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HIGH-GAIN CONTROL OF NONLINEAR MINIMUM-PHASE SYSTEMS

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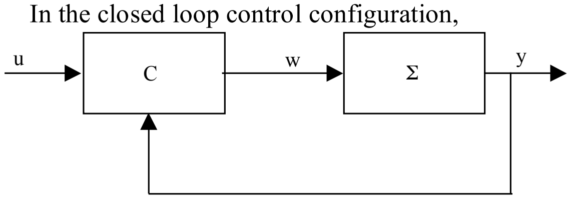
Abstract **— The classical design principle of using high forward gain to improve performance and accuracy of linear control systems is examined in a nonlinear context. By introducing a certain nonlinear gain function into the control loop, this principle is extended to nonlinear minimum-phase systems. The results include a solution to a class of nonlinear approximate model matching problems.**

I. INTRODUCTION

In the closed loop control configuration,



(1)

Σ is the system being controlled, C is a controller, and Σ_c is the system represented by the closed loop. For notational convenience, assume that C is constructed so that Σ_c has the same input space as Σ.

Stability of the Configuration 1 is, of course, critically important. To investigate stability, we limit our attention to bounded input signals u. We define the *bounded-input image* $\mathrm{Im}_b\,\Sigma_c$ as the set of output signals of Σ_c generated by bounded input signals. Similarly, the bounded-input image $\mathrm{Im}_b\,\Sigma$ of Σ is formed by all responses of Σ to bounded input sequences. As the output of Σ_c is the output of Σ, one has

$\mathrm{Im}_b\,\Sigma_c \subset \mathrm{Im}_b \subset$

Assume that a norm $|\bullet|$ is defined over our spaces, and consider the following problem.

(2) DEFINITION. THE APPROXIMATE MODEL MATCHING PROBLEM. Let Φ and Φ be two systems with a common input space and a common output space. Given a bounded domain S and a real number $\varepsilon > 0$, determine whether there is a controller C such that

$|\varepsilon\, u - \varepsilon_c u| \le \varepsilon$ for all $u \in S$. (3)

If such a controller exists, then $\in_c$ is an *ε-approximant* of $\in$ over S, and $\in$ is the *model*. ♦

When the model is the identity system ♦ = I, the approximate model matching problem reduces to the classical problem of designing a tracking system, i.e., a system whose output closely follows its input.

In the present note, we consider the approximate model matching problem for nonlinear minimum phase systems. We show that there is a simple solution based on the use of high forward gain. The solution is easy to visualize and to implement, and it generalizes to nonlinear systems the classical control principle of using high forward gain.

Alternative approaches to the control of nonlinear systems can be found in [11], [12], [6], [7], [8], [9], [4], [19], [18], [3], [15], [20], [17], [16], [1], [5], [13], and in other publications.

II. BASIC CONSIDERATIONS

A. Preliminaries

We consider here discrete-time systems, but similar principles apply to continuous-time systems as well. Let R be the set of real numbers, let R^m be the set of all m-dimensional real vectors, and let $S(R^m)$ be the set of all sequences $u = \{u_0, u_1, u_2, ...\}$ of m-dimensional real vectors, where u_i I R^m, $i = 0, 1, 2, ...$ A system with specified initial conditions induces a map I : $S(R^m) \to S(R^p)$ that transforms input sequences of m dimensional real vectors into output sequences of p dimensional real vectors. We write $y = \to u$ to represent the output sequence y generated by the input sequence u. It will be convenient to assume that $\to 0 = 0$.

As usual, a system $\to$ is *causal* if its present response does not depend on future input values. The system is *strictly causal* if there is a delay of at least one step before input changes are reflected in its response. Finally, the system $\to$ is *bicausal* if it is invertible, and if $\to$ and its inverse $\to^{1}$ are both causal systems.

We consider systems with a *state representation*

$$x_{k+1} = f(x_k, u_k)$$
$$y_k = h(x_k),\ k = 0, 1, 2, ... \qquad (4)$$

Here, $x_k \to R^n$ is the *state* of the system at step k, while u_k and y_k are the input value and the output value, respectively, at that step; $f : R^n \times R^m \times R^n$ is the *recursion function* and $h : R^n \times R^p$ is the *output function*. For convenience, we use the initial condition $x_0 = 0$ for our system. A system described by (4) is strictly causal, since the output function h does not depend on the input value u_k (e.g., [6]).

For a real number a, let $|a|$ be the absolute value of a.

Given a vector $r = (r^1, r^2, ..., r^q) \in R^q$, set
$|r| := \max \{|r^i|, i = 1, ..., q\}$.
The ℓ^∞-norm of an element $s \infty S(R^q)$ is given by
$|s| := \sup_{i\geq 0} |s_i|$,
where $|s| := \infty$ when the supremum does not exist. A subset $S \subset S(R^q)$ is *bounded* if there is a real number $M \geq 0$ such that $|s| \leq M$ for all $s \subset S$; when the latter holds, we write $|S| \leq M$. For a real number $\theta \leq 0$, denote by $S(\theta^q)$ the set of all sequences $s\ \theta\ S(R^q)$ satisfying $|s| \leq \theta$, i.e., the set of all sequences bounded by θ.

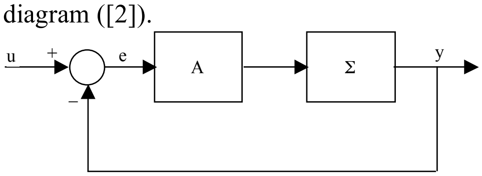
A system $\Sigma : S(R^m) \to S(R^p)$ is *BIBO-stable* (Bounded-Input Bounded-Output stable) if, for every real number $M \leq 0$, there is a real number $N \leq 0$ such that $|\to u| \leq N$ whenever $|u| \leq M$. The notion of BIBO-stability underlies all other stability notions.

When a system $\to: S(R^p) \to S(R^p)$ is invertible as a set mapping, its inverse is $\to^1$, and $\to^1\to = \to\to^1 = I$, the identity. If $\to^1$ is BIBO-stable, then $\to$ is a *BIBO-minimum phase system*. A system $\to$ is *BIBO-unimodular* if it is both BIBO-stable and BIBO-minimum phase.

For composite systems, a stronger notion of stability is required. Consider a composite system Ψ that consists of q subsystems. Add an external signal to the output of each subsystem. This results in a composite system with $q+1$ external input signals - the original input signal and the q newly added signals. The composite system Ψ is *internally BIBO-stable* if the following holds for each one of the $(q+1)$ external input signals: the map from the external signal to any signal within the configuration forms a BIBO-stable system. Internal BIBO-stability guaranties that a composite system is implementable.

B. Ideas from classical control theory

The classical solution of the tracking problem is based on the use of high forward gain in the following Black diagram ([2]).



(5)

Here, Ψ is the system being controlled and A represents a constant gain. The use of unity feedback requires that the number of outputs of Ψ be equal to its number of inputs, say $\Psi : S(R^m)\ \Psi\ S(R^m)$ ($m = 1$ is used in classical control). Let Ψ_A denote the input/output relation of the closed loop (5). A simple calculation yields

$y = \Psi[(1/A)I + \Psi]^{-1}u$, or $\Psi_A = \Psi[(1/A)I + \Psi]^{-1}$. (6)

Note that (6) remains valid when Ψ is a nonlinear system. Ignoring for a moment mathematical rigor (sections 3 and 4 present a rigorous discussion), one may presume that

$\lim_{A\Psi\Psi} [(1/A)I + \Psi] = \Psi$. (7)

If (7) is accepted as correct and substituted into (6), and if Ψ is continuous and invertible, one arrives at the conclusion

$\lim_{A\Psi\Psi} \Psi[(1/A)I + \Psi]^{-1} = \Psi\Psi^{-1} = I$. (8)

In other words, when the gain A is sufficiently large, then
$y \approx u$,
i.e., for large gain A, Configuration 5 becomes an accurate tracking system irrespective of the nature of $\approx$, as long as $\approx$ is invertible and strictly causal.

This conclusion is, in general, incorrect. A brief examination of (8) reveals a major difficulty in case $\approx$ is not a BIBO-minimum phase system. Indeed, the expression $\approx\approx^{-1}$ implies that, for large gain A, the input signal of $\approx$ in (5) is (almost) equal to the output signal of $\approx^{-1}$. When $\approx$ is not BIBO-minimum phase, this means that the input of $\approx$ in (5) is unbounded for at least some input signals u, invalidating the internal stability of the configuration. This qualitative argument indicates that Configuration 5 cannot be used with large gain A when $\approx$ is not BIBO-minimum phase. We show below that, with a small modification, the Black diagram is an effective tracking configuration for all linear and nonlinear BIBO-minimum phase systems.

III. SUBBOUNDED SYSTEMS

Let R^+ be the set of all non-negative real numbers. A *bound function* is a strictly increasing continuous function $\alpha : R^+\ \alpha\ R$, whose image includes R^+. Note that the restriction $\alpha : R^+\ \alpha\ \mathrm{Im}\,\alpha$ is an isomorphism with an inverse $\alpha^{-1} : \mathrm{Im}\,\alpha\ \alpha\ R^+$. Examples of bound functions include the functions $\alpha(\alpha) = a\alpha$, where a is a positive constant; $\alpha(\alpha) = a\alpha^2$; or $\alpha(\alpha) = a\alpha^{1/2}$; or, more generally, $\alpha(\alpha) = a\alpha^b$, where $a, b > 0$ are constants.

Of course, bound functions may take other forms as well. The following is the basic concept of this section.

(11) DEFINITION. A system $\alpha : S(R^m)\ \alpha\ S(R^m)$ is a *sub-bounded system* if there is a bound function α satisfying $|\alpha w| \leq \alpha(|w|)$ for all $w\ \alpha\ S(R^m)$. The function α is then called a *lower bound function* of α. ♦

The next statement indicates that (invertible) sub-bounded systems are BIBO-minimum phase systems.

(12) THEOREM. Let ♦ : $S(R^m)$ ♦ $S(R^m)$ be a BIBO-stable invertible system satisfying ♦0 = 0. Then, ♦ is a BIBO-minimum phase system if and only if it is sub-bounded.

Proof. Assume first that ♦ : $S(R^m)$ ♦ $S(R^m)$ is a sub-bounded invertible system, so there is a bound function ♦ satisfying |♦u| ≤ ♦(|u|) for all u ♦ $S(R^m)$. Now, consider a sequence w ♦ $S(R^m)$, and let u := ♦$^{-1}$w. The previous inequality then yields |♦♦$^{-1}$w| ≤ ♦(|♦$^{-1}$w|), or

|w| ≤ ♦(|♦$^{-1}$w|). (13)

Further, the fact that ♦ is strictly increasing implies that ♦$^{-1}$ is also strictly increasing. Consequently, applying ♦$^{-1}$ to both sides of (13) yields ♦$^{-1}$(|w|) ≤ ♦$^{-1}$♦(|♦$^{-1}$w|), so that

$\alpha^{-1}(|w|) \geq |\Sigma^{-1}w|$. This shows that $|\Sigma^{-1}w|$ is bounded whenever $|w|$ is bounded, namely, that Σ is a BIBO-minimum phase system. The converse direction is proved in [10]. ♦

It is easy to show that every linear BIBO-minimum phase system $♦ : S(R^m) \to S(R^m)$ satisfies

$$|\to w| \geq \frac{1}{|\to^{-1}|}|w| \text{ for all } w \in S(R^m), \tag{14}$$

where $|\in^{-1}|$ denotes the norm of the inverse system. Consequently, $\in$ has the lower bound function

$$\in(\theta) := \frac{1}{|\theta^{-1}|}\theta. \tag{15}$$

IV. NONLINEAR MINIMUM PHASE SYSTEMS

A. Tracking

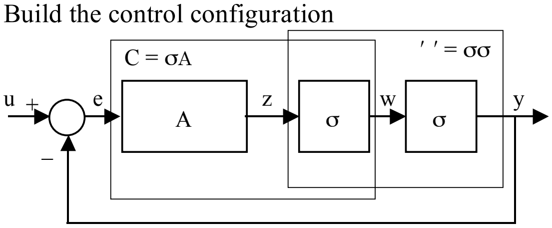
Let $\theta : S(R^m)\ \theta\ \ S(R^m)$ be a BIBO-stable and BIBO-minimum phase system. In view of Theorem 12, θ has a lower bound function θ. We define a new function $\gamma : R^+\ \gamma\ \ R^+$, called the *lower gain function* of γ, as follows. Select a real number $\gamma_0 > 0$ for which $\gamma(\gamma_0) > 0$, and set

$$\gamma(\gamma) := \begin{cases} \lceil(\lceil)/\lceil & \text{for } \lceil \geq \lceil_0, \\ \lceil(\lceil_0)/\lceil_0 & \text{for } 0 \leq \lceil < \lceil_0. \end{cases} \tag{16}$$

Note that the lower gain function is not unique, as it depends on the lower bound function $\lceil$ and on the number $\lceil_0$. For example, in the case of a linear BIBO-minimum phase system, it follows by (15) that a lower gain function is given by the constant function

$$\lceil(\lceil) = 1/|\lceil^{-1}|. \tag{17}$$

Consider now a strictly causal BIBO-stable and BIBO-minimum phase system $\lceil : S(R^m)\ \lceil\ \ S(R^m)$ having the lower bound function $\lceil$ and a lower gain function $\lceil$. Build the control configuration



(18)

where $A > 0$ represents a constant gain, and where σ represents a static controller given by

$$w_k = \sigma(z_k) := \left[\frac{1}{\rceil(\lceil^{-1}(|z_k|))}\right] z_k,\ k = 0, 1, 2, \ldots \tag{19}$$

(For unstable systems, the block "$\rceil$" in (18) must be replaced by a closed loop configuration stabilizing $\rceil$; see [9].)

In the special case when $\rceil$ is a linear system, it follows by (17) that $\rceil$ is simply a constant gain controller, in which case it can be combined with the constant gain controller A and eliminated from the diagram. However, when $\rceil$ is a nonlinear system, the compensator $\rceil$ may not represent constant gain. We start our investigation of $\rceil$ by examining its stability properties, showing that it is BIBO-unimodular. This will require the following auxiliary technical result. (Note that, by definition, a bound function $\rceil : R^+ \rceil\ \ R$ satisfies $R^+ \subset \operatorname{Im} \subset$.)

(20) LEMMA. Let $\subset : R^+ \subset\ R$ be a bound function and let $\mathbb{C}(\subset)$ be a lower gain function of the form (16). Then, the following are true.

(i) For every vector $s \subset R^m$, there is a unique vector $t \subset R^m$ satisfying the relation $s = t\mathbb{C}(|t|)$.

(ii) $s = t\mathbb{C}(|t|)$ if and only if $t = s/\mathbb{C}(\subset^{-1}(|s|))$.

(iii) For a sequence $w = \{w_0, w_1, w_2, \ldots\} \subset S(R^m)$, set $z_k := w_k\mathbb{C}(|w_k|)$, $k = 0, 1, 2, \ldots$ Then, the ℓ^∞-norms satisfy $|z| = |w|\propto(|w|)$.

Proof. (i) Let $t_1, t_2 \infty R^m$ be two vectors satisfying $t_1\propto(|t_1|) = t_2\propto(|t_2|)$. Calculating norms on both sides, we get $|t_1|\propto(|t_1|) = |t_2|\propto(|t_2|)$. We consider now several cases. (a) $|t_1|, |t_2| < \infty_0$, where ∞_0 is from (16); then, $\propto(|t_1|) = \propto(|t_2|) = \infty(\infty_0)/\infty_0$, so the equality $t_1\propto(|t_1|) = t_2\propto(|t_2|)$ clearly implies that $t_1 = t_2$. (b) $|t_1| < \infty_0$ while $t_2 \leq \infty_0$; then, $|t_1|\propto(|t_1|) = |t_1|\infty(\infty_0)/\infty_0$, while $|t_2|\propto(|t_2|) = \infty(|t_2|)$. Now, since $|t_1| < \infty_0$, it follows that $|t_1|\propto(|t_1|) = |t_1|\infty(\infty_0)/\infty_0 < \infty(\infty_0) \leq \infty(|t_2|)$, where the last inequality follows from the relation $\infty_0 \leq |t_2|$. Thus, $|t_1|\propto(|t_1|) \neq |t_2|\propto(|t_2|)$, so that $t_1\propto(|t_1|) \neq t_2\propto(|t_2|)$, and case (b) is not possible under our assumption. (c) $|t_1|, |t_2| \neq \infty_0$; then, by (16), the equality $|t_1|\propto(|t_1|) = |t_2|\propto(|t_2|)$ implies that $\infty(|t_1|) = \infty(|t_2|)$. From the invertibility of ∞, we conclude that $|t_1| = |t_2|$, so that $\propto(|t_1|) = \propto(|t_2|)$. The equality $t_1\propto(|t_1|) = t_2\propto(|t_2|)$ implies then that $t_1 = t_2$, proving part (i).

Turning to part (ii), assume that $s = t\propto(|t|)$ and $|t| < \infty_0$. Then, $s = t\infty(\infty_0)/\infty_0$, so that $|s| = |t|\infty(\infty_0)/\infty_0 < \infty(\infty_0)$. Using the fact that ∞ is strictly increasing, the last inequality implies that $\infty^{-1}(|s|) < \infty_0$. Thus, $s/\propto(\infty^{-1}(|s|)) = s(\infty_0/\infty(\infty_0)) = t$, so that (ii) is valid when $|t| < \infty_0$.

Next, assume that $s = t\propto(|t|)$ and $|t| \neq \infty_0$. Then, calculating norms on both sides, we obtain $|s| = |t\propto(|t|)| = |t|\propto(|t|) = \infty(|t|)$, according to (16). Consequently, $|t| = \infty^{-1}(|s|)$; substituting into the equation $s = t\propto(|t|)$, and using the fact that $\propto(\infty) > 0$ for all $\infty > \infty_0$, we can write $t = s/\propto(|t|) = s/\propto(\infty^{-1}(|s|))$. This proves one direction of part (ii).

For the converse direction of part (ii), assume that

$$t = s/\propto(\infty^{-1}(|s|)). \tag{21}$$

Consider first the case $\infty^{-1}(|s|) < \infty_0$. Then, since ∞^{-1} is strictly increasing, it follows that $|s| < \infty(\infty_0)$. Consequently, in this case, $|t| = (\infty_0/\infty(\infty_0))|s| < \infty_0$, so that $\propto(|t|) = (\infty(\infty_0)/\infty_0)$. The equation $s = (\infty(\infty_0)/\infty_0)t$, which follows directly from (21), implies then that $s = t\propto(|t|)$. This proves (ii) when $\infty^{-1}(|s|) < \infty_0$.

In continuation, assume that $\infty^{-1}(|s|) \neq \infty_0$. Then, $|t| = |s|/\propto(\infty^{-1}(|s|)) = |s|\infty^{-1}(|s|)/[\infty(\infty^{-1}(|s|))] = \infty^{-1}(|s|)$, so that $|t| = \infty^{-1}(|s|)$, or $|s| = \infty(|t|)$. Substituting into (21), we obtain $t = s/\propto(\infty^{-1}(\infty(|t|))) = s/\propto(|t|)$, or $s = t\propto(|t|)$, which completes the proof of (ii).

Finally, regarding part (iii), we again distinguish between two possibilities. First, if $|w_k| < \theta_0$ for all k, then $|w| \leq \theta_0$, and we obtain $z_k = (\alpha(\alpha_0)/\alpha_0)w_k$ for all k, which implies that $|z| = (\alpha(\alpha_0)/\alpha_0)|w| = |w|\gamma(|w|)$. Next, let K be the set of all integers k for which $|w_k| \geq \gamma_0$, and assume that $K \neq \emptyset$. Then, $|w| \neq \emptyset_0$, and, for $k \in K$, it follows by (16) that $|z_k| = |w_k| \in (|w_k|) = \in(|w_k|)$, $k \in K$.

Now, fix an integer $k \in K$, and consider the positive real numbers $\in(|w_0|)$, $\in(|w_1|)$, ..., $\in(|w_k|)$. Let $\in(|w_j|)$ be the largest of these numbers, that is, $\in(|w_j|) \neq \in(|w_i|)$ for all i = 0, ..., k. The fact that $\in$ is a monotone strictly increasing function implies that $|w_j| \neq |w_i|$ for all i = 0, ..., k, so that $|w_j| \neq \in_0$. These facts lead to the following chain of equalities

$|z_0^k| = \max_{i=0,...,k} |z_i| = \max_{i=0,...,k} \in(|w_i|) = \in(|w_j|) = \in(\max_{i=0,...,k} |w_i|) = \in|w_0^k|)$ for all $k \in K$.

Thus, $|z| = \in(|w|)$, and our proof concludes. ♦

(22) LEMMA. The controller σ of (19) is BIBO-unimodular.

Proof. Combining (19) with part (ii) of Lemma 20 yields $z_k = w_k\alpha(|w_k|)$, k = 0, 1, 2, ... Then, using part (iii) of Lemma 20, yields $|z_k| = \sigma(|w_k|)$ for all k = 0, 1, 2, ... By the definition of σ, this implies that the sequence w is bounded if and only if the sequence z is bounded, and the proof is complete. σ

Defining the system

Σ′ := ′ ′ ,

we have ′ ′ A = ′ ′A. In other words, instead of controlling the system ′ with the controller ′ A, we can control the system ′ ′ with the constant gain controller A. The interest in this interpretation arises from the fact that the combination ′ ′ = ′ ′ has the following feature, which is critical to "high-gain" tracking.

(23) DEFINITION. A BIBO-minimum phase system ′ : $S(R^m) \rightarrow S(R^m)$ is *linearly sub-bounded* if there are constants $c > 0$ and $d \neq 0$ such that $|\rightarrow z| \neq c|z|$ for all bounded input sequences satisfying $|z| \neq d$. →

In view of (14), every linear minimum phase system is also linearly sub-bounded. In general, however, a nonlinear BIBO-minimum phase system may not be linearly sub-bounded. We proceed to show that the Black diagram 5 achieves accurate tracking for all linearly sub-bounded systems, as long as the gain A is sufficiently large. For a system → that is sub-bounded, but not linearly sub-bounded, we show that the combination →→:= →→ is linearly sub-bounded. Thus, accurate tracking can be achieved with → by adding the compensator → to the Black diagram.

(24) PROPOSITION. Let →: $S(R^m) \rightarrow S(R^m)$ be a BIBO-stable and sub-bounded system with a lower gain function →, and let →: $S(R^m) \rightarrow S(R^m)$ be given by (19). Then, the composition →→:= →→ is internally BIBO-stable and linearly sub-bounded.

Proof. Lemma 22 implies that the combination →→= →→ is internally BIBO-stable. Thus, it only remains to show that →→ is a linearly sub-bounded system. Referring to Configuration 18, note that z is the input signal and w is the output signal of →, i.e., w = →z. In addition, w is also the input signal of →. Using Lemma 20, we can write $z_k = w_k$→$(|w_k|)$, k = 0, 1, 2, ... and $|z| = |w|$→$(|w|)$.

Letting → be the lower bound function of → corresponding to →, and recalling definition (16) of the gain function, this yields $|z| =$ →$(|w|)$ for all $|w| \neq$ →$_0$. Consequently, $|$→→$z| = |$→→$z| = |$→$w| \neq$ →$(|w|) = |z|$ for all $|w| \neq$ →$_0$. We can then write

$|$→→$z| \neq |z|$ for all $|z| \neq$ →(→$_0$). (25)

Whence, →→ is linearly sub-bounded. →

The following notions refer to stability properties that are preserved when a design parameter grows to infinity.

(26) DEFINITION. Let $\Psi(A) : S(R^m)\ \Psi\ S(R^p) : u \mapsto \Psi(A)u$ be a system that depends on a real parameter A. The system Ψ(A) is *uniformly BIBO-stable* if there is a real number A_0 such that the following is true for all $A \neq A_0$: for every real number $M \neq 0$, there is a real number $N \neq 0$ such that $|\Psi(A)u| \neq N$ for all input sequences of norm $|u| \neq M$.

The system Ψ(A) is *uniformly BIBO-minimum phase* if there is a real number B_0 such that Ψ(A) is invertible and $\Psi^{-1}(A)$ is uniformly BIBO-stable for all $A \neq B_0$.

Finally, the system Ψ(A) is *uniformly BIBO unimodular* if it is both uniformly BIBO-stable and uniformly BIBO-minimum phase. Ψ

(27) DEFINITION. Let Φ(A) be a composite system composed of subunits $\Phi^1(A)$, ..., $\Phi^q(A)$ that depend on a parameter A. Insert an adder at the output of each subunit, and add an external signal u^i to the output sequence of $\Phi^i(A)$, i = 1, ..., q. Denote by u^0 the input sequence of the composite system. For a given value of the parameter A, let $v^0(A)$ be the output sequence of the composite system, and let $v^i(A)$ be the output sequence of the subunit $\Phi^i(A)$, i = 1, ..., q. The composite system Φ(A) is *uniformly internally BIBO-stable* if there is a real number A_0 such that the following is true for all $A \neq A_0$: for every real number $M > 0$ there is a real number $N > 0$ such that $|v^i(A)| \neq N$ for all i = 0, ..., q, whenever $|u^i| \neq M$ for all i = 0, ..., q. Φ

Applying these notions to the closed loop configuration (18), we show subsequently that perfect tracking is achieved at the limit A Φ ∞, without disturbing internal stability. To this end, we start with the following.

(28) PROPOSITION. Let $\infty : S(R^m)\ \infty\ S(R^m)$ be a strictly causal, BIBO-stable, and BIBO-minimum phase system with lower bound function ∞ and lower gain function ∞, and let ∞ be given by (19). Then, Configuration 18 is

uniformly internally BIBO-stable.

Proof. Denote $\Sigma' := '\sigma$, where σ is given by (19). Using the notation of (18), we obtain $e = u - y$, $z = Ae$, $y = \sigma\alpha z$, so that

$u = [(1/A)I + \sigma\alpha]z$. (29)

Now, define the system $\Xi := [(1/A)I + \Xi\Xi] : S(R^m) \rightarrow S(R^m)$, so

$\rightarrow z = [(1/A)I + \rightarrow]z = u$, or (30)

$z = \rightarrow^{-1}u = [1/A)I + \rightarrow]^{-1}u$. (31)

In Configuration 18, we then have

$y = \rightarrow z = \rightarrow[1/A)I + \rightarrow]^{-1}u$.

From (29), we get

$|u| = |\rightarrow z| = |(1/A)z + \rightarrow z| \leq |z|/A + |\rightarrow z|$. (32)

Now, fix a real number $\theta > 0$, and consider a sequence $z \in S(\in^m)$. As $\in\in$ is BIBO-stable by Lemma 22, there is a real number $D \geq 0$ such that $|\in z| \geq D$. Then, by (32),

$|u| \geq \in/A + D \geq \in + D$

for all $A \geq 1$. This shows $\in$ is uniformly BIBO-stable.

Next, we show that $\in$ is uniformly BIBO-minimum phase. Rewriting (30) in the form

$u = \in z = z/A + \in z$,

we have

$|u| = |\in z| = |z/A + \in z| \geq \big||\in z| - |z|/A\big|$. (33)

Now, by (25), we have $|\in z| \geq |z|$ for all $|z| \geq \alpha(\alpha_0)$. Consequently, when $A > 1$ and $|z| \geq \alpha(\alpha_0)$, we can write $\big||\alpha z| - |z|/A\big| \geq |z| - |z|/A = (1 - 1/A)|z|$. Recalling (33), $|u| \geq (1 - 1/A)|z|$ for all $A > 1$ and $|z| \geq \alpha(\alpha_0)$. Thus,

$|z| \geq 2|u|$ for all $A \geq 2$ and $|z| \geq \alpha(\alpha_0)$.

Specifically, when $|u| \geq \alpha$, we get $|z| = |\alpha^{-1}u| \geq 2\alpha$ for all $A \geq 2$ and $|z| \geq \alpha(\alpha_0)$. Consequently,

Either $|z| < \alpha(\alpha_0)$ or $|z| = |\alpha^{-1}u| \geq 2\alpha$ for all $A \geq 2$. (34)

This proves that α^{-1} is uniformly BIBO-stable, so that α is uniformly BIBO-minimum phase. Combining with our earlier observation, we conclude that α is uniformly BIBO-unimodular.

We can now prove that Configuration 18 is uniformly internally BIBO-stable. To this end, note that additive signals added at the points u, e, and y in (18) all have similar effects on the output of the configuration's adder. Also, a signal $z\alpha$ added to z is equivalent to a signal $z\alpha/A$ added to u. Furthermore, since α is BIBO-unimodular (Lemma 22) and independent of A, it follows that w is uniformly bounded if and only if z is uniformly bounded. The last sentence implies that it suffices to investigate the impact of the signal z and of signals added to z; the effects of the signal w and of signals added to w do not need to be considered separately (see [8] for more details).

Additionally, since $e = z/A$, it is clear that e is uniformly bounded as $A \alpha \infty$ if the signal z is uniformly bounded as $A \infty \infty$. Finally, since $y = \infty\infty z$ and ∞ and ∞ are both BIBO-stable and independent of A, it follows that the transmission from u to y is uniformly BIBO-stable if so is the transmission from u to z. Thus, in order to prove that (18) is uniformly internally BIBO-stable, it only remains to show that the transmission from u to z is uniformly BIBO-stable. However, the latter is a direct consequence of (31) and our earlier conclusion that ∞^{-1} is uniformly BIBO-stable. ♦

We examine next the tracking capabilities of Configuration 18 with high gain A. The *tracking error* for a gain A is given by $\tau(A) := |u - y| = |e|$. The next statement shows that Configuration 18 yields accurate tracking for BIBO-minimum phase systems, linear or not.

(35) THEOREM. Let $\tau : S(R^m) \tau S(R^m)$ be a strictly causal, BIBO-stable, and BIBO-minimum phase system, having the lower bound function τ and the lower gain function γ. Enclose γ in Configuration 18 with the controller $C = \gamma A$, where γ is given by (19). Then, for every bounded input sequence u, the tracking error satisfies $\lim_{A\gamma\,\gamma} \gamma(A) = 0$.

Proof. Let $u \gamma S(R^m)$ be a bounded input sequence of Configuration 18 with norm $|u| = \gamma$. Then, $e = y - u = z/A$, so that $\gamma(A) = |e| = |z|/A$ for all $A > 0$. Using (34), this implies that $\gamma(A) \geq (1/A)\max\{\gamma(\gamma_0), 2\gamma\}$ for all $A \geq 2$, so that $\lim_{A\gamma\,\gamma} \gamma(A) = 0$. γ

Thus, Configuration 18 extends the tracking prowess of the Black diagram to nonlinear BIBO-minimum phase systems.

(36) EXAMPLE. Consider the (scalar) system

$x_{k+1} = x_k/2 + u_k$, $x_0 = 0$,

$y_k = \exp(|x_k|)$.

A simple calculation shows that, in this case, we can use the lower bound function $\gamma(\gamma) = \exp(2\gamma/3) - 1$. The corresponding lower gain function can then be taken as

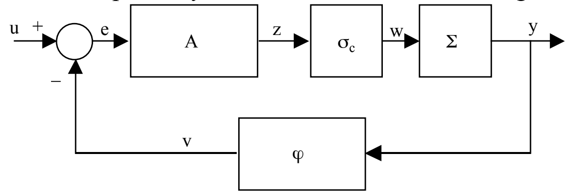
$\gamma(\gamma) = [\exp(2\gamma/3) - 1]/\gamma$, $\gamma \geq 0$,

using the continuous extension at $\gamma = 0$. The compensator γ, in this case, is given by

$$\gamma(z_k) = \frac{\frac{3}{2}\log(|z_k| + 1)}{|z_k|} z_k. \ \gamma$$

B. Approximate model matching

Let γ and φ be two causal BIBO-stable and BIBO-minimum phase systems, and consider the configuration



(37)

Here, φ is the strictly causal system being controlled, φ is used as a feedback compensator, A represents a constant gain amplifier, and φ_c is a static compensator. Denoting by φ_A the input/output relation of the diagram, we have

$\varphi_A = \varphi\varphi_c A(I + \varphi\varphi\varphi_c A)^{-1}$.

As φ and φ are both BIBO-minimum phase systems, so is the combination $\varphi\Sigma$. It follows then by Theorem 12 that $\Sigma\Sigma$ has a lower bound function α_c. Let γ_c be a lower gain function corresponding to γ_c. In analogy with (19), define the compensator σ_c by

$$w_k = \sigma_c(z_k) := \frac{1}{\alpha_c(\sigma_c^{-1}(|z_k|))} z_k \text{ , } k = 0, 1, 2, \ldots \qquad (38)$$

The next statement shows that, with this compensator, Configuration 37 matches the model σ^{-1} as $A \to \infty$.

(39) THEOREM. Let $\infty : S(R^m) \infty S(R^m)$ and $\infty : S(R^m) \infty S(R^m)$ be two BIBO-stable and BIBO-minimum phase systems, where ∞ is strictly causal and ∞ is bicausal. Assume that the inverse system ∞^{-1} is continuous with respect to the ℓ^∞-norm. Then, with the compensator ∞_c of (38), Configuration 37 has the following properties:

(i) It is uniformly BIBO-internally stable, and

(ii) $\lim_{A\infty\infty} |\infty_A u - \infty^{-1}u| = 0$ for every bounded input sequence u; moreover, the limit converges uniformly over all input sequences of norm $|u| \leq \theta$, where θ is any positive real number. ♦

The proof of Theorem 39 is similar to the proof of Theorem 35 (see [10] for details).

To summarize, we have seen that the Black diagram, with the modification described by Configuration 37, facilitates tracking and approximate model matching for nonlinear BIBO-minimum phase systems.

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