2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Monotonic Convergence of Iterative Learning Control for Uncertain Systems Using a Time-Varying Q-filter

Douglas A. Bristow and Andrew G. Alleyne, *Senior Member, IEEE*
Department of Mechanical and Industrial Engineering
University of Illinois at Urbana-Champaign
Urbana, IL 61801

Abstract—**Time-varying Q-filtering in Iterative Learning Control (ILC) has demonstrated potential performance benefits over time-invariant Q-filtering. In this paper, LTV Q-filtering of ILC is considered for uncertain systems. Sufficient conditions for stability and the important monotonic convergence property are developed for the uncertain system. A class of LTV Q-filters that has particular benefit for rapid motion trajectories is presented, and monotonic convergence conditions are developed. The developed conditions highlight a relationship that the bandwidth can be increased locally and decreased elsewhere to localize high performance at specific times. These conditions are also iteration-length invariant and allow for significant design freedom *after* analysis enabling online modification of the LTV Q-filter.**

I. Introduction

Iterative Learning Control (ILC) is a technique for improving the performance of systems that execute the same operation multiple times. The ILC algorithm "learns" from previous iterations of the repeated operation to generate a customized feedforward control signal. ILC was first introduced to the wider academic community in 1984 [1] and has since grown into a large body of literature including several textbooks [2-5].

Much of the work on ILC has focused on converged performance, though in [6] it was shown that, under ideal circumstances, the simplest P-type ILC can be used to obtain zero error tracking for an LTI discrete-time system. This same work, however, discusses the problem of "learning transients" where stable ILC systems may undergo extremely large transients before eventual convergence. These learning transients may be orders of magnitude larger than initial errors, which is often impractical for implementation on physical systems. For this reason, many learning algorithms have been developed for monotonic convergence. Monotonic convergence not only ensures that performance improves each iteration, but it can also be easily related to a convergence rate that indicates how quickly the ILC will effectively converge.

This work was supported by NSF DMI-0140466 and an NSF Graduate Research Fellowship.

D. A. Bristow is a Ph.D. student at the University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA (e-mail: dbristow@uiuc.edu).

A. G. Alleyne is a Professor at the University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA (phone: 217-244-9993; fax: 217-244-6534; e-mail: alleyne@uiuc.edu).

When the actual system is a perturbation of the modeled system, however, stability and monotonic convergence may no longer hold. For this reason a number of authors have advocated the use of a low-pass Q-filter in the learning algorithm to enhance robustness [7-9]. In [10] a μ-synthesis design was presented to explicitly guarantee monotonic convergence for a set of uncertain plant dynamics. The added robustness of the Q-filter, however, comes at the cost of performance. While large transient-inducing high frequencies are filtered by Q, so is the potential to learn the high frequency components of the reference.

Recently, it is has been shown that a time-varying Q-filter may have potential to improve performance over the traditional time-invariant Q-filter [11,12]. In this work the same time-varying Q-filtering scheme will be considered, though model uncertainty is explicitly considered. An extension of the stability conditions for time-varying Q-filtering in [11] is developed for the uncertain model case. A class of time-varying Q-filters is also presented that allows the large monotonic convergence analysis problem to be separated into decoupled, short-time analyses problems. Conditions for monotonic convergence of this class of LTV Q-filters are obtained which provide insight into TV Q-filter bandwidth tradeoffs. These results also have a number of useful consequences including the ability to extend and make modifications to an existing LTV Q-filter without requiring a complete re-analysis of the system.

The rest of this paper is organized as follows. The uncertain system is described and the ILC problem defined in Section II. In Section III, conditions for stability and monotonic convergence of the uncertain system are presented. In Section IV, the class of monotonically converging TV Q-filters is presented. A discussion of the results is given in Section VI.

II. Uncertain System Description and ILC Problem Setup

Let an uncertain SISO discrete-time LTI system have the frequency domain input-output description

$$Y(z) = z^{-m} P(z) U(z) \tag{1}$$

with

$$P(z) = \hat{P}(z)(1 + W(z)\Delta(z)), \tag{2}$$

where m is the delay of the system, $\hat{P}(z)$ is a known, stable and rational polynomial with relative degree zero, $W(z)$ is known and stable, and $\Delta(z)$ is unknown, but stable and bounded as $\|\Delta(z)\|_\infty \equiv \sup_{\omega\in[0,\pi]} |\Delta(e^{i\omega})| < 1$. Without loss of generality, it is assumed that m=1, as is often the case for sampled systems. Let $y_d(k)$ be the desired output of the system for $k \in [1, N]$. The actual system output on $k \in [1, N]$ is given by

$$y(k) = P(q)u(k-1) + d(k) , \tag{3}$$

where q is the forward time shift operator defined as $qf(k) = f(k+1)$. The combined effects of any disturbances, initial conditions, and other inputs on the system are contained in $d(k)$. The time domain input-output relationship in (3) can be written equivalently as the convolution

$$y(k) = \sum_{\tau=1}^{k} p(k-\tau)u(\tau-1) + d(k) , \tag{4}$$

where $p(k)$ are the discrete unit impulse response of $P(q)$ and are sometimes referred to as the Markov parameters of $P(q)$ [6]. The Markov parameters can be obtained from $P(q)$ by dividing the denominator into the numerator to obtain the infinite series.

The ILC problem is to find a sequence in j, the iteration index, of inputs $u_j(k)$ with $k \in [0, N-1]$ so that the error $e_j(k) \equiv y_d(k) - y_j(k)$ becomes small as $j \to \infty$. All disturbances acting on the system and the system's initial conditions are assumed to be iteration-invariant so that $d_j(k) = d(k)$. The 2-dimensional system evolving in time and iteration can be written equivalently and compactly as the "lifted system"

$$\underbrace{\begin{bmatrix} y_j(1) \\ y_j(2) \\ \vdots \\ y_j(N+1) \end{bmatrix}}_{\mathbf{y}_j} = \underbrace{\begin{bmatrix} p(0) & 0 & 0 & 0 \\ p(1) & p(0) & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ p(N) & \cdots & p(1) & p(0) \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} u_j(0) \\ u_j(1) \\ \vdots \\ u_j(N) \end{bmatrix}}_{\mathbf{u}_j} + \underbrace{\begin{bmatrix} d(1) \\ d(2) \\ \vdots \\ d(N+1) \end{bmatrix}}_{\mathbf{d}} \tag{5}$$

where $\mathbf{d}$ is the vector of disturbances, $\mathbf{u}_j$ is the vector of ILC inputs, $\mathbf{y}_j$ is the vector of plant outputs and $\mathbf{P}$ is a lower triangular Toeplitz matrix of the plant's Markov parameters.

Let $\hat{p}(k)$, $w(k)$, and $\delta(k)$ be the Markov parameters of $\hat{P}(q)$, $W(q)$, and $\Delta(q)$ respectively. Then the NxN matrices $\hat{\mathbf{P}}$, $\mathbf{W}$, and $\mathbf{\Delta}$ can be constructed in the same manner as $\mathbf{P}$ from $\hat{p}(k)$, $w(k)$, and $\delta(k)$, respectively, to yield the relationship

$$\mathbf{P} = \hat{\mathbf{P}}(\mathbf{I} + \mathbf{W}\mathbf{\Delta}) . \tag{6}$$

Therefore, $\hat{\mathbf{P}}$ and $\mathbf{W}$ are known, while $\mathbf{\Delta}$ and, by extension, $\mathbf{P}$ are unknown. The following theorem from [13] offers some relationships between the time and frequency domain representations that will be useful for characterizing $\mathbf{\Delta}$.

Theorem 1: Let $F(q)$ be stable, causal, and SISO. Then if

$$\sup_{\omega\in[0,\pi]} |F(e^{i\omega})| < c ,$$

the first Markov parameter satisfies

$$|f(0)| < c .$$

Further, the largest singular value of the NxN lifted system matrix of Markov parameters, $\mathbf{F}$, satisfies

$$\overline{\sigma}(\mathbf{F}) < c .$$

Proof: See [13].

Theorem 1 allows us to infer that $|\delta(0)| < 1$ and $\overline{\sigma}(\mathbf{\Delta}) < 1$ for any stable $\Delta(z)$ with $\|\Delta(z)\|_\infty < 1$.

Let $\mathbf{y_d}$ be the vector of desired outputs $y_d(k)$ on $[1,N]$, and define the error vector as

$$\mathbf{e}_j = \mathbf{y_d} - \mathbf{y}_j \tag{7}$$

The most commonly used ILC learning law is the first-order law given by

$$\mathbf{u}_{j+1} = \mathbf{Q}(\mathbf{u}_j + \mathbf{L_e}\mathbf{e}_j) \tag{8}$$

where $\mathbf{Q} \in R^{NxN}$ is referred to as the Q-filter and $\mathbf{L_e} \in R^{NxN}$ is referred to as the learning function. Here the learning function is selected as the inverted plant model,

$$\mathbf{L_e} = \alpha\hat{\mathbf{P}}^{-1} \tag{9}$$

where $\alpha \in (0,1]$ is the geometric rate of convergence under perfect plant knowledge. Note that $\hat{\mathbf{P}}^{-1}$ always exists because $\hat{\mathbf{P}}$ has full rank and also that $\hat{\mathbf{P}}^{-1}$ will be lower triangular Toeplitz.

The Q-filter of interest is causal and time-varying [11]. This results in $\mathbf{Q}$ having the lower triangular, but not Toeplitz, structure given by

$$\mathbf{Q} = \begin{bmatrix} q_1(0) & 0 & 0 & 0 & 0 \\ q_2(1) & q_2(0) & 0 & 0 & 0 \\ q_3(2) & q_3(1) & q_3(0) & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ q_N(N-1) & q_N(N-2) & \cdots & q_N(1) & q_N(0) \end{bmatrix} \tag{10}$$

where $q_k(i)$ is the i^{th} Markov parameter of the LTV filter "frozen" at time k. The elements $q_k(i)$ should not be confused with the forward time shift operator q.

III. ILC Stability and Convergence

The iteration-domain dynamics of $\mathbf{u}_j$ can be found with substitution of (5-7,9) into (8) yielding

$$\mathbf{u}_{j+1} = -\mathbf{Q}\left((1-\alpha)\mathbf{I} - \alpha\mathbf{W\Delta}\right)\mathbf{u}_j + \mathbf{Q}\hat{\mathbf{P}}^{-1}\left(\mathbf{y_d} - \mathbf{d}\right). \quad (11)$$

Theorem 2: The uncertain ILC system described by (5-10) is stable for all $\mathbf{\Delta}$ with $\|\Delta(z)\|_\infty < 1$ if

$$\left|q_k(0)\right| < \frac{1}{1+\alpha\left(|w(0)|-1\right)}, \text{ for all } k \in [1, N]. \quad (12)$$

Proof: The term $\mathbf{Q}\hat{\mathbf{P}}^{-1}(\mathbf{y_d} - \mathbf{d})$ is constant and bounded, so $\mathbf{u}_j$ converges if and only if

$$\left|\lambda_k\left(\mathbf{Q}\left((1-\alpha)\mathbf{I} - \alpha\mathbf{W\Delta}\right)\right)\right| < 1\ \forall k \quad (13)$$

where λ_k is the k^{th} eigenvalue. $\mathbf{Q}$, $\mathbf{W}$, and $\mathbf{\Delta}$ are lower triangular, so the k^{th} eigenvalue is given by the diagonal elements

$$(1-\alpha)q_k(0) + \alpha q_k(0)w(0)\delta(0).$$

From Theorem 1, $|\delta(0)| < 1$. Then

$$\begin{aligned}&\left|(1-\alpha)q_k(0) + \alpha q_k(0)w(0)\delta(0)\right| \\ &\quad \le |q_k(0)|(1-\alpha) + \alpha|q_k(0)| \cdot |w(0)| \cdot |\delta(0)| \\ &\quad \le |q_k(0)| \cdot \left[1-\alpha+\alpha|w(0)|\right].\end{aligned}$$

If $|q_k(0)|$ satisfies (12), then

$$\left|(1-\alpha)q_k(0) + \alpha q_k(0)w(0)\delta(0)\right| < 1. \quad (14)$$

Therefore $\mathbf{u}_j$ converges and convergence of $\mathbf{y}_j$ and $\mathbf{e}_j$ follows from (5) and (7). ■

Remark 1: The restrictions placed on the TV Q-filter by the stability condition in (12) will depend on the specific type of digital filter used and also on the uncertainty weighting. These restrictions can be classified into three situations:

1. If the Q-filter has a zero-order hold, then $q_k(0) = 0$. In this case, the system will always be stable, but performance may suffer due to the delayed response.
2. If the Q-filter does not have a zero-order hold ($q_k(0) \ne 0$), but $|w(0)| < 1$, then the system will always be stable for any LTV low-pass filter without ripple in the pass band. The reason is that when $|w(0)| < 1$, then (12) reduces to $|q_k(0)| \le 1$. If the frozen filter does not have ripple in the pass band, then $\|Q_k(z)\|_\infty = 1$ and by Theorem 1, $|q_k(0)| \le 1$.
3. Otherwise, (12) will result in a maximum bandwidth restriction on the frozen filters. For many digital filters, $|q_k(0)|$ increases monotonically with increasing bandwidth. Therefore, above some upper bandwidth $|q_k(0)|$ will no longer satisfy (12). This effect can be somewhat alleviated by choosing a small α in the learning function.

If the ILC system is stable, the fixed point that the ILC control converges to can be found by replacing $\mathbf{u}_j$ and $\mathbf{u}_{j+1}$ in (5-8) with $\mathbf{u}_\infty$, and manipulating the equations yields

$$\mathbf{u}_\infty = \alpha\left[\mathbf{I} - \mathbf{Q}\left[(1-\alpha) - \alpha\mathbf{W\Delta}\right]\right]\mathbf{Q}\hat{\mathbf{P}}^{-1}\left(\mathbf{y_d} - \mathbf{d}\right). \quad (15)$$

Further, the converged error can be found as

$$\begin{aligned}\mathbf{e}_\infty = \Big\{&\mathbf{I} - \alpha\hat{\mathbf{P}}(\mathbf{I} + \mathbf{W\Delta}) \\ &\cdot\left[\mathbf{I} - \mathbf{Q}\left[(1-\alpha)\mathbf{I} - \alpha\mathbf{W\Delta}\right]\right]^{-1}\mathbf{Q}\hat{\mathbf{P}}^{-1}\Big\}\left(\mathbf{y_d} - \mathbf{d}\right).\end{aligned} \quad (16)$$

Of particular interest may be the necessary conditions for convergence of the uncertain plant to zero error. If the system converges, then from (16) it can be shown that a necessary condition for $\mathbf{e}_\infty=\mathbf{0}$ is that $\mathbf{Q}=\mathbf{I}$. Therefore this result, along with Theorem 2, demonstrates that zero error convergence can be achieved for the uncertain system if $|w(0)| < 1$, or equivalently $\|W(z)\|_\infty < 1$

As was discussed earlier, stability may be of little value if learning transients are impractically large. To avoid large transients, monotonic convergence conditions can be imposed.

Definition 1: The uncertain ILC system is monotonically convergent if

$$\left\|\delta\mathbf{u}_{j+1}\right\|_2 < \left\|\delta\mathbf{u}_j\right\|_2 \quad (17)$$

for all $\mathbf{\Delta}$ with $\|\Delta(z)\|_\infty < 1$, where $\delta\mathbf{u}_j \equiv \mathbf{u}_\infty - \mathbf{u}_{j+1}$.

Using the solution for the fixed point in (15) and the $\mathbf{u}_j$ dynamics in (11), the $\delta\mathbf{u}_j$ dynamics can be found as

$$\delta\mathbf{u}_{j+1} = -\mathbf{Q}\left((1-\alpha)\mathbf{I} - \alpha\mathbf{W\Delta}\right)\delta\mathbf{u}_j. \quad (18)$$

Theorem 3: The uncertain ILC system described by (5-10) is monotonically convergent for all $\mathbf{\Delta}$ with $\|\Delta(z)\|_\infty < 1$ if

$$\|\mathbf{QW}\|_2 \le \frac{1-(1-\alpha)\|\mathbf{Q}\|_2}{\alpha}. \quad (19)$$

Proof: From (18), the ILC system is monotonically convergent if and only if

$$\left\|\mathbf{Q}\left((1-\alpha)\mathbf{I} - \alpha\mathbf{W\Delta}\right)\right\|_2 < 1. \quad (20)$$

From Theorem 1, $\|\mathbf{\Delta}\|_2 < 1$. Therefore, by (18),

$$\begin{aligned}\left\|\mathbf{Q}\left((1-\alpha)\mathbf{I} - \alpha\mathbf{W\Delta}\right)\right\|_2 &\le (1-\alpha)\|\mathbf{Q}\|_2 + \alpha\|\mathbf{QW}\|_2\|\mathbf{\Delta}\|_2 \\ &< (1-\alpha)\|\mathbf{Q}\|_2 + \alpha\|\mathbf{QW}\|_2 < 1\end{aligned}.$$ ■

Remark 2: Because $\mathbf{Q}$ is an LTV filter whose frozen behavior is such that $\|Q_k(z)\|_\infty = 1$, then one would expect that in general, $\|\mathbf{Q}\|_2 > 1$. Then, from (19), the least restrictive monotonic convergence condition occurs when $\alpha = 1$. Therefore, for the remainder of this work, it is assumed that α is selected as 1.

Remark 3: Suppose that $\|W(z)\|_\infty \leq 1$ and $\mathbf{Q} = \mathbf{I}$ for zero error convergence. Then by Theorem 1, $\|\mathbf{W}\|_2 \leq 1$ and by Theorem 3 with $\alpha = 1$, the system is monotonically convergent. Therefore the Q-filter design problem is trivial if the multiplicative uncertainty is less than 100% at all frequencies. However, for most physical systems, this will not be the case, particularly at high frequencies. In these cases it is likely that performance must be sacrificed to gaurantee good learning transients.

While the monotonic convergence condition in (19) is a simple expression, numerical evaluation of $\|\mathbf{QW}\|_2$ may be difficult in practice because of the large size of the matrices involved. For long time durations such as those encountered in robotic manufacturing applications, N may be on the order of $O(10^5)$. Calculations involving matrices this large is challenging with today's computational machinery. This motivates the class of LTV Q-filters that is presented in the next section which can be separated into short-time blocks yielding a decoupled small matrix analysis.

IV. A Monotonically Convergent Time-Varying Q-Filter

Consider an LTV Q-filter that consists of long segments of a constant bandwidth and short segments of a non-constant, TV bandwidth like that shown in Fig. 1. Such a profile may be particularly useful for stepping motion profiles like that shown in Fig. 2 where high frequencies might be necessary to capture the short-time stepping motions. At other locations where no motion is desired, a lower bandwidth may be sufficient. In this section we seek to determine the relationship between the low-bandwidth, long-time segments and the high-bandwidth, short-time segments under a monotonic convergence constraint.

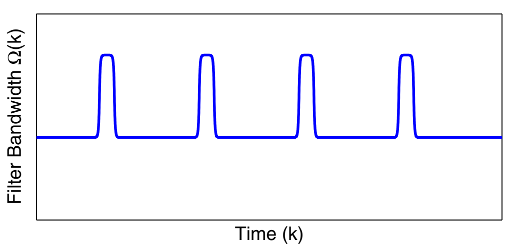

Fig. 1. Time-varying bandwidth profile with long segments of constant bandwidth.

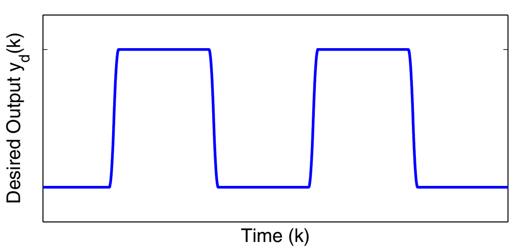

Fig. 2. Stepping motion profile.

Mathematically, a TV bandwidth profile like that shown in Fig. 1 can be written as

$$\Omega(k) = \begin{cases} \Omega_0 & , \ k \leq N_1 \\ \Omega_1(k) & , \ T_1 < k \leq T_2 \\ \Omega_0 & , \ T_2 < k \leq T_3 \\ \Omega_2(k) & , \ T_3 < k \leq T_4 \\ \Omega_0 & , \ T_4 < k \leq T_5 \\ \vdots & \quad \vdots \\ \Omega_m(k) & , \ T_{2m-1} < k \leq T_{2m} \\ \Omega_0 & , \ T_{2m} < k \end{cases} \tag{21}$$

with

$$T_{2i} - T_i \geq \overline{N} \text{, for } i \text{ odd,} \tag{22}$$

where $\overline{N}$ is the minimum length of the constant bandwidth, long-time segments. Note that while Fig. 1 shows the constant bandwidth segments having the same length and the non-constant bandwidth segments being identical, this need not be the case. Let the total length of the time-varying, short-time segments be given by

$$N_{TV} = \sum_{i=1}^{m} T_{2i} - T_{2i-1} \,. \tag{23}$$

Let the "template filter" be an LTI filter which accepts its bandwidth as one of its arguments. This template will be used to generate the LTV Q-filter. The template filter is assumed to be a finite impulse response (FIR) filter and is defined as

$$F(q,\Omega) = f_\Omega(0) + f_\Omega(1)q^{-1} + \cdots f_\Omega(N_Q - 1)q^{-(N_Q-1)} \tag{24}$$

where $N_Q << N$ is the length of the impulse response and Ω is the bandwidth of the filter. Assuming that this low-pass filter has unit DC gain, then

$$\sum_{i=0}^{N_Q-1} f(i,\Omega) = 1 \,. \tag{25}$$

Then, the TV Q-filter can be expressed as

$$Q(q,k) = F(q,\Omega(k)) \tag{26}$$

and the Q-filter matrix generated by $Q(q,k)$ is given by

$$\mathbf{Q} = \begin{bmatrix} f_{\Omega(1)}(0) & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 & 0 \\ f_{\Omega(N_Q)}(N_Q - 1) & \cdots & f_{\Omega(N_Q)}(0) & 0 & 0 \\ 0 & \ddots & & \ddots & 0 \\ 0 & 0 & f_{\Omega(N)}(N_Q - 1) & \cdots & f_{\Omega(N)}(0) \end{bmatrix} . \tag{27}$$

When the time-varying segments are spaced sufficiently far apart, they no longer have a combined effect on the monotonicity of the learning. Instead, it will be shown that only the "worst" segment determines whether the system converges monotonically. Therefore, if the ILC system converges monotonically with one time-varying segment, then it will converge monotonically for multiple time-varying segments. This feature is important because it gives the designer the ability to easily extend or modify a bandwidth profile that is already known to be monotonically convergent without having to re-analyze the system. Further, this separation has the additional benefit that the numerical calculation of the impact that each time-varying segment has is reduced to a small, computationally manageable size. First some matrix manipulation is necessary before this decoupling feature becomes apparent.

Let an LTI filter with bandwidth Ω_0 be defined by

$$Q_{LTI}(q) = F(q, \Omega_0). \tag{28}$$

The lifted system representation of $Q_{LTI}(q)$ is then

$$\mathbf{Q_{LTI}} = \begin{bmatrix} f_{\Omega_0}(0) & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 & 0 \\ f_{\Omega_0}(N_Q - 1) & \cdots & f_{\Omega_0}(0) & 0 & 0 \\ 0 & \ddots & & \ddots & 0 \\ 0 & 0 & f_{\Omega_0}(N_Q - 1) & \cdots & f_{\Omega_0}(0) \end{bmatrix}. \tag{29}$$

Now define the "delta Q-filter" as given below in (30-32).

$$\mathbf{Q_\delta} = \mathbf{Q} - \mathbf{Q_{LTI}} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \mathbf{Q}_{\boldsymbol{\delta},1} & 0 & & & & \vdots \\ \vdots & 0 & 0 & 0 & & & \vdots \\ \vdots & & 0 & \mathbf{Q}_{\boldsymbol{\delta},2} & 0 & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & 0 & \mathbf{Q}_{\boldsymbol{\delta},m} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix} \tag{30}$$

$$\mathbf{Q}_{\boldsymbol{\delta},i} = \begin{bmatrix} \left[\mathbf{f}_{\Omega(N_{2i-1}+1)} - \mathbf{f}_{\Omega_0}\right] & 0 & \cdots & \cdots & 0 \\ 0 & \left[\mathbf{f}_{\Omega(N_{2i-1}+2)} - \mathbf{f}_{\Omega_0}\right] & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & & 0 & \left[\mathbf{f}_{\Omega(N_{2i}-1)} - \mathbf{f}_{\Omega_0}\right] & 0 \\ 0 & \cdots & \cdots & 0 & \left[\mathbf{f}_{\Omega(N_{2i})} - \mathbf{f}_{\Omega_0}\right] \end{bmatrix} \tag{31}$$

$$\mathbf{f}_\Omega = \left[f_\Omega(N_Q - 1) \;\; \cdots \;\; f_\Omega(1) \;\; f_\Omega(0)\right]. \tag{32}$$

The matrices $\mathbf{Q}_{\boldsymbol{\delta},i}$ have size $(T_{2i} - T_{2i-1})\text{x}(N_Q + T_{2i} - T_{2i-1})$.

To decouple the short-time TV segments in the monotonic convergence analysis, it is necessary to approximate the IIR $W(q)$ dynamics as FIR dynamics. Because $W(q)$ is assumed stable, there exists a convergent geometric sequence that upper bounds the impulse response of $W(q)$ as

$$|w(k)| \le \kappa_W (\gamma_W)^k \text{, for } k \in [N_W + 1, N], \tag{33}$$

where $\kappa_W > 0$, $\gamma_W \in [0,1)$, and N_W is the length of the FIR approximation. We may write $\mathbf{W}$ as the sum of the FIR portion of the filter and the remaining portion as

$$\mathbf{W} = \mathbf{W_{FIR}} + \mathbf{W}_{\boldsymbol{\varepsilon}}, \tag{34}$$

where $\mathbf{W_{FIR}}$ and $\mathbf{W}_{\boldsymbol{\varepsilon}}$ are NxN and given in (35) and (36), respectively.

$$\mathbf{W_{FIR}} = \begin{bmatrix} w(0) & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 & 0 \\ w(N_W - 1) & \cdots & w(0) & 0 & 0 \\ 0 & \ddots & & \ddots & 0 \\ 0 & 0 & w(N_W - 1) & \cdots & w(0) \end{bmatrix} \tag{35}$$

$$\mathbf{W}_{\boldsymbol{\varepsilon}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 & 0 & 0 \\ 0 & & \ddots & 0 & 0 & 0 \\ w(N_W) & \ddots & & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & & \ddots & 0 \\ w(N) & \cdots & w(N_W) & 0 & \cdots & 0 \end{bmatrix} \tag{36}$$

Define a short-time matrix for each short-time segment as

$$\widetilde{\mathbf{W}}_{\mathbf{FIR},i} = \begin{bmatrix} w(0) & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & 0 & 0 \\ w(N_W - 1) & \cdots & w(0) & 0 & 0 \\ 0 & \ddots & & \ddots & 0 \\ 0 & 0 & w(N_W - 1) & \cdots & w(0) \end{bmatrix} \tag{37}$$

with size $(N_W + N_Q + T_{2i} - T_{2i-1} - 1)\text{x}(N_W + N_Q + T_{2i} - T_{2i-1} - 1)$. Also define a short-time delta Q-filter matrix for each short-time segment as

$$\widetilde{\mathbf{Q}}_{\boldsymbol{\delta},i} = \left[\mathbf{0}_{(T_{2i} - T_{2i-1})x(N_W - 1)} \quad \mathbf{Q}_{\boldsymbol{\delta},i}\right] \tag{38}$$

with size $(T_{2i} - T_{2i-1})\text{x}(N_W + N_Q + T_{2i} - T_{2i-1} - 1)$.

The following theorem establishes monotonic convergence for the proposed class of TV Q-filters.

Theorem 4: Let an ILC system be described by (5-9) with $\alpha = 1$, the TV Q-filter (21-32) and the FIR approximation (33-37). Then the ILC system is stable and monotonically convergent if

$$\overline{N} \ge N_Q + N_W + 1, \tag{39}$$

and

$$\underbrace{\left\| Q_{LTI}(z) W(z) \right\|_\infty}_{LTI\ filter} + \underbrace{\max_{i \in [1,m]} \left\{ \left\| \widetilde{\mathbf{Q}}_{\boldsymbol{\delta},i} \widetilde{\mathbf{W}}_{\mathbf{FIR}} \right\|_2 \right\}}_{LTV\ segments} + \underbrace{\frac{2\kappa_W (\gamma_W)^{N_W+1} \sqrt{N_{TV}}}{1 - \gamma_W}}_{W(q)\ FIR\ approximation\ error} \le 1, \tag{40}$$

Proof: Monotonic convergence implies stability because $|\rho(\mathbf{A})| < \overline{\sigma}(\mathbf{A})$, so it will be sufficient to show only monotonic convergence. From Theorem 3, monotonic convergence is assured if $\|\mathbf{QW}\|_2 \le 1$. Now,

$$\|\mathbf{QW}\|_2 = \Bigg\| \mathbf{Q_{LTI}W} + \underbrace{(\mathbf{Q}-\mathbf{Q_{LTI}})}_{\mathbf{Q_\delta}} \cdot \underbrace{(\mathbf{W_{FIR}}+\mathbf{W_\varepsilon})}_{\mathbf{W}} \Bigg\|_2 \le \|\mathbf{Q_{LTI}W}\|_2 + \|\mathbf{Q_\delta W_{FIR}}\|_2 + \|\mathbf{Q_\delta W_\varepsilon}\|_2 .$$

The system $\mathbf{Q_{LTI}W}$ is LTI and therefore equivalent to the filter $Q_{LTI}(q)W(q)$. By Theorem 1,

$$\|\mathbf{Q_{LTI}W}\|_2 \le \|Q_{LTI}(z)W(z)\|_\infty . \tag{41}$$

It can be shown that if $\mathbf{A}$ is an NxN matrix with M non-zero rows, then $\|\mathbf{A}\|_2 \le \sqrt{M}\|\mathbf{A}\|_\infty$ where $\|\mathbf{A}\|_\infty = \max_i\left(\sum_{j=1}^{N}|a_{ij}|\right)$. The NxN matrix $\mathbf{Q_\delta}$ has a non-zero row for each time instant where the frozen filter bandwidth is not Ω_0. That is, the number of non-zero rows in $\mathbf{Q_\delta}$ is equal to the sum of the lengths of the time-varying segments as given by N_{TV} in (38). Therefore,

$$\|\mathbf{Q_\delta W_\varepsilon}\|_2 \le \sqrt{N_{TV}}\|\mathbf{Q_\delta W_\varepsilon}\|_\infty . \tag{42}$$

Then,

$$\begin{aligned}
\|\mathbf{Q_\delta W_\varepsilon}\|_\infty &\le \big\| |\mathbf{Q_\delta}| \cdot |\mathbf{W_\varepsilon}| \big\|_\infty \\
&= \max_i \sum_{j=0}^{\min\{N_Q-1,i-1\}} \left| f_{\Omega(i)}(j) - f_{\Omega_0}(j) \right| \\
&\qquad \cdot \sum_{k=\max\{i-j-N_Q+1, N_W+1\}}^{i-j} |\kappa_W| \cdot |\gamma_W|^k \\
&\le \max_i \sum_{j=0}^{N_Q-1} \left| f_{\Omega(i)}(j) - f_{\Omega_0}(j) \right| \\
&\qquad \cdot \kappa_W \sum_{k=N_W+1}^{N} (\gamma_W)^k \\
&\le \max_i \frac{\kappa_W (\gamma_W)^{N_W+1}}{1-\gamma_W} \cdot \sum_{j=0}^{N_Q-1} \left| f_{\Omega(i)}(j) \right| + \left| f_{\Omega_0}(j) \right| \\
&= \frac{2\kappa_W (\gamma_W)^{N_W+1}}{1-\gamma_W} .
\end{aligned}$$

Therefore,

$$\|\mathbf{Q_\delta W_\varepsilon}\|_2 \le \frac{2\kappa_W (\gamma_W)^{N_W+1}\sqrt{N_{TV}}}{1-\gamma_W} . \tag{43}$$

Finally, $\mathbf{Q_\delta W_{FIR}}$ has the block diagonal structure

$$\begin{bmatrix}
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \tilde{\mathbf{Q}}_{\delta,1}\tilde{\mathbf{W}}_{\mathbf{FIR},1} & 0 & & & & \vdots \\
\vdots & 0 & 0 & 0 & & & \vdots \\
\vdots & & 0 & \tilde{\mathbf{Q}}_{\delta,2}\tilde{\mathbf{W}}_{\mathbf{FIR},2} & 0 & & \vdots \\
\vdots & & & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & & 0 & \tilde{\mathbf{Q}}_{\delta,m}\tilde{\mathbf{W}}_{\mathbf{FIR},m} & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 0
\end{bmatrix} .$$

Therefore, $\|\mathbf{Q_\delta W_{FIR}}\|_2$ is the largest singular value of the individual blocks, or

$$\|\mathbf{Q_\delta W_{FIR}}\|_2 \le \max_{i\in[1,m]} \left\|\tilde{\mathbf{Q}}_{\delta,i}\tilde{\mathbf{W}}_{\mathbf{FIR},i}\right\|_2 . \tag{44}$$

Therefore, $\|\mathbf{QW}\|_2$ is less than or equal to the left side of the expression in (37) and, hence, (37) implies monotonic convergence and stability of the ILC system. ■

Remark 4: While (37) is a more complicated expression than (17), it is numerically much easier to solve for long iteration durations. The most computationally taxing part of the expression is the 2-norm of the short-time matrices. These matrices are approximately the size of the FIR length of the Q-filter which is typically no larger than $O(10^2)$. Calculations involving this size matrix are quite tractable on current computational machinery.

Remark 5: The condition in (37) offers insight into several tradeoffs that occur with this class of LTV Q-filters. First, the larger the baseline bandwidth Ω_0 is, the larger $\|Q_{LTI}(z)W(z)\|_\infty$ will be. Therefore the constraint on the short-time bandwidth profile will become more restrictive. Consider a situation where the bandwidth Ω_0 is increased until $\|Q_{LTI}(z)W(z)\|_\infty \to 1$. In this case, the short-time bandwidth deviations must decrease until they are 0. Therefore, the maximum bandwidth LTI Q-filter is a special case of the presented LTV Q-filter class. As the baseline bandwidth is reduced, it can be allocated into the short-time segments to improve performance locally.

A second tradeoff occurs in the FIR approximation of $W(z)$. Note that this term can be made arbitrarily small by choosing larger FIR lengths N_W. Larger N_W, however, increase the amount of spacing required between the short-time segments (36).

The relationship that monotonicity depends only on the largest short-time filter and not a combination of the short-time filters offers a number of interesting possibilities. Conceivably a controls engineer could design an array of special purpose short-time filters with each one satisfying (37). For instance, some filters could have very high frequency, but very short duration while others could have mid-level frequency, but longer durations. These short-time filters can then be assembled to rapidly generated a customized LTV Q-filter for a given desired trajectory. The only requirement that would need to be fulfilled is the minimum filter spacing (36).

V. Conclusions

This paper considered the use of LTV Q-filters in ILC of uncertain systems. Sufficient conditions for stability and monotonic convergence of the uncertain system were developed. Additionally, more specific and computationally feasible monotonic convergence conditions were developed for a specific class of LTV Q-filters. This class was composed of filters that have a baseline fixed bandwidth with short-time deviations that may be particularly useful for rapid motion applications. The monotonic convergence condition highlighted the tradeoff between the baseline bandwidth and the maximum short-time deviation. Additionally, if the short-time deviations are sufficiently separated from one another, then they do not have a compounding effect. Instead, only the "worst" deviation is used in determining monotonicity. This result has important implications in design by allowing a virtually limitless number of short-time deviations, and by allowing the ability to rearrange the bandwidth profile online or extend the iteration length and add additional short-time deviations without additional analysis.

An important question that remains open regards the theoretically maximum performance enhancement that the proposed LTV Q-filter class might have over LTI Q-filters for uncertain systems. Investigation of this question will be the focus of future work.

APPENDIX

NOMENCLATURE

Symbol	Meaning
$d(k)$	Output disturbance at time k
$\mathbf{d}$	Vector of output disturbances
$e_j(k)$	Error on iteration j at time k
$\mathbf{e}_j$	Vector of errors for iteration j
$\mathbf{e}_\infty$	Vector of converged error
$f_\Omega(i)$	i^{th} Markov parameter of template filter for bandwidth Ω
$\boldsymbol{f}_\Omega$	Vector of Markov parameters of template filter
j	Iteration index
k	Time index
$\mathbf{L_e}$	Lifted system representation of LTI learning function
N	Length of iteration
N_Q	Length of FIR of Q-filter
$\boldsymbol{N_{TV}}$	Total length of time-varying, short-time segments
N_W	Length of approximation of W
$\overline{N}$	Minimum length of long-time segments
$P(z)$	Transfer function of actual plant
$p(k)$	Markov parameters of P
$\mathbf{P}$	Lifted system representation of P
$\hat{P}(z)$	Transfer function of plant model
$\hat{p}(k)$	Markov parameters of P
$\hat{\mathbf{P}}$	Lifted system representation of $\hat{P}$
$\mathbf{Q}$	Lifted system representation of LTV Q-filter
$Q_{LTI}(z)$	Transfer function of LTI Q-filter with bandwidth Ω_0
$\mathbf{Q_{LTI}}$	Lifted system representation of $Q_{LTI}(z)$
$\mathbf{Q}_\delta$	Mathematical difference of $\mathbf{Q}$ and $\mathbf{Q_{LTI}}$
$\mathbf{Q}_{\delta,i}$	Truncated $\mathbf{Q}_\delta$ matrix for the i^{th} short-time segment
$\tilde{\mathbf{Q}}_{\delta,i}$	Extended $\mathbf{Q}_{\delta,i}$ matrix
$q_k(i)$	i^{th} Markov parameter of frozen Q-filter at time k
T_i	Time index marking beginning of filter segments
$u_j(k)$	Control input on iteration j at time k
$\mathbf{u}_j$	Vector of control inputs for iteration j
$\mathbf{u}_\infty$	Vector of converged control inputs
$\delta\mathbf{u}_j$	Mathematical difference of $\mathbf{u}_j$ and $\mathbf{u}_\infty$
$W(z)$	Transfer function of uncertainty weighting
$w(k)$	Markov parameters of W
$\mathbf{W}$	Lifted system representation of W
$\mathbf{W_{FIR}}$	Lifted system representation of FIR approximation of W
$\tilde{\mathbf{W}}_{\mathbf{FIR},i}$	Truncated $\mathbf{W_{FIR}}$ matrix for i^{th} short-term segment
$\mathbf{W}_\varepsilon$	Mathematical difference of $\mathbf{W}$ and $\mathbf{W_{FIR}}$
$y_j(k)$	Output on iteration j at time k
$\mathbf{y}_j$	Vector of outputs for iteration j
$y_d(k)$	Desired output at time k
$\mathbf{y_d}$	Vector of desired outputs for iteration j
α	Learning rate
γ_W	Constant in bounding function for $w(k)$
$\Delta(z)$	Transfer function of uncertainty
$\delta(k)$	Markov parameters of Δ
$\boldsymbol{\Delta}$	Lifted system representation of Δ
κ_W	Constant in bounding function for $w(k)$
Ω_0	Baseline bandwidth for long-time segments of Q-filter
$\Omega(k)$	LTV Q-filter bandwidth at time k
$\Omega_i(k)$	Bandwidth for i^{th} short-time segment at time k

ACKNOWLEDGMENT

The authors would like to express their gratitude to Marina Tharayil for insightful discussions on time-varying Q-filtering.

REFERENCES

[1] Arimoto, S., S. Kawamura, and F. Miyazaki, "Bettering Operation of Robots by Learning," *Journal of Robotic Systems*, vol. 1, pp. 123-140, 1984.

[2] Moore, K.L., *Iterative Learning Control for Deterministic Systems*, Germany: Springer-Verlag, 1993.

[3] Bien, Z. and J.-X. Xu, eds., *Iterative Learning Control: Analysis, Design, Integration, and Applications*, Boston: Kluwer Academic Publishers, 1998.

[4] Chen, Y. and C. Wen, *Iterative Learning Control: Convergence, Robustness and Applications*, London: Springer-Verlag, 1999.

[5] Xu, J.-X. and Y. Tan, *Linear and Nonlinear Iterative Learning Control*, Berlin: Springer, 2003.

[6] Huang, Y.-C. and R.W. Longman, "The Source of the Often Observed Property of Initial Convergence Followed by Divergence in Learning and Repetitive Control," *Advances in Astronautical Sciences*, vol. 90, no. 1, pp. 555-572, 1996.

[7] Longman, R.W., "Iterative Learning Control and Repetitive Control for Engineering Practice," *International Journal of Control*, vol. 73, no. 10, pp. 930-954, 2000.

[8] Norrlöf, M., "Comparative Study on First and Second Order ILC – Frequency Domain Analysis and Experiments," *Proc. of the 39th IEEE Conf. on Decision and Control*, pp. 3415-3420, 2000.

[9] Kavli, T., "Frequency Domain Synthesis of Trajectory Learning Controllers for Robot Manipulators," *Journal of Robotic Systems*, vol. 9, no. 5, pp. 663-680, 1992.

[10] de Roover, D. and O.H. Bosgra, "Synthesis of Robust Multivariable Iterative Learning Controllers with Application to a Wafer Stage Motion System," *International Journal of Control*, Vol. 73, No. 10, pp. 968-979, 2000.

[11] Tharayil, M. and A. Alleyne, "A Time-Varying Iterative Learning Control Scheme", *Proc. of the American Control Conference*, vol. 4, pp. 3782-3787, 2004.

[12] Tharayil, M. and A. Alleyne, "Design and Convergence of a Time-Varying Iterative Learning Control Law," *Proc. of the Int. Mech. Eng. Congress and RD&D Expo*, 2004.

[13] Norrlöf, M. and S. Gunnarsson, "Time and Frequency Domain Convergence Properties in Iterative Learning Control," *International Journal of Control*, vol. 75, no. 14, pp. 1114-1126, 2002.