2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

On the Use of Empirical Gramians for Controllability and Observability Analysis

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Abstract—**This short paper illustrates the use of empirical gramians for controllability/observability analysis of nonlinear systems and compares the extracted information to results obtained from linear gramians and nonlinear observability matrices. It is shown that empirical gramians can more accurately represent controllability/observability of a nonlinear system over an operating region than if linear gramians are used. At the same time the information contained in empirical gramians is easier to extract and interpret than if Lie algebra-based controllability or observability matrices are used.**

I. INTRODUCTION

Empirical controllability and observability gramians have been used for several years for model reduction of nonlinear systems [1]-[3]. However, no results of using empirical gramians for controllability and observability analysis have been reported in the literature. This work addresses this point by comparing observability information from linear gramians [4], lie-algebra-based observability matrices [5], and empirical gramians [1]. The presentation is limited to observability due to the allocated space; however, the results for controllability are similar in nature.

II. PRELIMINARIES

A. Linear Gramians

For a linear system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

the linear observability ($W_{O,Linear}$) and controllability ($W_{C,Linear}$) gramians [4]

$$W_{O,Linear} = \int_0^\infty e^{A^T t} C^T C e^{At} dt \quad W_{C,Linear} = \int_0^\infty e^{At} B B^T e^{A^T t} dt \tag{2}$$

can be used to compute observability and controllability of states of the system.

B. Nonlinear Controllability and Observability Matrices

For nonlinear systems

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{3}$$

it is possible to determine if a system is locally observable or controllable using concepts from differential geometry [5].

C. Empirical Gramians

Empirical gramians [1], [2]

$$W_C = \sum_{i=1}^{p}\sum_{l=1}^{r}\sum_{m=1}^{s} \frac{1}{rsc_m^2} \int_0^\infty \Phi^{ilm}(t) dt \quad W_O = \sum_{l=1}^{r}\sum_{m=1}^{s} \frac{1}{rsc_m^2} \int_0^\infty T_l \Psi^{lm}(t) T_l^T dt \tag{4}$$

have been recently introduced for nonlinear systems of the form of equation (3). Refer to [2] for the exact definition of the variables from equation (4).

III. EMPIRICAL GRAMIANS FOR CONTROLLABILITY/OBSERVABILITY ANALYSIS

Empirical gramians have been used for model reduction of nonlinear systems and the benefit of using empirical gramians over conventional linear gramians for model reduction has been illustrated [2]. However, no current results exist in the literature that directly compare the information contained in empirical gramians to linear gramians or nonlinear geometric methods for controllability and observability analysis. An illustrative example is presented in this section and a comparison of the conclusions that can be drawn about observability from the available information is made.

A. Illustrative Example

Consider the following nonlinear system

$$\begin{aligned} \dot{x}_1 &= 1 - x_1 \\ \dot{x}_2 &= 1 - x_2 + (x_1 - 1)(x_2 - 1)^2 \\ y &= x_2 \end{aligned} \tag{5}$$

which has an equilibrium point at $(x_1,x_2) = (1,1)$. An example without inputs is used due to the space constraint and, therefore, only observability of the system will be analyzed. The linearization of the system (5) at the equilibrium point is given (in deviation variables) by:

Manuscript received September 10, 2004.

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$$\dot{x}_1' = -x_1' \qquad \text{where} \qquad x_1' = x_1 - 1$$
$$\dot{x}_2' = -x_2' \qquad\qquad\qquad x_2' = x_2 - 1 \tag{6}$$
$$y = x_2' + 1$$

Accordingly, the linear observability gramian results in

$$W_{O,Linear} = \begin{pmatrix} 0 & 0 \\ 0 & 0.5 \end{pmatrix} \tag{7}$$

which suggests that only the state x'_2 of the system (5) is observable, and x'_1 is unobservable. While this result is correct locally, it fails to describe that the system (5) is locally observable everywhere except along the manifold given by $x_2 = 1$. This can be concluded from the observability matrix computed for the nonlinear system using concepts from differential geometry:

$$Q(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ (x_2 - 1)^2 & -1 + 2(x_1 - 1)(x_2 - 1) \end{pmatrix} \tag{8}$$

As every system has a region of operation, it is important to take information about the property of the system over a region into account, which is not possible using linear gramians. At the same time it is not currently feasible to apply the analysis used for deriving and interpreting equation (8) to systems of medium or even large scale. Both of these points can be addressed by analyzing controllability and observability of a system using empirical gramians, as they are naturally computed for a region of operation. At the same time, they have been successfully applied to systems with dozens [2] or even hundreds of states and extracting information from empirical gramians is no more challenging than it is for linear systems.

Because it has been proven that empirical gramians (4) reduce to the linear gramians (2) if the investigated system is linear [1], it can be ensured that the observability results will locally match the ones derived for a linearized system. However, as empirical gramians provide information about the controllability/observability of a system over a region of operation, they are able to return results of more than just local value. For example, the empirical observability gramian for the system given by equation (5) for an operating region of ±10% around the equilibrium point results in

$$W_o = \begin{pmatrix} 3.4667 \times 10^{-7} & 0.0002 \\ 0.0002 & 0.5066 \end{pmatrix} \tag{9}$$

The rank of the empirical observability gramian for this operating region is equal to two indicating that the states of the system can be reconstructed. These results have been validated in numerical experiments where an observer has been designed. Additionally, they are also backed up by the conclusions drawn from the observability matrix for the nonlinear system shown in equation (8). The condition number and the magnitude of the smallest eigenvalue for empirical observability gramians for varying magnitudes of the operating region are shown in Table 1.

TABLE I

COMPARISON OF EMPIRICAL GRAMIANS FOR DIFFERENT OPERATING REGIONS

Operating Region (% around (1,1))	Smallest Eigenvalue	Condition Number
0.1	0.3675×10^{-13}	1.3549×10^{13}
0.5	0.1632×10^{-11}	3.0677×10^{11}
1	0.2600×10^{-10}	1.9251×10^{10}
5	0.1626×10^{-7}	3.0799×10^{7}
10	0.2601×10^{-6}	1.9249×10^{6}
20	0.4162×10^{-5}	1.2030×10^{5}
30	0.2180×10^{-4}	2.3756×10^{4}
40	0.6669×10^{-4}	7.5105×10^{3}
50	0.1632×10^{-3}	3.0711×10^{3}

It can be concluded from the condition number and the smallest eigenvalue, that the smaller the operating region, the more ill-conditioned the empirical observability gramian becomes. This is expected as the empirical gramian will be rank deficient if it is computed for an infinitesimal small operating region around the equilibrium point. At the same time, the results for a realistic operating region allow conclusions similar to the ones for the observability matrix of the nonlinear system. However, empirical gramians are easier to compute and interpret than Lie-algebra based observability matrices, and may be a promising alternative to conventional methods.

IV. CONCLUSIONS

This paper investigates the use of empirical gramians for controllability and observability analysis of nonlinear systems. While empirical gramians will reduce to linear gramians for linear systems, it is shown that they contain more accurate information about the behavior of a system over an operating region than if linear gramians are used. This observation has been illustrated with an example, where the same conclusions can be drawn from empirical gramians and nonlinear observability matrices, whereas local results derived from linear gramians can be misleading if a system is investigated over a region of operation.

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