Petri net and $\mathbf{V} = \{\mathbf{v}_j : t_j \in T_D\}$ is a clock structure.

B. A Queueing System in $(\text{Max}, +)$-Algebra

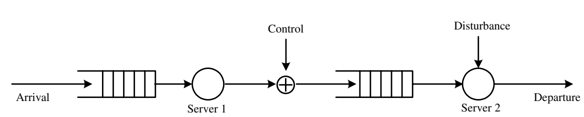


Fig. 2. A queueing system with two servers.

If we consider a queueing system with two servers as shown in Fig. 2, where the second server has a controller for the customer arrival times and a service disturbance, such as service breakdown. The timed Petri net model for this queueing system is shown in Fig. 3. There are four places for the server $i = 1, 2$: A_i (arrival), Q_i (queue), I_i (idle) and B_i (busy). For the server 2, there are two more places: C (server 1 completion) and D (disturbance). So $P = \{A_1, Q_1, I_1, B_1; A_2, C, Q_2, I_2, B_2, D\}$. The transitions (events) for each server are a_i (customer arrives), s_i (service starts) and c_i (service completes and customer departs). For the server 2, there are transitions u (control input) and d (disturbance). The timed transition $T_D = \{a_1, a_2, c_1, c_2\}$. The clock structure of this model has constant sequences $\mathbf{v}_{a_1} = \{k_1, k_1, \cdots\}$, $\mathbf{v}_{c_1} = \{k_2, k_2, \cdots\}$, $\mathbf{v}_{a_3} = \{k_3, k_3, \cdots\}$ and $\mathbf{v}_{c_2} = \{k_4, k_4, \cdots\}$. The rectangles present the timed transitions. The initial marking is $x_0 = \{1, 0, 1, 0, 1, 1, 1, 1, 0, 1\}$.

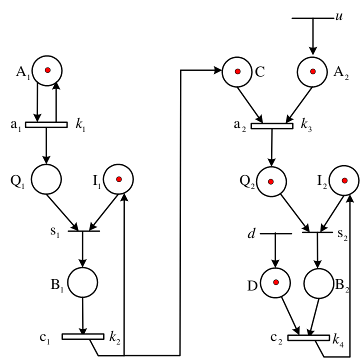


Fig. 3. The timed Petri net model for this queueing system.

(Max,+)-algebra is to replace the traditional addition and multiplication into max and $+$ operations for the set $\mathbb{R} \cup \{-\inf\}$, i.e.

$$a \oplus b \stackrel{\text{def}}{=} \max\{a, b\},$$
$$a \otimes b \stackrel{\text{def}}{=} a + b.$$

The set $\mathbb{R}_{\text{Max}} = (\mathbb{R} \cup \{-\inf\}, \oplus, -\inf, \otimes, 0)$ is a semifield, because there are inverses with respect to the $\otimes$ operator. Denote $\epsilon = -\inf$ and $e = 0$. We define a_k^i as the k-th arrival time of customers for service i, s_k^i as the k-th service starting time for service i and c_k^i as the k-th service completion and the customer departure time, where $i = \{1, 2\}$. If we define $\mathbf{x}_k = [a_k^1, s_k^1, c_k^1, a_k^2, s_k^2, c_k^2]^T$ and assume the output is the customer arrival time of the second server, then we can write the system equation using (Max,+)-algebra as

$$\begin{aligned} x_{k+1} &= A\, x_k \,\oplus\, B\, u_k \,\oplus\, S\, d_k, \\ y_k &= C\, x_k, \end{aligned}$$

where the system matrices are

$$A = \begin{bmatrix} k_1 & \epsilon & k_2 & \epsilon & \epsilon & \epsilon \\ k_1 & \epsilon & k_2 & \epsilon & \epsilon & \epsilon \\ k_1 & \epsilon & k_2 & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & k_2 & \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon & k_3 & \epsilon & k_4 \\ \epsilon & \epsilon & \epsilon & k_3 & \epsilon & k_4 \end{bmatrix}, \; B = \begin{bmatrix} \epsilon \\ \epsilon \\ \epsilon \\ e \\ \epsilon \\ \epsilon \end{bmatrix} \text{ and}$$

$S = [\epsilon,\ \epsilon,\ \epsilon,\ \epsilon,\ \epsilon,\ e]^T$ and $C = [\epsilon,\ \epsilon,\ \epsilon,\ e,\ \epsilon,\ \epsilon]$. The transfer function $G(\delta) = CB\delta^{-1} \oplus CAB\delta^{-2} \oplus \cdots = \delta^{-1}$. Therefore, $G^{-1}(\Omega Y) = u_{poly} = u_1\delta \oplus u_2\delta^2 \oplus \cdots$ for any $u_i \in U$. Then $\widetilde{B}u_{poly} = p_1(Z_1) = ABu_1 \oplus A^2Bu_2 \oplus \cdots = \text{Im}(\mathbf{e}_5, \mathbf{e}_6)$, where $\mathbf{e}_1 = [e,\ \epsilon,\ \ldots,\ \epsilon]^T, \cdots, \mathbf{e}_6 = [\epsilon,\ \epsilon,\ \cdots,\ e]^T$. We can find a feedback $F : X \to U$ such that $p_1(Z_1)$ is $(A \oplus BF)$-invariant. For instance, we can pick $F = [f_1,\ f_2,\ f_3,\ f_4,\ \epsilon,\ \epsilon]$ for any $f_i \in \mathbb{R}_{\text{Max}}$, $i = 1, \cdots 4$. Therefore V_{FB}^* is equal to $p_1(Z_1)$. $\mathcal{K} = \ker C = \text{Im}(\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{e}_5,\ \mathbf{e}_6)$, therefore, there exists a V_{BF} such that $\text{Im } S \subset V_{FB} \subset \mathcal{K}$. Hence, DDP for this queueing system is solvable.

VIII. Conclusion

In this paper, a zero semimodule for a linear system over a semiring is introduced. Unlike a linear system over a field, an (A, B)-controlled invariant sub-semimodule V is not equivalent to an (A, B)-controlled invariant sub-semimodule of feedback type V_{FB}. The solvability condition for the disturbance decoupling problem (DDP) cannot be established by knowing the maximal (A, B)-controlled invariant sub-semimodule. In this paper, an extended zero semimodule is used to find a tight upper bound for the maximal (A, B)-controlled invariant sub-semimodule of feedback type V_{FB}^*, if one exists. Thus a connection is established between the geometric control method and the frequency domain method. This connection implies a necessary condition for the solvability of DDP. For example systems over some special semirings, this tight upper bound is equal to V_{BF}^*; then we have a necessary and sufficient condition for the solvability of DDP. A queueing system, described by (Max,+)-algebra, is studied to illustrate the main results. This new computational method offers interesting possibilities for the DDP of linear systems over a semiring. Future research is to focus on more complicated control problems and applications.

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