Application of Normal Form in Chaotic Synchronization

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Abstract—**Two criterions for converting a chaotic system to a normal form via coordinate transformation are presented. Firstly, it was proved that a chaotic system can be converted to a normal form if and only if there exists a single-input control system which treats the vector field of the chaotic system as drift vector field and can be fully linearized by state feedback. Secondly, near the non-singular point, there always exists some coordinate transformation to perform the converting; and near the singular point, the converting can be found if and only if, at this point, the eigenpolynomial of the Jacobian matrix of the vector field is equal to the minimal polynomial of the same matrix. Moreover, the condition, under which the synchronization between the normal form of the drive system and the Brunovsky canonical form of the response system implies the synchronization between the drive system and the response system, was discussed. Finally, synchronizing two Rössler chaotic systems with difference in two parameters was taken as a concrete example to illustrate the new method.**

I. Introduction

SINCE Lorenz found the first Chaotic System in 1963 and especially after Ott et al. presented OGY method to control the chaotic system in 1990 [1], chaotic control has received wide attention [2]. Recently, some achievements have been made in chaotic systems which can be converted to the following normal form (not Poincaré's Normal Form) via coordinate transformation.

$$\begin{aligned} &\dot{z}_1 = z_2 \quad \dot{z}_2 = z_3 \quad \cdots \quad \dot{z}_{n-1} = z_n \\ &\dot{z}_n = F(z_1, z_2, \cdots, z_n) \end{aligned} \tag{1}$$

It has been found that Lur'e system [3] and strict-feedback chaotic systems [4,5] are the chaotic systems which can be converted to normal forms via coordinate transformation. Reference [3] uses normal form in chaotic anti-control; Reference [5] proposes a method to implement synchronization between two different normal forms; References [6,7] use chaotic system which can be converted to normal form and nonlinear system which can be linearized by state feedback as driving system and response system respectively, convert the former to normal form and the latter to Brunovsky canonical form (4), and then discusses the synchronization between normal form and canonical form by the same method as [5] except that uncertainty is taken into account in [6,7]. However, the synchronization between normal form after coordinate transformation and canonical form cannot imply the synchronization between the original driving system and the original response system, even in the case of identical synchronization. [4,5] take into account this limitation of [6,7] and use the method of inverse design, rather than normal form, to implement the synchronization in the practical systems (Chua's system and Rössler system).

References [8,9] combine the driving system and the respond system to implement the synchronization by full or partial feedback linearization of error dynamic, using the same differential geometry method as introduced in [6,7]. Reference [9] presents the condition under which error dynamic can be fully linearized by feedback; however the algorithm is complex due to the combination of systems.

The paper discusses the sufficient and necessary condition in the transformation from equation to normal form in two aspects. One is from the input-state linearization; the other is from the local point. Then we discuss the application of normal form in chaotic synchronization. In contrast with the inverse design method [4,5], we use differential geometry method; Meanwhile in contrast with [8,9], we discuss the condition under which the synchronization between the normal form after coordinate transformation and Brunovsky canonical form implies the synchronization between the original driving system and the respond system. Finally, we realize the synchronization between two Rössler chaotic systems with difference in two parameters to illustrate the new method.

II. Normal Form

In order to explain the relationship between normal form and input-state linearization, we shall study affine nonlinear control system with single input $u \in \mathbb{R}$ and state $\boldsymbol{x} = (x_1, \cdots, x_n)$ described by means of a set of equations of the following type

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u \tag{2}$$

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where $\boldsymbol{f}(\boldsymbol{x})=(f_1,f_2,\cdots,f_n)^T$ and $\boldsymbol{g}(\boldsymbol{x})=(g_1,g_2,\cdots,g_n)^T$ are smooth vector fields on $\mathbb{R}^n$. In the language of differentiable manifold, the smooth vector field $\boldsymbol{f}$ can be written as $f_1\frac{\partial}{\partial x_1}+f_2\frac{\partial}{\partial x_2}+\cdots+f_n\frac{\partial}{\partial x_n}$. Suppose the following problem is set: given a point $\boldsymbol{x}_0$, find (if possible) a neighborhood U of $\boldsymbol{x}_0$, a feedback

$$u=\alpha(\boldsymbol{x})+\beta(\boldsymbol{x})v \quad \beta(\boldsymbol{x})\neq 0 \tag{3}$$

defined on U, and a coordinates transformation $\boldsymbol{z}=\boldsymbol{\Phi}(\boldsymbol{x})$ also defined on U, such that the corresponding closed loop system in the coordinates $\boldsymbol{z}=\boldsymbol{\Phi}(\boldsymbol{x})$, can be described as Brunovsky canonical form

$$\begin{aligned}&\dot{z}_1=z_2 \quad \dot{z}_2=z_3 \quad \cdots \quad \dot{z}_{n-1}=z_n\\&\dot{z}_n=v.\end{aligned} \tag{4}$$

This problem is the "single-input" version of the so-called state space exact linearization problem.

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be smooth vector fields on an open set U of $\mathbb{R}^n$, $ad_{\boldsymbol{X}}^0\boldsymbol{Y}:=\boldsymbol{Y}$, $ad_{\boldsymbol{X}}\boldsymbol{Y}:=ad_{\boldsymbol{X}}^1\boldsymbol{Y}:=\left(\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{x}}\right)\boldsymbol{X}-\left(\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}\right)\boldsymbol{Y}$, and $ad_{\boldsymbol{X}}^i\boldsymbol{Y}:=ad_{\boldsymbol{X}}(ad_{\boldsymbol{X}}^{i-1}\boldsymbol{Y})$ for $i=1,2,\cdots$.

Theorem 1[10, 11]*:* The state space exact linearization problem of system (2) is solvable near a point $\boldsymbol{x}_0$ if and only if the following conditions are satisfied in an open neighborhood U of $\boldsymbol{x}_0$

(i) the matrix $[\boldsymbol{g},ad_{\boldsymbol{f}}\boldsymbol{g},\cdots,ad_{\boldsymbol{f}}^{n-1}\boldsymbol{g}]$ has rank n;

(ii) the distribution $\mathrm{span}\{\boldsymbol{g},ad_{\boldsymbol{f}}\boldsymbol{g},\cdots,ad_{\boldsymbol{f}}^{n-2}\boldsymbol{g}\}$ is involution.

Assuming chaotic system is expressed as:

$$\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x}), \tag{5}$$

where $\boldsymbol{x}=(x_1,\cdots,x_n)$, and $\boldsymbol{f}=(f_1,\cdots,f_n)^T$ is a smooth vector field. The relation between state space exact linearization problem of (2) and the problem that (5) can be transformed to normal form is showed by the following theorem:

Theorem 2: (5) can be converted to normal form (1) in the neighborhood U of $\boldsymbol{x}_0$ if and only if there exists a smooth vector field $\boldsymbol{g}(\boldsymbol{x})=(g_1,\cdots,g_n)^T$, such that single-input controlled system $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})u$ can be state space exactly linearized by state feedback in U.

Proof: (Necessity): If (5) can be converted to normal form via coordinate transformation $\boldsymbol{z}=\boldsymbol{\Phi}(\boldsymbol{x})$ in U, by choosing

$$\boldsymbol{g}(\boldsymbol{x})=\boldsymbol{\Phi}^{-1}(\boldsymbol{B}_c),\ \boldsymbol{B}_c=(0,\cdots,0,1)^T \tag{6}$$

and defining

$$u=-F(\boldsymbol{\Phi}(\boldsymbol{x}))+v, \tag{7}$$

(2) is converted to (4) via the same coordinate transformation.

(Sufficiency): If there exists a neighborhood of $\boldsymbol{x}_0$, coordinate transformation $\boldsymbol{z}=\boldsymbol{\Phi}(\boldsymbol{x})$ in U and feedback $u=\alpha(\boldsymbol{x})+\beta(\boldsymbol{x})v$ which can convert (2) to (4). Let

$$v=-\frac{\alpha(\boldsymbol{\Phi}^{-1}(z))}{\beta(\boldsymbol{\Phi}^{-1}(z))}, \tag{8}$$

then we conclude that $u=0$, and (5) have been converted to normal form (1). □

According to [3], suppose that a chaotic system can be transformed to normal form in the neighborhood U of $\boldsymbol{x}_0$, a arbitrary control system which is feedback linearizable in U can generate this chaotic system via state feedback.

Corollary 1: Suppose that chaotic system (5) exists smooth vector field $\boldsymbol{g}(\boldsymbol{x})=(g_1,\cdots,g_n)^T$ such that single-input control system $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})u$ is global feedback linearizable, and an arbitrary control system Σ is also global feedback linearizable, Σ can generate chaotic system (5) via state feedback.

Remark 1: Many chaotic system, such as Duffing system, Van der Pol system, Rössler system and some Chua's system, can be written as low triangular form

$$\begin{aligned}&\dot{x}_1=f_1(x_1,x_2) \quad \dot{x}_2=f_2(x_1,x_2,x_3) \ \cdots\\&\dot{x}_n=f_n(x_1,x_2,\cdots,x_n),\end{aligned} \tag{9}$$

where $f_1,f_2,\cdots,f_n$ are smooth functions. If $\frac{\partial f_i}{\partial x_{i+1}}\neq 0$ $(i=1,2,\cdots,n-1)$ is satisfied, the control systems constructed by adding a scalar input to the right side of the last equation of (9) are global feedback linearizable [12]. So (9) can be converted to normal form.

We now present the conditions for singular point and nonsingular point respectively, under which equation (5) can be transformed to normal form near a point.

Theorem 3: If (5) satisfies $\boldsymbol{f}(\boldsymbol{x}_0)\neq\boldsymbol{0}$, there always exist a neighborhood U of $\boldsymbol{x}_0$ and a local coordinate transformation $\boldsymbol{z}=\boldsymbol{\Phi}(\boldsymbol{x})$ in U such that (5) is represented as normal form (1) in the new coordinate.

Proof: $\boldsymbol{f}(\boldsymbol{x}_0)\neq\boldsymbol{0}$ implies that $\boldsymbol{f}(\boldsymbol{x}_0)\neq\boldsymbol{0}$ in some neighborhood of $\boldsymbol{x}_0$. By Frobenius theorem, $\boldsymbol{f}$ becomes $\boldsymbol{f}^{\boldsymbol{w}}=(0,0,\cdots,1)^T$ in the neighborhood W of $\boldsymbol{x}_0$ and a local coordinate $(W;w_i)$. By theorem 2, if (5) can be converted to normal form in a neighborhood of $\boldsymbol{x}_0$, there exists a appropriate vector field $\boldsymbol{g}^{\boldsymbol{w}}(\boldsymbol{x})=(g_1^{\boldsymbol{w}},g_2^{\boldsymbol{w}},\cdots,g_n^{\boldsymbol{w}})^T$ such that the system $\dot{\boldsymbol{w}}=\boldsymbol{f}^{\boldsymbol{w}}(\boldsymbol{w})+\boldsymbol{g}^{\boldsymbol{w}}(\boldsymbol{w})u$ is feedback linearizable. Compute the following vector fields,

$$ad_{\boldsymbol{f}^{\boldsymbol{w}}}\boldsymbol{g}^{\boldsymbol{w}} = \frac{\partial \boldsymbol{g}^{\boldsymbol{w}}}{\partial \boldsymbol{x}}\boldsymbol{f}^{\boldsymbol{w}} - \frac{\partial \boldsymbol{f}^{\boldsymbol{w}}}{\partial \boldsymbol{x}}\boldsymbol{g}^{\boldsymbol{w}} = \frac{\partial \boldsymbol{g}^{\boldsymbol{w}}}{\partial \boldsymbol{x}}\boldsymbol{f}^{\boldsymbol{w}}$$
$$= \left(\frac{\partial g_1^{\boldsymbol{w}}}{\partial x_n}, \frac{\partial g_2^{\boldsymbol{w}}}{\partial x_n}, \cdots \frac{\partial g_n^{\boldsymbol{w}}}{\partial x_n}\right)^T$$
$$ad_{\boldsymbol{f}^{\boldsymbol{w}}}^2\boldsymbol{g}^{\boldsymbol{w}} = \frac{\partial (ad_{\boldsymbol{f}^{\boldsymbol{w}}}\boldsymbol{g}^{\boldsymbol{w}})}{\partial \boldsymbol{x}}\boldsymbol{f}^{\boldsymbol{w}} - \frac{\partial \boldsymbol{f}^{\boldsymbol{w}}}{\partial \boldsymbol{x}}ad_{\boldsymbol{f}^{\boldsymbol{w}}}\boldsymbol{g}^{\boldsymbol{w}} \tag{10}$$
$$= \frac{\partial (ad_{\boldsymbol{f}^{\boldsymbol{w}}}\boldsymbol{g}^{\boldsymbol{w}})}{\partial \boldsymbol{x}}\boldsymbol{f}^{\boldsymbol{w}} = \left(\frac{\partial^2 g_1^{\boldsymbol{w}}}{\partial x_n^2}, \frac{\partial^2 g_2^{\boldsymbol{w}}}{\partial x_n^2}, \cdots \frac{\partial^2 g_n^{\boldsymbol{w}}}{\partial x_n^2}\right)^T$$
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$$ad_{\boldsymbol{f}^{\boldsymbol{w}}}^{n-1}\boldsymbol{g}^{\boldsymbol{w}} = \left(\frac{\partial^{n-1} g_1^{\boldsymbol{w}}}{\partial x_n^{n-1}}, \frac{\partial^{n-1} g_2^{\boldsymbol{w}}}{\partial x_n^{n-1}}, \cdots \frac{\partial^{n-1} g_n^{\boldsymbol{w}}}{\partial x_n^{n-1}}\right)^T.$$

Construct those vector fields mentioned above

$$\left(\boldsymbol{g}^{\boldsymbol{w}}, ad_{\boldsymbol{f}^{\boldsymbol{w}}}\boldsymbol{g}^{\boldsymbol{w}}, \cdots, ad_{\boldsymbol{f}^{\boldsymbol{w}}}^{n-1}\boldsymbol{g}^{\boldsymbol{w}}\right) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ * & * & \cdots & 1 \end{pmatrix}, \tag{11}$$

where "*" means the function is arbitrary if no collision with the right side entry in the same line of the matrix, e.g. $\frac{\partial g_2^{\boldsymbol{w}}}{\partial x_n} = 1$, then $g_2^{\boldsymbol{w}} = x_n + k(x_1, x_2, \cdots, x_{n-1})$, where $k(x_1, x_2, \cdots, x_{n-1})$ is a arbitrary smooth function. By this way, $\left\{\boldsymbol{g}^{\boldsymbol{w}}, ad_{\boldsymbol{f}^{\boldsymbol{w}}}\boldsymbol{g}^{\boldsymbol{w}}, \cdots, ad_{\boldsymbol{f}^{\boldsymbol{w}}}^{n-1}\boldsymbol{g}^{\boldsymbol{w}}\right\}$ is full rank, and span$\left\{\boldsymbol{g}^{\boldsymbol{w}}, ad_{\boldsymbol{f}^{\boldsymbol{w}}}\boldsymbol{g}^{\boldsymbol{w}}, \cdots, ad_{\boldsymbol{f}^{\boldsymbol{w}}}^{n-1}\boldsymbol{g}^{\boldsymbol{w}}\right\}$ is involution. So $\dot{\boldsymbol{w}} = \boldsymbol{f}^{\boldsymbol{w}}(\boldsymbol{w}) + \boldsymbol{g}^{\boldsymbol{w}}(\boldsymbol{w})u$ is feedback linearizable in a neighborhood of $\boldsymbol{x}_0$. Let $\boldsymbol{g}$ denote $\boldsymbol{g}^{\boldsymbol{w}}$ in the original coordinate system. Because involution and rank don't change after coordinate transformation, $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u$ is feedback linearizable in some neighborhood of $\boldsymbol{x}_0$. □

Let d denote exterior differential mapping. Let $\boldsymbol{X}(\boldsymbol{x})$ and $\lambda(\boldsymbol{x})$ be smooth vector fields. Lie derivatives were defined as $L_{\boldsymbol{X}}\lambda = L_{\boldsymbol{X}}^1\lambda = \langle d\lambda, \boldsymbol{X} \rangle$, $L_{\boldsymbol{X}}^i\lambda = L_{\boldsymbol{X}}L_{\boldsymbol{X}}^{i-1}\lambda$ for $i = 2,3\cdots$. Denote $\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T} = \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right)^T$.

Theorem 4: If (5) satisfies $\boldsymbol{f}(\boldsymbol{x}_0) = \boldsymbol{0}$, it can be represented as normal form in some neighborhood of $\boldsymbol{x}_0$ via coordinate transformation if and only if the minimal polynomial of $\left.\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\right|_{\boldsymbol{x}=\boldsymbol{x}_0}$ is equal to its eigenpolynomial.

Proof: (5) can be convert to form (1) via coordinate transformation $\boldsymbol{z} = \boldsymbol{\Phi}(\boldsymbol{x})$, i.e. $z_1, z_2, \cdots, z_n$ are independent and the following equations are solvable,

$$\frac{\partial z_1}{\partial x_1}f_1 + \frac{\partial z_1}{\partial x_2}f_2 + \cdots + \frac{\partial z_1}{\partial x_n}f_n = L_f z_1 = z_2$$
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$$\frac{\partial z_{n-1}}{\partial x_1}f_1 + \frac{\partial z_{n-1}}{\partial x_2}f_2 + \cdots + \frac{\partial z_{n-1}}{\partial x_n}f_n = L_f z_2 = z_n \tag{12}$$
$$\frac{\partial z_{n-1}}{\partial x_1}f_1 + \frac{\partial z_{n-1}}{\partial x_2}f_2 + \cdots + \frac{\partial z_{n-1}}{\partial x_n}f_n = L_f z_n = F(z_1, z_2, \cdots, z_n),$$

where $F(z_1, z_2, \cdots, z_n)$ is a arbitrary smooth function. Assume

$$z_1 = \phi(\boldsymbol{x}), \tag{13}$$

then the following equations hold

$$z_2 = L_{\boldsymbol{f}}\phi(\boldsymbol{x}) \quad \cdots \quad z_n = L_{\boldsymbol{f}}^{n-1}\phi(\boldsymbol{x}). \tag{14}$$

Consider a set of covectors

$$\{d\phi, d(L_{\boldsymbol{f}}\phi), \cdots, d(L_{\boldsymbol{f}}^{n-1}\phi)\}. \tag{15}$$

If the rank of (15) is n in a neighborhood of $\boldsymbol{x}_0$, (15) can be transformed to (1) by using the last equation of (12) to compute $F(z_1, z_2, \cdots, z_n)$ in the neighborhood. And if (5) can be converted to (1) in a neighborhood of $\boldsymbol{x}_0$, the rank of (15) is n in a neighborhood of $\boldsymbol{x}_0$ obviously. Therefore, (5) can be transformed to (1) via coordinate transformation if and only if the rank of (15) is n in a neighborhood of $\boldsymbol{x}_0$. Considering the continuity of the following determinant with regard to $\boldsymbol{x}$,

$$\det\begin{pmatrix} \frac{\partial \phi}{\partial x_1} & \frac{\partial L_{\boldsymbol{f}}\phi}{\partial x_1} & \cdots & \frac{\partial L_{\boldsymbol{f}}^{n-1}\phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} & \frac{\partial L_{\boldsymbol{f}}\phi}{\partial x_2} & \cdots & \frac{\partial L_{\boldsymbol{f}}^{n-1}\phi}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \phi}{\partial x_n} & \frac{\partial L_{\boldsymbol{f}}\phi}{\partial x_n} & \cdots & \frac{\partial L_{\boldsymbol{f}}^{n-1}\phi}{\partial x_n} \end{pmatrix} \tag{16}$$

the rank of (15) is n in some neighborhood of $\boldsymbol{x}_0$ if and only if it is n at $\boldsymbol{x}_0$.

Noting that $\boldsymbol{f}(\boldsymbol{x}_0) = \boldsymbol{0}$,

$$d(L_{\boldsymbol{f}}\phi)\big|_{\boldsymbol{x}=\boldsymbol{x}_0} = d < d\phi, \boldsymbol{f} >\big|_{\boldsymbol{x}=\boldsymbol{x}_0} = d(\sum_{i=1}^{n} f_i \frac{\partial \phi}{\partial x_i})\Big|_{\boldsymbol{x}=\boldsymbol{x}_0}$$

$$= \sum_{i=1}^{n}(\frac{\partial \phi}{\partial x_i} df_i + f_i d\frac{\partial \phi}{\partial x_i})\Big|_{\boldsymbol{x}=\boldsymbol{x}_0} = \sum_{i=1}^{n}\frac{\partial \phi}{\partial x_i} df_i\Big|_{\boldsymbol{x}=\boldsymbol{x}_0}$$

$$= \sum_{i=1}^{n}\left(\frac{\partial \phi}{\partial x_i}\left(\sum_{j=1}^{n}\frac{\partial f_i}{\partial x_j}dx_j\right)\right)\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0} \tag{17}$$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_j}\frac{\partial \phi}{\partial x_i}\right)dx_j\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0}$$

$$= \left(dx_1, dx_1, \cdots, dx_n\right)\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \cdots, \frac{\partial \phi}{\partial x_n}\right)^T\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0}.$$

Further,

$$d(L_{\boldsymbol{f}}^2\phi)\big|_{\boldsymbol{x}=\boldsymbol{x}_0}$$

$$= \left(dx_1, \cdots, dx_n\right)\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\left(\frac{\partial L_{\boldsymbol{f}}\phi}{\partial x_1}, \cdots, \frac{\partial L_{\boldsymbol{f}}\phi}{\partial x_n}\right)^T\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0}$$

$$= \left(dx_1, \cdots, dx_n\right)\left(\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\right)^2\left(\frac{\partial \phi}{\partial x_1}, \cdots, \frac{\partial \phi}{\partial x_n}\right)^T\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0} \tag{18}$$

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$$d(L_{\boldsymbol{f}}^{n-1}\phi)\big|_{\boldsymbol{x}=\boldsymbol{x}_0}$$

$$= \left(dx_1, \cdots, dx_n\right)\left(\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\right)^{n-1}\left(\frac{\partial \phi}{\partial x_1}, \cdots, \frac{\partial \phi}{\partial x_n}\right)^T\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0}.$$

Denote $\boldsymbol{G} = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \cdots, \frac{\partial \phi}{\partial x_n}\right)^T$, the rank of (15) at $\boldsymbol{x}_0$ equals that of (19).

$$\left\{\boldsymbol{G}, \frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\boldsymbol{G}, \cdots, \left(\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\right)^{n-1}\boldsymbol{G}\right\} \tag{19}$$

(Sufficiency): Suppose that the rank of (19) is less than n at $\boldsymbol{x}_0$, then there exists a set of real numbers, not all of them equal to zero, $b_0, b_1, \cdots, b_{n-1}$.

$$\begin{aligned} & b_{n-1}\left(\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0}\right)^{n-1}\boldsymbol{G}\big|_{\boldsymbol{x}=\boldsymbol{x}_0} + \cdots + \\ & b_1\left(\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0}\right)\boldsymbol{G}\big|_{\boldsymbol{x}=\boldsymbol{x}_0} + b_0\boldsymbol{I}_n\boldsymbol{G}\big|_{\boldsymbol{x}=\boldsymbol{x}_0} = \boldsymbol{0} \end{aligned} \tag{20}$$

where $\boldsymbol{I}_n$ is n-order identity matrix. Thus the rank of minimal polynomial of $\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\Big|_{\boldsymbol{x}=\boldsymbol{x}_0}$ is less than n, namely the minimal polynomial of $\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\Big|_{\boldsymbol{x}=\boldsymbol{x}_0}$ is not the eigenpolynomial of the same matrix, which is contradictive.

(Necessity): Suppose that the minimal polynomial of $\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\Big|_{\boldsymbol{x}=\boldsymbol{x}_0}$ is not its eigenpolynomial. There exist real numbers $c_0, c_1, \cdots, c_{i-1}$ and an integer i ($0 < i \le n-1$) satisfies

$$\begin{aligned} & \left(\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0}\right)^i + c_{i-1}\left(\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0}\right)^{i-1} + \cdots + \\ & c_1\left(\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0}\right) + c_0\boldsymbol{I}_n = \boldsymbol{0}. \end{aligned} \tag{21}$$

For an arbitrary $\boldsymbol{G}\big|_{\boldsymbol{x}=\boldsymbol{x}_0}$,

$$\begin{aligned} & \left(\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0}\right)^i \boldsymbol{G}\big|_{\boldsymbol{x}=\boldsymbol{x}_0} + c_{i-1}\left(\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0}\right)^{i-1}\boldsymbol{G}\big|_{\boldsymbol{x}=\boldsymbol{x}_0} \\ & + \cdots + c_1\left(\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_0}\right)\boldsymbol{G}\big|_{\boldsymbol{x}=\boldsymbol{x}_0} + c_0\boldsymbol{I}_n\boldsymbol{G}\big|_{\boldsymbol{x}=\boldsymbol{x}_0} = \boldsymbol{0}. \end{aligned} \tag{22}$$

So the rank of the first $i+1$ vectors of (19) less than $i+1$, which is contradictive. □

According to the theorem 4, the following two control systems as follows can not be transformed to normal form near origin.

$$\dot{x}_1 = x_1,\ \dot{x}_2 = x_2,\ \dot{x}_3 = x_3 \tag{23}$$

$$\dot{x}_1 = x_2^2,\ \dot{x}_2 = x_1^2 + x_2^2,\ \dot{x}_3 = x_2^2 + x_3^2 \tag{24}$$

However the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2x_2 + x_3 \\ \dot{x}_2 &= -x_1 + x_2 + x_3 \\ \dot{x}_3 &= x_2 - (x_1 - x_2)(x_1 - x_2 - x_3) \end{aligned} \tag{25}$$

can be transformed to normal form near origin. In fact, $\boldsymbol{z} = \boldsymbol{\Phi}(x_1, x_2, x_3) = (x_1 - x_2, x_2, -x_1 + x_2 + x_3)^T$ is a coordinate transformation to do this work.

By theorem 2, 3and 4, we have

Corollary 2: For system (5), when $\boldsymbol{f}(\boldsymbol{x}_0) \ne \boldsymbol{0}$, there always exists a n-dimensional smooth vector field $\boldsymbol{g}(\boldsymbol{x}) \ne \boldsymbol{0}$ in a neighborhood U of $\boldsymbol{x}_0$ such that single-input system

$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u$ is feedback linearizable in U ; But when $\boldsymbol{f}(\boldsymbol{x}_0) = \boldsymbol{0}$, there exists the same conclusion only if the eigenpolynomial of $\left.\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}^T}\right|_{\boldsymbol{x}_0}$ is minimal polynomial of this matrix.

III. Chaotic Synchronization Using Normal Form

If driving system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_1(\boldsymbol{x}) \tag{26}$$

can be converted to normal form(1) via global coordinate transformation of $\overline{\boldsymbol{z}} = \boldsymbol{\Phi}(\boldsymbol{x})$; Response system is a single-input control system

$$\dot{\boldsymbol{y}} = \boldsymbol{f}_2(\boldsymbol{y}) + \boldsymbol{g}_2(\boldsymbol{y})u, \tag{27}$$

and can be linearized via the feedback of $u = \alpha(\boldsymbol{y}) + \beta(\boldsymbol{y})v$ and the same global coordinate transformation of $\boldsymbol{z} = \boldsymbol{\Phi}(\boldsymbol{y})$. (The two coordinate transformations are same. Next section, we will give an example). When $\boldsymbol{f}_1 = \boldsymbol{f}_2$, i.e., the case of identical synchronization, if (27) is feedback linearizable, by theorem 2, (26) becomes normal form via the same coordinate transformation. Let $\boldsymbol{e}' = (e_1', e_2', \cdots, e_n')^T = \overline{\boldsymbol{z}} - \boldsymbol{z}$, then

$$\begin{aligned} &\dot{e}_1' = e_2' \quad \dot{e}_2' = e_3' \quad \cdots \quad \dot{e}_{n-1}' = e_n' \\ &\dot{e}_n' = F(\overline{z}_1, \overline{z}_2, \cdots, \overline{z}_n) - v. \end{aligned} \tag{28}$$

Choose

$$v = F(\overline{z}_1, \overline{z}_2, \cdots, \overline{z}_n) + k_1 e_1' + k_2 e_2' + \cdots + k_n e_n', \tag{29}$$

and assign poles by selecting proper real numbers $k_1, k_2, \cdots, k_n$ to make it global asymptotically stable. Error dynamic in original coordinate is $\boldsymbol{e} = (e_1, e_2, \cdots, e_n)^T = \overline{\boldsymbol{x}} - \boldsymbol{x}$ or

$$\boldsymbol{e} = \boldsymbol{\Phi}^{-1}(\overline{z}) - \boldsymbol{\Phi}^{-1}(z). \tag{30}$$

When $t \to \infty$, $\boldsymbol{e}'(t) \to 0$ holds by using (29); but on the other hand, influenced by $\boldsymbol{\Phi}^{-1}$, $\boldsymbol{\Phi}^{-1}(\overline{z}) - \boldsymbol{\Phi}^{-1}(\boldsymbol{z})$ does not always converge to zero, i.e., synchronization after coordinate transformation does not imply synchronization in original coordinate. For example, if choosing global coordinate transformation

$$\boldsymbol{\Phi}^{-1}(\boldsymbol{a}) = \begin{pmatrix} a_1^3 + a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \tag{31}$$

then

$$\begin{aligned} &\| \boldsymbol{\Phi}^{-1}(\overline{z}) - \boldsymbol{\Phi}^{-1}(z) \| \geq \sqrt{[\overline{z}_1^3 + \overline{z}_1 - z_1^3 - \overline{z}]^2} \\ &= \left| (\overline{z}_1 - z_1)(\overline{z}_1^2 + \overline{z}_1 z_1 + z_1^2 + 1) \right|. \end{aligned} \tag{32}$$

Obviously, when $\overline{z}_1 - z_1 \to 0$, the right side of the last equal mark of (32) does not always converge to zero.

However, in the following two cases, $\boldsymbol{e}'(t) \to 0$ means $\boldsymbol{e}(t) \to 0$:

i) If there exists a real number $L > 0$ satisfied

$$\| \boldsymbol{\Phi}^{-1}(\overline{z}) - \boldsymbol{\Phi}^{-1}(z) \| \leq L \| \overline{z} - z \| = L \| \boldsymbol{e}' \|, \tag{33}$$

namely the coordinate transformation $\boldsymbol{\Phi}^{-1}$ is uniformly continuous. Whereupon $\boldsymbol{\Phi}^{-1}(\overline{z}) - \boldsymbol{\Phi}^{-1}(z) \to 0$ when $\boldsymbol{e}'(t) \to 0$. So (26) can synchronize with (27).

ii) If system (26) is a dissipative system, denote the trajectory of system (19) as $\boldsymbol{x}(t; \boldsymbol{x}_0)$ with initial value $\boldsymbol{x}_0$, then, there exists d satisfied [13]

$$\limsup_{t \to \infty} \| \boldsymbol{x}(t; \boldsymbol{x}_0) \| < d, \tag{34}$$

i.e., trajectory with initial value $\boldsymbol{x}_0$ always lies in a compact set. Considering that transformation $\boldsymbol{\Phi}^{-1}$ is global diffeomorphism, in an arbitrary compact set in $\mathbb{R}^n$, there must exist a real $L > 0$ satisfying (33). So we also conclude that (26) can synchronize with (27).

Remark 2: Whatever $L > 0$ is, synchronization is always implemented by (29) independent of L .(Of course, the speed of running to synchronization is influenced.) So do other parameters $\boldsymbol{x}_0$ and d .

Conclusively, we have the following theorem:

Theorem 5: If (26) can be converted to normal form (1) via global coordinate transformation $\boldsymbol{\Phi}(\boldsymbol{x})$, (27) is global feedback linearizable via the same coordinate transformation and feedback $u = \alpha(\boldsymbol{x}) + \beta(\boldsymbol{x})v$, and at the same time, $\boldsymbol{\Phi}^{-1}$ is uniformly continuous or (26) is dissipative, then (26) can synchronize with (27).

IV. Example

We now present an example of Rössler system [14] to illustrate the method mentioned in the prior section. Assuming the driving system is

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 + a x_2 \\ \dot{x}_3 &= b_1 + x_3(x_1 - c_1) \end{aligned} \tag{35}$$

where a , b_1 and c_1 are parameters. The response system is

$$\begin{aligned} \dot{y}_1 &= -y_2 - y_3 \\ \dot{y}_2 &= y_1 + a y_2 \\ \dot{y}_3 &= b_2 + y_3(y_1 - c_2) + u. \end{aligned} \tag{36}$$

where u is scalar input, b_2 and c_2 are parameters. Define $\boldsymbol{f} = (-y_2 - y_3)\frac{\partial}{\partial y_1} + (y_1 + a y_2)\frac{\partial}{\partial y_2} + (b_2 + y_3(y_1 - c_2))\frac{\partial}{\partial y_3}$ and $\boldsymbol{g}(\boldsymbol{x}) = \frac{\partial}{\partial y_3}$, then the rank of

$$(\boldsymbol{g}, ad_{\boldsymbol{f}}\boldsymbol{g}, ad_{\boldsymbol{f}}^2\boldsymbol{g}) = \begin{pmatrix} 0 & 1 & -y_1+c_2 \\ 0 & 0 & 1 \\ 1 & -y_1+c_2 & y_2+(y_1-c_2)^2 \end{pmatrix} \quad (37)$$

is 3, and Lie bracket $[\boldsymbol{g}, ad_{\boldsymbol{f}}\boldsymbol{g}] = 0 \in span\{\boldsymbol{g}, ad_{\boldsymbol{f}}\boldsymbol{g}\}$, so system (36) is global feedback linearizable. In fact, it is easy to find coordinate transformation [10-12]

$$\boldsymbol{\Phi}(\boldsymbol{w}) = \begin{pmatrix} w_2 \\ w_1 + aw_2 \\ aw_1 + (a^2-1)w_2 - w_3 \end{pmatrix}^T, \quad \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}^T, \quad (38)$$

and feedback

$$u = (a^2-1)y_1 + a(a^2-2)y_2 + (c_2-a)y_3 - y_1y_3 - b_2 - v \quad (39)$$

Using the transformation and feedback, (36) becomes

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= v, \end{aligned} \quad (40)$$

and the inverse transformation of the above coordinate transformation is

$$\boldsymbol{\Phi}^{-1}(\phi_1,\phi_2,\phi_3) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}^T = \begin{pmatrix} -a\phi_1+\phi_2 \\ \phi_1 \\ -\phi_1 + a\phi_2 - \phi_3 \end{pmatrix}^T. \quad (41)$$

This inverse transformation is linear, so there exists L satisfied (33), whatever the value of L is. According to the proof of theorem 2, if set $v = (a^2-1)y_1 + a(a^2-2)y_2 + (c_2-a)y_3 - y_1y_3 - b_2$, then $u=0$ and in the new coordinate (41), (35) can be represented as normal form (1). Since (35) is similar to (36), (35) is converted to (42) via the same coordinate transformation.

$$\begin{aligned} \dot{\bar{z}}_1 &= \bar{z}_2 \\ \dot{\bar{z}}_2 &= \bar{z}_3 \\ \dot{\bar{z}}_3 &= (a^2-1)x_1 + a(a^2-2)x_2 + (c_1-a)x_3 - x_1x_3 - b_1 \end{aligned} \quad (42)$$

where x_1, x_2 and x_3 in the third equation of (42) can substitute by $\bar{z}_1$, $\bar{z}_2$ and $\bar{z}_3$. But the control law should be described by a function of x_1, x_2, x_3, y_1, y_2 and y_3, we do not use this substitution. Define $\boldsymbol{e}' = (e'_1, e'_2, \cdots, e'_n)^T = \bar{\boldsymbol{z}} - \boldsymbol{z}$, then

$$\begin{aligned} \dot{e}'_1 &= e'_2 \\ \dot{e}'_2 &= e'_3 \\ \dot{e}'_3 &= (a^2-1)x_1 + a(a^2-2)x_2 + (c_1-a)x_3 - x_1x_3 - b_1 - v \end{aligned} \quad (43)$$

Obviously, choosing

$$\begin{aligned} v = {} & (a^2-1)x_1 + a(a^2-2)x_2 + (c_1-a)x_3 - x_1x_3 - b_1 \\ & + e'_1 + 3e'_2 + 3e'_3 \end{aligned} \quad (44)$$

can make (43) global asymptotically stable. Because the inverse transformation is uniformly continuous, the error dynamic $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{y}(t) \to 0$ when $t \to \infty$. Using (39) and (44), the following control law can realize the synchronization

$$\begin{aligned} u = {} & b_1 - b_2 - (a^2+3a+2)(x_1-y_1) - (a^3+3a^2+a-2)\cdot \\ & (x_2-y_2) - (c_1-a-3)x_3 + (c_2-a-3)y_3 + x_1x_3 - y_1y_3 \end{aligned} \quad (45)$$

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