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Nonlinear interconnected systems described by integro-differential equations

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Abstract— **In this paper, a class of multi-input and multi-output nonlinear systems described by integro-differential equations are studied. Some frequency domain conditions are established for estimation of the period and frequency of periodic solutions of the second kind which is a close cycle in cylindrical phase space for a class input-output interconnected pendulum-like systems. The effects of interconnections are shown through a permutation matrix, and then upper bounds for the frequency of periodic solutions of the second kind can be presented. Numerical examples are given to illustrate the main results of the paper. Examples show that input and output interchange presented here can result in some great changes in some practical systems. Chaotic phenomenon appears in partial variables of pendulum-like systems in Example 1.**

I. INTRODUCTION

The frequency domain method has got a great success on study of systems described by differential equation[1][2][3][4][5][6][7]. Many classical results were established such as Yakubovich-Kalman frequency domain theorem in which the well-known Popov criterion and Circle criterion for absolute stability can be viewed as special cases. In fact, the investigation of asymptotic properties of differential equation may be easily reduced to the investigation of asymptotic properties of systems described by integro-differential equations[1][8][9]. And the aggregate of control systems which can be described by integro-differential equation is much wider and the mathematical description of control systems with time-delay or partial derivatives can often be reduced to such systems[10].

In this paper, we consider the problem of estimation of the frequency of periodic solutions of the second kind for multi-input and multi-output systems described by integro-differential equation with multiple equilibrium points. In fact, the global properties of solutions of systems with multiple equilibrium points are much more complicated than that with single equilibrium point. In the past, most attention focus on the convergence of the solutions. However, there also exist other global properties such as the existence of various kinds of cycles, to describe systems with multiple equilibrium points. The problem of estimation of the frequency of periodic solutions of the second kind was first stated in 1948[11] and solved by a rough harmonic balance method. Furthermore, various asymptotic methods were developed to solve this problem and most of them turned out to be the averaging method[12] and the method of slowly varying energy. All these methods are not rigorous and even may give false results for concrete systems. A rigorous mathematical method based on frequency domain theory for this problem was presented by [1].

The system considered in this paper can be viewed as interconnected system generated by some subsystems through linear and nonlinear interconnections. It is an interesting problem to know what roles the interconnections play in large scale systems[13]. In order to discuss the effects of some kind of nonlinear interconnections, a class of nonlinear input and output intercross is presented for pendulum-like systems described by integro-differential equation. The actions of input and output interconnections are shown through a permutation matrix. Combining with the frequency-domain method developed in [1], the frequency domain conditions for estimation of frequency of a periodic solution of the second kind are given. Because linear matrix inequalities(LMIs) can be solved numerically using efficient optimization algorithms, we convert the frequency-domain conditions into LMIs-based conditions by using Kalman-Yakubovich-Popov Lemma.

Throughout this paper, $A<0$ means that A is a Hermitian and negative definite matrix. The superscript $*$ means transpose for real matrices or conjugate transpose for complex matrices. $\mathrm{Re}\{Y\}$ means $\frac{1}{2}(Y+Y^*)$ for any real or complex square matrix Y.

II. NONLINEAR INTERCONNECTED INTEGRO-DIFFERENTIAL EQUATIONS

A class of integro-differential equations were studied thoroughly in [1],

$$\dot{\sigma}(t)=\alpha(t)+R\varphi(\sigma(t-h))-\int_0^t \gamma(t-\tau)\varphi(\sigma(\tau))d\tau. \quad (1)$$

Here h is a nonnegative number, $\alpha(t)=(\alpha_1(t),\cdots,\alpha_n(t))^*$, $\sigma(t)=(\sigma_1(t),\cdots,\sigma_n(t))^*$, $\varphi(\tau)=(\varphi_1(\tau),\cdots,\varphi_n(\tau))^*$ are n-vector functions with $\sigma_i:[0,+\infty)\to\mathbb{R}, \varphi_i:[0,+\infty)\to\mathbb{R}$, $\gamma(t)=(\gamma_{ij})(i,j=1,\cdots,n)$ is $n\times n$-matrix function with $\gamma_{ij}:[0,+\infty)\to\mathbb{R}$; R is a constant $n\times n$ matrix.

Before giving the main results in this section, the following Kalman-Yakubovich-Popov (KYP) Lemma is needed. Please refer to [6] for this important lemma which establishes the relationship between frequency domain method and time domain method for linear systems.

Lemma 1[6] *Given $A\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times m}, M=M^*\in\mathbb{R}^{(n+m)\times(n+m)}$, with $\det(iwI-A)\neq 0$ for $w\in\mathbb{R}$ and (A,B) is controllable. The following two statements are equivalent:*

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(i) $\begin{bmatrix} (iwI-A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (iwI-A)^{-1}B \\ I \end{bmatrix} \leq 0, \forall w \in \mathbb{R}$.

(ii) there is a real symmetric matrix P such that $M + \begin{bmatrix} PA + A^*P & PB \\ B^*P & 0 \end{bmatrix} \leq 0$.

The corresponding equivalence for strict inequalities holds even if (A,B) is not controllable.

In this paper, we will see what can be resulted in by some nonlinear interconnections. First we consider the following permutation, let P be an $n \times n$ permutation matrix (got by exchanging the columns of unit matrix I_n),

$$(i_1, \cdots, i_n)^* = P(1, \cdots, n)^*, \psi_P(\sigma) = (\varphi_1(\sigma_{i_1}), \cdots, \varphi_n(\sigma_{i_n}))^*. \tag{2}$$

substituting $\varphi(\sigma)$ in (1) by $\psi_P(\sigma)$, we can get a new interconnected system

$$\dot{\sigma}(t) = \alpha(t) + R\psi_P(\sigma(t-h)) - \int_0^t \gamma(t-\tau)\psi_P(\sigma(\tau))d\tau. \tag{3}$$

Similar to [1], we need the following assumptions for φ_i:

(i) $\varphi_i(\tau)$ is Δ-periodic, $i = 1, \cdots, n$;

(ii) $\varphi_i(\tau) \in C^1[0, +\infty)$, and there exist numbers $\mu_i > 0$ such that

$$|d\varphi_i(\tau)/d\tau| < \mu_i < +\infty, \quad \forall \tau \in \mathbb{R}, \quad i = 1, \cdots, n; \tag{4}$$

(iii) $\varphi_i(\tau)$ has two zeros $\tau_i^1 < \tau_i^2$ on $[0, \Delta)$ and

$$[\varphi_i(\tau_i^k)]^2 + [\varphi_i'(\tau_i^k)]^2 \neq 0, \quad k = 1, 2, \quad i = 1, \cdots, n. \tag{5}$$

Furthermore, we assume that $\gamma(t)$ and $\alpha(t)$ acquire the following properties:

(i) $\alpha_i(t)$ is continuous on $[0, +\infty)$ and $\alpha_i(t) \to 0$ as $t \to +\infty$, $i = 1, \cdots, n$;

(ii) there exists $c_1 > 0$ such that

$$\alpha_i e^{c_1 t} \in L^2[0, +\infty), \quad i = 1, \cdots, n; \tag{6}$$

(iii) $\gamma_{ij}(t)$ is measurable and there exists $c_2 > 0$ such that

$$\gamma_{ij} e^{c_2 t} \in L^2[0, +\infty), \quad i, j = 1, \cdots, n; \tag{7}$$

It is easy to demonstrate, as done in [1], that for any initial data

$$\sigma_i = \sigma_i^0(t) \in C^1[-h, 0], \, t \in [-h, 0], \quad (i = 1, \cdots, n) \tag{8}$$

system (3) has a solution which can be continued on any interval $[0, \theta)$ under the assumptions above. The function $\dot{\sigma}(t) (i = 1, \cdots, n)$ is bounded and uniformly continuous on $[0, +\infty)$.

Define the transfer function matrix of the linear part of system (1) from the input φ to the output $\dot{\sigma}$ as follows

$$K(s) = Re^{-sh} - \int_0^\infty e^{-st}\gamma(t)dt \quad (s \in \mathbb{C}, \quad \text{Re}s > -c_2). \tag{9}$$

Definition 1[1] *The solution of (1) or (3) is called periodic solution of the second kind if there exist a number $T \neq 0$ and integers k_i such that*

$$\sigma_i(t+T) = \sigma_i(t) + k_i\Delta, \qquad i = 1, \cdots, n.$$

The number $w = 2\pi/T$ is called the frequency of periodic solution.

Remark 1 Since the periodic solution defined by Definition 1 is a close cycle in cylindrical phase space, we call it a periodic solution of the second kind. Note that the natural condition which must be satisfied by a phase space of a mathematical model of a certain technical system is that every physical state of the system corresponds to one and only one point of the phase space. From this point of view, for a class of nonlinear systems with multiple equilibrium points such as pendulum-like systems, the cylindrical phase space can serve more satisfactory as a phase space than plane one.

Let

$$\begin{aligned} &v_i = \textstyle\int_0^\Delta \varphi_i(\tau)d\tau / (\int_0^\Delta |\varphi_i(\tau)|d\tau), \\ &F_i(\tau) = \varphi_i(\tau) - v_i|\varphi_i(\tau)|, \qquad i = 1, \cdots, n, \end{aligned}$$

it is clear that

$$\int_0^\Delta F_i(\tau)d\tau = 0. \tag{10}$$

Further define $M = diag(\mu_1^2, \cdots, \mu_n^2)$, $N = diag(v_1, \cdots, v_n)$. In order to give the main result, the following Lemma is also needed.

Lemma 2[1] *Suppose that there exist real diagonal matrices $E = diag(\varepsilon_1, \cdots, \varepsilon_n) > 0$, $D = diag(\delta_1, \cdots, \delta_n) > 0$, $L = diag(\tau_1, \cdots, \tau_n) \geq 0$, $G = diag(\kappa_1, \cdots, \kappa_n)$ such that the following requirements are fulfilled:*

(i) $4ED \geq (GN)^2$*;*

(ii) $Re\{GK(0) + K^*(0)(E + LM)K(0) + D\} < 0$;

(iii) there exists a nonnegative number w_0 such that the following inequality is fulfilled

$$\begin{aligned} &Re\{GK(iw) + K^*(iw)(E + LM)K(iw) \\ &-w^2L + D\} < 0, \, \forall w \geq w_0. \end{aligned} \tag{11}$$

Then system (1) has no periodic solutions of the second kind with the frequency $w \geq w_0$.

Then for interconnected system (3) we can get the following theorem.

Theorem 1 *Suppose that there exist real diagonal matrices $E = diag(\varepsilon_1, \cdots, \varepsilon_n) > 0$, $D = diag(\delta_1, \cdots, \delta_n) > 0$, $L = diag(\tau_1, \cdots, \tau_n) \geq 0$, $G = diag(\kappa_1, \cdots, \kappa_n)$ such that the following requirements are fulfilled:*

(i) $4ED \geq (GN)^2$*;*

(ii) $Re\{GPK(0) + K^*(0)P^*(E + LM)PK(0) + D\} < 0$;

(iii) there exists a nonnegative number w_0 such that the following inequality is fulfilled

$$\begin{aligned} &Re\{GPK(iw) + K^*(iw)P^*(E + LM)PK(iw) \\ &-w^2L + D\} < 0, \, \forall w \geq w_0. \end{aligned} \tag{12}$$

Then system (3) has no periodic solutions of the second kind with the frequency $w \geq w_0$.

Proof : let $z = P\sigma$, by equation(3) we have

$$\dot{z}(t) = P\alpha(t) + PR\varphi(z(t-h)) - \int_0^t P\gamma(t-\tau)\varphi(z(\tau))d\tau. \tag{13}$$

where $\varphi(z) = \psi_P(\sigma) = (\varphi_1(\sigma_{i_1}), \cdots, \varphi_n(\sigma_{i_n}))^*$. It's obvious that systems (3) and (13) have the same global properties. Since the transfer function from $\varphi(z)$ to $\dot{z}$ is $PK(s)$, then we can prove the theorem easily. □

Remark 2 System (1) can be viewed as a special case of system (3) with permutation matrix $P = I$. The number of all the permutations in (2) is $n!$. That is, the number of nonlinear combinations in system (3) is $n!$. If $w_0 = 0$ in Theorem 1, system (3) has no any periodic solution.

III. A CLASS OF PENDULUM-LIKE SYSTEMS

Using the method above we can study the following pendulum-like systems.

$$\begin{cases} \frac{dx}{dt} = Ax + B\psi_P(\sigma), \\ \frac{d\sigma}{dt} = Cx + R\psi_P(\sigma), \end{cases} \tag{14}$$

where $A \in \mathbb{R}^{m\times m}, B \in \mathbb{R}^{m\times n}, C \in \mathbb{R}^{n\times m}, R \in \mathbb{R}^{n\times n}$, $\psi_P(\sigma) = (\varphi_1(\sigma_{i_k}), \cdots, \varphi_n(\sigma_{i_n}))^*$. $\varphi_i (i = 1, \cdots, n)$ are periodic functions which satisfy the assumptions given in the section above. P is a permutation matrix given as in (2).

Suppose that A is Hurwitzian. System (14) can be transformed into the following system just like (1) with $\alpha(t) = Ce^{At}x(0), h = 0, \gamma(t) = -Ce^{At}B$.

$$\dot{\sigma}(t) = Ce^{At}x(0) + R\psi_P(\sigma(t)) + \int_0^t Ce^{A(t-\tau)}B\psi_P(\sigma(\tau))d\tau. \tag{15}$$

In this case $K(s) = R - \int_0^\infty e^{-st}\gamma(t)dt = R + C(sI - A)^{-1}B$. Obviously the assumptions in the section above for $\alpha(t), \gamma(t)$ are all satisfied for system (15) under the condition that A is Hurwitzian.

Obviously Theorem 1 can be used to system (14) or (15). In order to use LMIs method to determine the parameters in Theorem 1, we give some sufficient conditions for system (15) having no periodic solutions.

Theorem 3 *Suppose that there exist real diagonal matrices $E = diag(\varepsilon_1, \cdots, \varepsilon_n) > 0$, $D = diag(\delta_1, \cdots, \delta_n) > 0$, $L = diag(\tau_1, \cdots, \tau_n) \geq 0$, $G = diag(\kappa_1, \cdots, \kappa_n)$ and a symmetric matrix X such that the following requirements are fulfilled:*

(i) $\begin{bmatrix} 2E & GN \\ GN & 2D \end{bmatrix} > 0;$

(ii) $Re\{GPK(0) + K^*(0)P^*(E + LM)PK(0) + D\} < 0;$

(iii) there exists a nonnegative number w_0 such that the following inequality is fulfilled

$$M_1 + \begin{bmatrix} XA + A^*X & XB \\ B^*X & 0 \end{bmatrix} < 0,$$

where

$$M_1 = \begin{bmatrix} C^*P^*(E + LM)PC & \Pi_{12} \\ \Pi_{12}^T & \Pi_{22} \end{bmatrix},$$

$$\Pi_{12} = \frac{1}{2}C^*P^*G + C^*P^*(E + LM)PR,$$

$$\Pi_{22} = D + R^*P^*(E + LM)PR + \frac{1}{2}(GPR + R^*P^*G) - w_0^2L$$

Then system (15) has no periodic solutions of the second kind with the frequency $w \geq w_0$.

Proof : By Schur complement, the conditions (i) and (ii) are equivalent to that the following hold:

(i) $4ED \geq (GN)^2$;

(ii) $\mathrm{Re}\{GPK(0) + K^*(0)P^*(E + LM)PK(0) + D\} < 0$.

By Lemma 1, condition (iii) of the present theorem is equivalent to that there exists a nonnegative number w_0 such that the following inequality is fulfilled

$$\mathrm{Re}\{GPK(iw) + K^*(iw)P^*(E + LM)PK(iw) \\ -w_0^2L + D\} < 0, \ \forall w \in \mathbb{R}.$$

Above inequality can derive the following inequality

$$\mathrm{Re}\{GPK(iw) + K^*(iw)P^*(E + LM)PK(iw) \\ -w^2L + D\} < 0, \ \forall w \geq w_0.$$

Then by Theorem 1, the conclusion holds obviously. □

Remark 3 Similarly we can get some sufficient conditions expressed in LMIs according to Theorem 2. By Theorem 3, for any given w, parameter matrices L, D, G can be got easily using LMI toolbox of MATLAB[14]. If $w_0 = 0$ in Theorem 3, system (15) has no periodic solution. In this case L can be taken as a zero matrix and condition (ii) is implied by condition (iii) in Theorem 3.

IV. EXAMPLES

Example 1. We consider the pendulum-like system defined in (14) with $A = \begin{bmatrix} -0.4 & 3 \\ -1 & -0.5 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -1.4 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$, $R = \begin{bmatrix} -2.3 & 1.2 \\ -2 & 1 \end{bmatrix}$, $\sigma = [\ \sigma_1 \quad \sigma_2\ ]^*$, $\varphi_1(\tau) = sin(\tau) - 0.2$, $\varphi_2(\tau) = sin(2\tau) - 0.1$, $\psi_p(\sigma) = [\ \varphi_1(\sigma_2) \quad \varphi_2(\sigma_1)\ ]^*$ and $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It is easy to test that there exists a number $w_0 = 2.7\pi$ such that the conditions of Theorem 3 are satisfied. According to the theorem, system (15) has no periodic solutions of the second kind with the frequency $w \geq 2.7\pi$. Referring to Fig.1 for the solutions σ of system (15) with three given initial values respectively, it shows that these solutions σ are convergent to a certain periodic solution. So, it is possible for the existence of some periodic solutions(in this case, $k_i = 0$ in Definition 1). Of course, periodic solutions can be viewed as special cases of periodic solutions of the second kind. The solution σ of system (15) with initial value $x(0) = [1, -0.5]^*$ and $\sigma(0) = [2, -2]^*$ is shown in Fig.2. From Fig.2 we can estimate the frequency of periodic solution satisfying $w < 2.7\pi$. This illustrative result coincides with the theorem given above. As shown in Fig.3 and Fig.4, the state variable x with initial value $x(0) = [1, -0.5]^*$ and $\sigma(0) = [2, -2]^*$ is also convergent to a certain periodic orbit. In order to show the variations of the solutions of system (15) clearly, we draw them in plane phase space rather than in cylindrical phase space.

In what follows, we see what will happen if interconnection in system (15) is disappeared. Let us substitute $\psi_p(\sigma)$ in (15) by $\varphi(\sigma) = [\ \varphi_1(\sigma_1) \quad \varphi_2(\sigma_2)\ ]^*$. Referring to Fig.5, it shows that the solution σ with the initial value

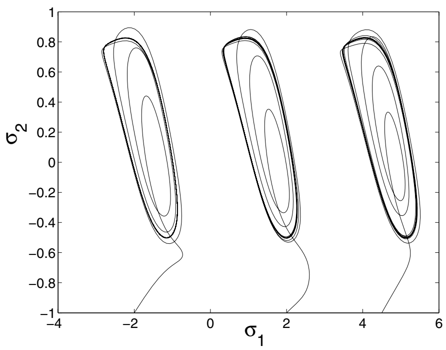

Fig. 1. The phase portrait of solution σ

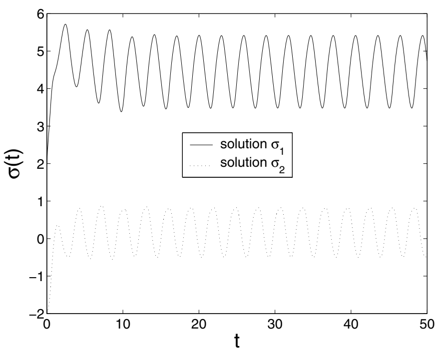

Fig. 2. The curve of the solution σ for system (15)

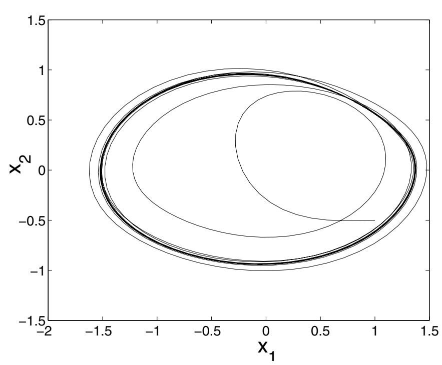

Fig. 3. The phase portrait of state variable x

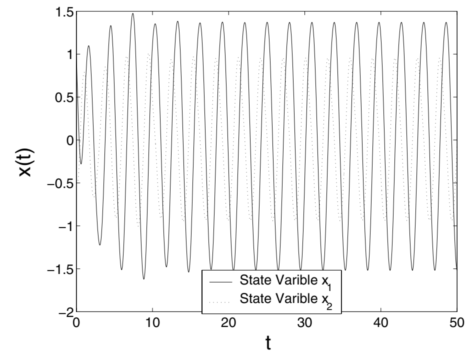

Fig. 4. The curve of the state variable x for system (15)

$x(0) = [1, -0.5]^*$ and $\sigma(0) = [2, -2]^*$ is unbounded and nonperiodic. However, as shown in Fig.6 and Fig.7, the state variable x appears to be a kind of chaotic phenomenon as $t \to +\infty$. Since the solution σ is unbounded, there is no chaotic phenomenon in plane phase space. However, chaotic phenomenon appears on the cylindrical surface of cylindrical phase space, here we call it the chaos on cylindrical surface, which was not considered by [1].

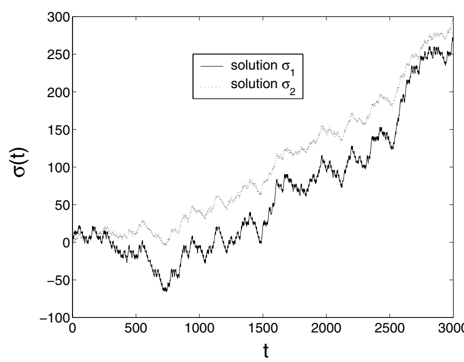

Fig. 5. The curve of the solution σ for (15) without interconnection

Example 2. Let us consider the pendulum-like system defined in (14) with $A = \begin{bmatrix} -3 & 1 & 1 \\ 1.5 & -1 & 0.5 \\ 0 & -2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & -0.6 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \end{bmatrix}$, $R = \begin{bmatrix} 2 & -2.5 \\ -5 & -3 \end{bmatrix}$, $\sigma = \begin{bmatrix} \sigma_1 & \sigma_2 \end{bmatrix}^*$, $\varphi_1(\tau) = sin(\tau) - 0.1$, $\varphi_2(\tau) = sin(2\tau) - 0.2$, $\psi_p(\sigma) = \begin{bmatrix} \varphi_1(\sigma_2) & \varphi_2(\sigma_1) \end{bmatrix}^*$ and $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It is also easy to test that there exists a number $w_0 = 5.6\pi$ such that the conditions of Theorem 3 are satisfied. According

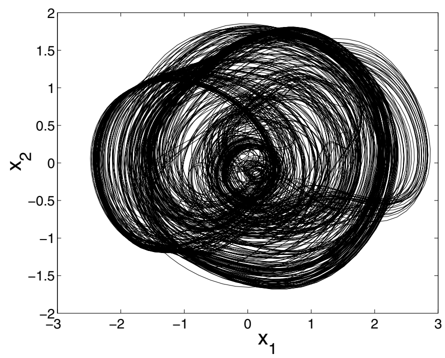

Fig. 6. The phase portrait of x for (15) without interconnection

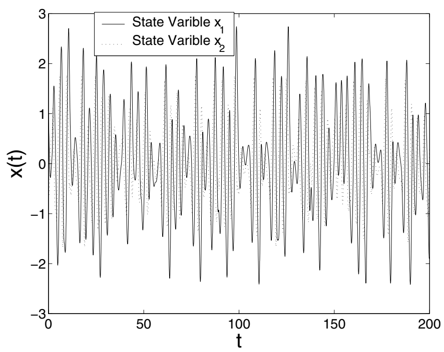

Fig. 7. The curve of the state variable x for (15) without interconnection

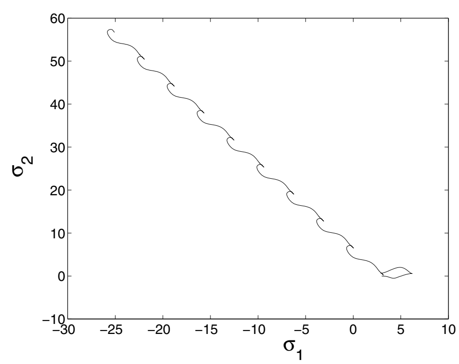

Fig. 8. The phase portrait of solution σ

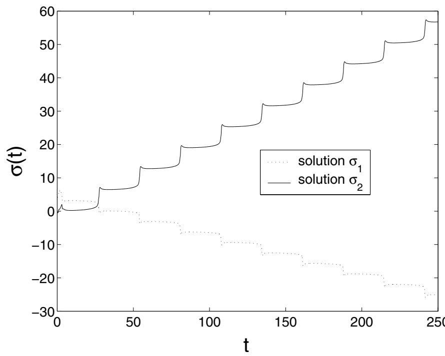

Fig. 9. The curve of the solution σ for system (15)

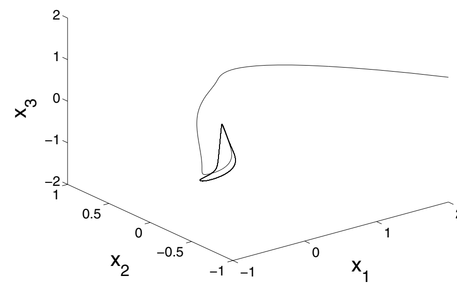

Fig. 10. The phase portrait of state variable x

to the theorem, system (15) has no periodic solutions of the second kind with the frequency $w \geq 5.6\pi$. Referring to Fig.8 for the solution σ of system (15) with initial value $x(0) = [2,-1,1]^*$ and $\sigma(0) = [3,0]^*$, it shows that it is possible for the existence of some periodic solutions of the second kind (in this case, $k_i \neq 0$ in Definition 1). From Fig.9 we can estimate the frequency of periodic solutions of the second kind satisfying $w < 5.6\pi$. As shown in Fig.10 and Fig.11, the state variable x with initial value $x(0) = [2,-1,1]^*$ and $\sigma(0) = [3,0]^*$ is also convergent to a certain periodic orbit.

Let us substitute $\psi_p(\sigma)$ in (15) by $\varphi(\sigma) = [\ \varphi_1(\sigma_1)\ \ \varphi_2(\sigma_2)\ ]^*$, then interconnection in system (15) is disappeared. Referring to Fig.12 and Fig.13, it shows that it is also possible for the existence of some periodic solutions of the second kind. Compare Fig.13 with Fig.9, it shows that the frequency of periodic solutions of the second kind increases after interconnection disappeared.

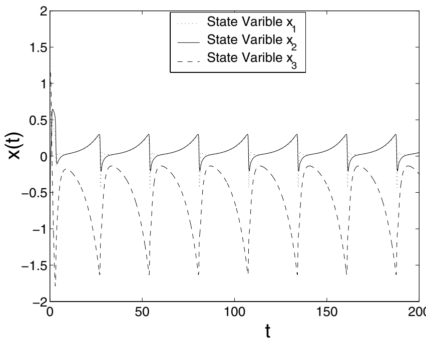

Fig. 11. The curve of the state variable x for system (15)

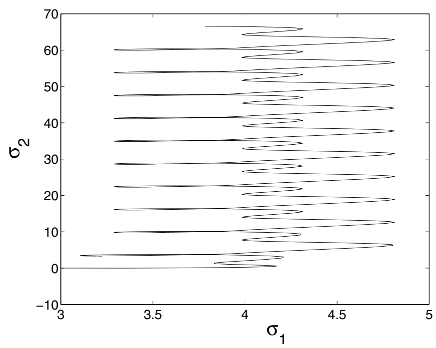

Fig. 12. The phase portrait of σ for system (15) without interconnection

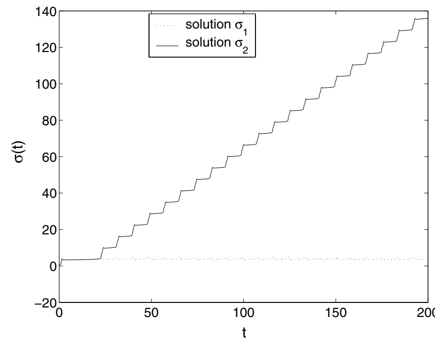

Fig. 13. The curve of σ for system (15) without interconnection

V. CONCLUSIONS

In this paper, some frequency domain conditions are established for estimation of the frequency of periodic solutions of the second kind in nonlinear interconnected systems described by integro-differential equations. In practical systems such as Phase-Locked Loop systems, there is the engineering requirement to study the relationship between parameters of systems and the existence of the cycle of the second kind. By the results given here, upper bounds for the frequency of periodic solutions of the second kind can be presented. The effects of interconnections are shown through a permutation matrix. Examples show that input and output interchange presented here can result in some great changes in some practical systems. In addition, in the example 1, chaotic phenomenon appears in partial variables and the other variables are unbounded, which can be viewed as the chaos on cylindrical surface. It shows the complexity of physical property in concrete systems. This also indicates that it is possible for the existence of chaotic attractors in pendulum-like systems. Chaotic solutions of pendulum-like systems were hardly studied by now.

VI. ACKNOWLEDGMENTS

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