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Root-Locus Dynamics

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Abstract

In this paper, the discussion of root locus is taken from the point of view of field theory by treating root locus as some kind of potential flows. This approach throws a new light on root locus and suggests a physical modeling of root locus in terms of the streamlines in flow field and electric field. Based on potential theory we derive the governing equations of root locus for time-varying systems, in terms of which the force interaction existing within root loci can be explained, and root sensitivity and root robustness can be defined. Furthermore, the superposition of elementary potential flows makes it possible to reconstruct open-loop transfer function from the desired root locus - the so-called inverse root-locus problem.

Key words: Root locus, Fluid Dynamics, Field Theory

I. Introduction

Though root locus method is so popular and familiar to control engineers, we still do not know theoretically why some rules of experience for constructing root locus should be. For example, a rule of thumb in plotting root locus says that adding zeros to the open-loop transfer function $G(s)$ has the effect of moving the root loci toward the left-half of the s-plane and adding a pole to $G(s)$ in the left half of the s-plane has the effect of pushing the original root loci toward the right-half plane. Indeed there still lacks a theoretical verification of the above phenomena using the existing knowledge about root locus.

For a control engineer who has some experiences in fluid visualization, he may find an interesting link between the flow pattern of fluids and the trajectory of root locus. In fluid laboratory we can see the phenomena that adding a source to a flow field will push the streamlines away from the source, while adding a sink will pull the streamlines toward the sink. The effect of adding a pole in root locus is very similar to the effect of adding a source in flow field, and the effect of a zero is similar to a sink. These analogies motivate the present study. The main concept we want to introduce here is that the movement of root locus is a phenomenon of potential flow. Like other fields of potential flow, such as electric field, magnetic field, fluid field, gravitational field, and temperature field, etc., the investigation of root locus can be generalized to field theory [2].

II.Generalized Root Locus (GRL)

Consider a general transfer function

$$G(s)=\frac{(s-b_1)^{\beta_1}(s-b_2)^{\beta_2}\cdots(s-b_m)^{\beta_m}}{(s-a_1)^{\alpha_1}(s-a_2)^{\alpha_2}\cdots(s-a_n)^{\alpha_n}}e^{\lambda_1 s+\lambda_2/s} \tag{1}$$

where, the parameters λ_i, a_i, b_i, α_i, and β_i are all assumed to be complex numbers with $\mathrm{Re}(\alpha_i)$, $\mathrm{Re}(\beta_i)\in R^+$. This class of transfer function may contain irrational terms such as $\sqrt{s+1}$, $(s+j)^j$, and $e^{s+1/s}$, etc., where $j=\sqrt{-1}$. The systems defined by Eq.(1) covers both lumped systems governed by ordinary differential equations and some distributed systems governed by partial differential equations.

Conventional root loci (**CRL**) of G(s) are defined by the following set

$$CRL=\left\{s \,\middle|\, 1+kG(s)=0,\ k\in R^+(R^-)\right\} \tag{2}$$

The major advantage of using root-locus method is that one can examine the effect of changing open-loop gain or plant parameters to aid in achieving best overall control design. Instead of merely changing the open-loop gain k, here we will consider a more practical and general case of changing the dynamics compensator $K(s)$. In this case the roots of $1+K(s)G(s)=0$ for varying dynamic compensator $K(s)$ are to be determined. In terms of Eq.(2) we can see that if the parameter k is replaced by the compensator $K(s)$, the magnitude and the phase of parameter k are varying simultaneously. With this consideration the conventional definition of root locus is modified as following.

$$GRL(k_\theta)=\left\{s \,\middle|\, 1+kG(s)=0,\ k=k_r e^{jk_\theta},\ k_r\in R^+\right\} \tag{3}$$

In this definition k is no longer restricted to be positive real or negative real; instead, k is released to be any complex number.

To establish this analogue, the complex potential function is introduced

$$\Omega(s) = -\ln G(s) \tag{4}$$

and define

$$\Omega(\sigma+j\omega) = \Phi(\sigma,\omega)+j\Psi(\sigma,\omega) \tag{5}$$

Let C_0 be the complex plane excluding the singularities contained in $\Omega(s)$. Since $\Omega(s)$ is analytical within C_0, Φ and Ψ must be conjugate harmonic functions in C_0, satisfying the Laplace equations.

$$\nabla^2\Phi = \nabla^2\Psi = 0 \tag{6}$$

The relations imposed by Cauchy-Riemann condition

$$\frac{\partial\Phi}{\partial\sigma} = \frac{\partial\Psi}{\partial\omega}, \quad \frac{\partial\Phi}{\partial\omega} = -\frac{\partial\Psi}{\partial\sigma} \tag{7}$$

allow the phase function and magnitude function to be derived from each other.

Laplace equations govern many physical phenomena, such as steady heat conduction, electrostatics, magnetostatics, gravitational field, and flow of an ideal fluid. For the present case, it can be seen that behavior of root locus is also characterized by Laplace equation.

III. Phaser and Phase flux

(A) "Phaser" : a moving particle

In this section we attempt to give a physical interpretation of **GRL**. Root locus **GRL** (k_θ) can be considered as a path line traced out by a moving medium which is called "phaser" here. Phaser conveys the phase k_θ of k from one point to another along root locus **GRL** (k_θ). One k_θ will determine one path line **GRL**(k_θ) as phase k_θ is varied, we then form a flow pattern consisting of many such path lines along which phasers carry different values of phase.

(B) Phase flux

Next we strive to introduce the concept of flux of phaser. Let $A(\sigma_0, \omega_0)$ be a fixed point in the $\sigma-\omega$ plane which has unit thickness and $P(\sigma, \omega)$ an arbitrary point in the same plane. (See Fig.1) Now, two arbitrary paths ABP and ACP are drawn between the points P and A. Assume that there is no singularity within the region bounded by these curves, i.e., there is no phase created or destroyed within the region, then the rate of phase flow entering the region across the curve ABP is equal to the flow rate across the curve ACP. The term flux for the rate of phase flow will be used hereafter. The flux across the curve ABP is equal to the flux across any curve joining A to P. Since the point *A* is fixed, the flux is a function of the position of *P*. If the root locus passing through *P* is **GRL**(Ψ_P), then the phase flux across *AP* is denoted by Ψ_P. It must be borne in mind that the existence of phase function is a consequence only of the conservation of phase, so a phase function is valid for nonlinear and time-varying systems.

Now consider two points P_1 and P_2 and two curves drawn from them to the fixed point *A*. Let **GRL**(Ψ_1), **GRL**(Ψ_2) represent the root loci passing through points P_1 and P_2, respectively (see Fig.2). Then, the flux across the curve AP_2 is equal to the flux across the curve AP_1 plus that across the curve P_1P_2. Hence, the flux across the curve P_1P_2 is $\Psi_1 - \Psi_2$. It can be easily seen that if the reference point *A* is replaced by another point A_1, the value of the stream function $\Psi_1 - \Psi_2$ changes by a constant, namely the flux across A_1A_2. Since definition the stream function is constant along a root locus, when the point P_1 and P_2 are points of the same root locus (not necessarily coincident), the flux across P_1P_2 is $\Delta\Psi = \Psi_1 - \Psi_2 = 0$. It turns out that there is no phase flow across a root locus.

The phase flow rate in any direction (i.e., velocity of phaser) can be derived from the phase function. In Cartesian coordinates the phaser velocity components q_σ and q_ω in the σ and ω-directions are found from the following consideration. Let Δs be an infinitesimal length of the curve *AP* whose components in the σ and ω-directions are $\Delta\sigma$ and $\Delta\omega$, respectively (Fig.3). The flux across the curve *AP* is

$$\Psi = \int_{AP} q \cdot n ds = \int_{AP} \left(q_\sigma d\omega - q_\omega d\sigma\right) \tag{8}$$

where $q = \left[q_\sigma \ \ q_\omega\right]$ and n is the outward unit vector perpendicular to ds. The flux across the arc Δs is

$$d\Psi = q_\sigma d\omega - q_\omega d\sigma \tag{9}$$

On the other hand, the phase function $\Psi = \Psi(\sigma, \omega)$ in general can be expressed by

$$d\Psi = \frac{\partial\Psi}{\partial\sigma}d\sigma + \frac{\partial\Psi}{\partial\omega}d\omega \tag{10}$$

Hence, the phaser velocity components q_σ, q_ω parallel to the axes are given by

$$q_\sigma = \frac{\partial\Psi}{\partial\omega}, \ q_\omega = -\frac{\partial\Psi}{\partial\sigma} \tag{11}$$

Using Cauchy-Riemann condition, flux rate in Eq.(11) can be expressed in terms of magnitude function Φ as

$$q_\sigma = \frac{\partial\Phi}{\partial\sigma}, \ q_\omega = \frac{\partial\Phi}{\partial\omega}, \tag{12}$$

or in a vector form

$$\mathbf{q} = \nabla\Phi \tag{13}$$

Along lines $\Psi(\sigma, \omega) =$ constant, $d\Psi = 0$ and the combination of Eq.(9) and Eq.(11) gives

$$\left(\frac{d\omega}{d\sigma}\right)_{\Psi = cons\tan t} = \frac{q_\omega}{q_\sigma} \tag{14a}$$

This result shows that the velocity of phaser **q** is everywhere tangent to curves in the $\sigma-\omega$ plane along which $\Psi(\sigma, \omega) =$ constant. This result is consistent with the previous assumption that $\Psi(\sigma, \omega) =$ constant (**GRL**) is the trajectory of a phaser. On the other hand, along $\Phi(\sigma, \omega) =$ constant

$$d\Phi = \frac{\partial\Phi}{\partial\sigma}d\sigma + \frac{\partial\Phi}{\partial\omega}d\omega = 0$$

Therefore

$$\left(\frac{d\omega}{d\sigma}\right)_{\Phi = cons\tan t} = -\frac{\partial\Phi/\partial\sigma}{\partial\Phi/\partial\omega} = -\frac{q_\sigma}{q_\omega} \tag{14b}$$

Since these slopes in Eq.(14a) and Eq.(14b) are negative reciprocals, the lines are perpendicular to one another. Thus, the constant magnitude lines and **GRL** form an orthogonal set of lines which completely describe the phase flow in a two dimensional field.

IV. Superposition of Elementary Root Loci

The Laplace equation or Poissin equation, governing phase function for two-dimensional potential flows is in such a simple form that some elementary solutions to these equations can easily be found. Each of these solutions represents a physically possible elementary **GRL**. The method of superposition is also used in solving problems in electromagnetism, heat conduction and fluid dynamics whose governing equations are in the form of either the Laplace or the Poisson equation. Some of the **GRL** represented by the linear combination of solutions may simulate the root loci originated from linear or nonlinear systems such as in Eq.(1). There are four elementary **GRL** that are commonly encountered in control problem : The time delay, the pole (or sink), the dipole, and the vortex. Although the **CRL** for pole, sink and time delay is well known, the concept of **GRL** provides us new physical interpretation for these loci.

1) **Time delay**

The transfer function of a time delay unit is

$$G_T(s) = e^{\lambda_1 s} \tag{15}$$

where $\lambda_1 = -Te^{-j\alpha}$. According to Eq.(5), the phase function and the magnitude function now become

$$\begin{aligned} \Psi(\sigma, \omega) &= T(\omega\cos(\alpha) - \sigma\sin(\alpha)) \\ \Phi(\sigma, \omega) &= T(\omega\cos(\alpha) + \omega\sin(\alpha)) \end{aligned} \tag{16}$$

The $GRL(\Psi)$ and the constant potential lines are plotted in Fig.4. The phaser velocity is calculated via Eq.(11) as

$$q_\sigma = T\cos(\alpha), \quad q_\omega = T\sin(\alpha) \tag{17}$$

This shows that the root loci are all parallel straight lines making an angle α with the σ axis.

2) **Poles and zeros**

For a pole at $s_p = \sigma_p + j\omega_p$ with multiplicity m_0 (positive and not limited to integer), the corresponding complex potential function is

$$\Omega(s) = m_0 \ln(s - s_p) = \Phi + j\Psi \tag{18}$$

Hence, phase function and potential function are obtained, respectively, as

$$\Psi(\sigma, \omega) = m_0 \tan^{-1}\frac{\omega - \omega_p}{\sigma - \sigma_p} = m_0\theta \tag{19a}$$

$$\Phi(\sigma, \omega) = m_0 \ln\left(\left(\sigma - \sigma_p\right)^2 + \left(\omega - \omega_p\right)^2\right)^{1/2} = m_0 \ln r \tag{19b}$$

We can verify by direction substitution that both Ψ and Φ satisfy Laplace equation and Poisson equation. The curves $\Psi = c$ are sketched in Fig.5. From the observation of this figure, a pole at point (σ_p, ω_p) can be conceived of as a point from which phaser emanates in equal amounts along radial paths. Hence the **GRL** in this case be straight radial lines, i.e.,

$$q_\theta = 0, \quad q_r = q_r(r) \tag{20}$$

where $q_r(r)$ can be found from Eq.(13) using polar coordinate expression.

$$q_r = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = \frac{m_0}{r} \tag{21}$$

3) **Dipole**

Consider a pole-zero pair (see Fig.6) which is assumed to be part of the open-loop transfer function Eq.(1). We are interested in the case where other poles and zeros of $G(s)$ are far removed from the pole-zero pair, in which case the distance $|s_0|$ of Fig.6 is much smaller than the distance between the pole-zero pair and any other pole or zero of $G(s)$. Such an isolated pole-zero pair is called a dipole. When dipoles occur on or very near the stability margin, the hidden modes may cause stability problems if not properly anticipated [4]. It is then worth taking a close look on this kind of locus. An special **GRL** results when the distance $|s_0|$ between pole and zero of equal multiplicity m_0 (not limited to integer) approaches zero while their multiplicity approached infinity in such a way that their product $m_0|s_0| = \kappa$ remains constant. In the limiting the resulting **GRL** is called a dipole of strength κ.

Let $s_0 = |s_0|e^{j\alpha}$, α is called the direction angle of the dipole. The superposition of the complex potential functions of a pole at origin and a zero at s_0 gives

$$\Omega(s) = \lim_{|s_0|\to 0} m_0 \ln\frac{s}{s - s_0} = \frac{\kappa e^{j\alpha}}{s} = \Phi + j\Psi \tag{22}$$

In polar coordinate we have

$$\Psi(r, \theta) = -\frac{\kappa}{r}\sin(\theta - \alpha), \quad \Phi(r, \theta) = -\frac{\kappa}{r}\cos(\theta - \alpha) \tag{23}$$

The equation above represent a family of circles which pass through origin with centers on the $\omega-$ axis for $\Psi = c$ and with centers on $\sigma-$ axis for $\Phi = c$. The **GRL** for a dipole with $\alpha = 0$ is shown in Fig.7. From Eq.(22) the transfer function corresponding to a dipole has the form, as appeared in Eq.(1),

$$G(s) = e^{\lambda_2/s} \tag{24}$$

where $\lambda_2 = \kappa e^{j\alpha}$ .Obviously, this transfer function is originated from a nonlinear element.

4) Vortex

This type of **GRL** stems from poles oz zeros with imaginary power in Eq.(1), i.e., from the existence of $\mathrm{Im}(\alpha_i)$ and $\mathrm{Im}(\beta_i)$. The complex potential function for vortex can be written as

$$\Omega(s) = -j\frac{\Gamma}{2\pi}\ln(s - s_0) = \Phi + j\Psi \tag{25}$$

where Γ is a real number and $s_0 = \sigma_0 + j\omega_0$ is the center of the vortex. Therefore, Ψ and Φ for vortex becomes

$$\Psi(\sigma, \omega) = -\frac{\Gamma}{2\pi}\ln r, \quad \Phi(\sigma, \omega) = \frac{\Gamma}{2\pi}\tan^{-1}\frac{\omega - \omega_0}{\sigma - \sigma_0} = \frac{\Gamma\theta}{2\pi} \tag{26}$$

It can be seen that **GRL** are circles, and equi-magnitude lines are radii (Fig.8). Note if we interchange Ψ and Φ, we obtain the **GRL** for pole. The phaser velocities are

$$q_r = \frac{\partial\Phi}{\partial r} = \frac{1}{r}\left(\frac{\partial\Psi}{\partial\theta}\right) = 0, \quad q_\theta = \frac{1}{r}\left(\frac{\partial\Phi}{\partial\theta}\right) = -\frac{\partial\Psi}{\partial r} = \frac{\Gamma}{2\pi r} \tag{27}$$

It turns out that phasers in this type of **GRL** moves in circular paths with the velocities being inversely proportional to the radii of the circles.

The **GRL** for any pole-zero configurations with delay or dipole can be obtained by linear combinations of this four elementary **GRL**. Hence, Eq.(3) can be rewritten in an alternate form for $GRL(k_\theta)$ as

$$\left\{(\sigma,\omega)\middle| l_1\Psi_{delay}+l_2\Psi_{pole}+l_3\Psi_{dipole}+l_4\Psi_{vortex}=k_\theta,\ \forall l_i\in R\right\} \tag{28}$$

Fig.9 shows part of the **GRL** for the delay system with a dipole at origin. The **CRL** is composed of the σ axis and the big circle centering at origin. It is seen that the loci of dipole are isolated from those of time delay by this big circle and thus form a phase flow pattern very similar to the fluid flow pattern caused by a uniform flow via a cylinder with infinite length [3, 6].

IV. Inverse Root-Locus problem

Let $G(s)=C(s)P(s)$, where $P(s)$ denotes the plant model and $C(s)$ denotes the compensator. The root-locus problem is that given the compensator $C(s)$, plot the roots of $1+kG(s)=0$ as k is varied, while the inverse root-locus problem is that given one segment of the root loci of $1+kG(s)=0$, find the compensator $C(s)$. Quite often, in designing regions or to follow some prescribed paths as k is varied. Thus the inverse root-locus problem is practically important. This problem can be solved by using the property of superposition of root loci, introduced in the last section where it has been shown that any shape of loci can be synthesized by the four elementary root loci.

In general, if root loci with closed boundary are to be synthesized, dipole (or pole-zero pair with equal multiplicity and finite distance apart) would be the best elementary loci to meet this purpose, while if loci with open boundary are to be synthesized, time delay and combination of isolated poles or zeros with different multiplicities would be appropriate. The former will be considered here. We use a continuous distribution of dipoles, of strength κ per unit length along the σ axis, within the range between $\sigma=a$ and $\sigma=b$, to synthesize the given loci $f(\sigma,\omega)=0$ by properly adjusting the distribution $\kappa(\sigma)$. As shown in Fig.10, at a point $P(\sigma,\omega)$ the dipoles contained within the small interval $d\eta$, located at a distance η from the origin, contribute $d\Psi$ to the phase function at that point. Use Eq.(23) with $\alpha=0$, we find that

$$d\Psi=-\frac{\kappa(\eta)\,\omega d\eta}{(\sigma-\eta)^2+\omega^2} \tag{29}$$

Then, the $CRL=GRL(\pi)$, obtained from the superposition of the dipoles distributed from $\sigma=a$ to $\sigma=b$, becomes

$$\Psi(\sigma,\omega)=-\int_a^b\frac{\kappa(\eta)\,\omega}{(\sigma-\eta)^2+\omega^2}d\eta=\pi \tag{30}$$

Where σ and ω is related by the relation $f(\sigma,\omega)=0$. Dipole distribution $\kappa(\sigma)$ is determined by the solving above integral equation, which is known as the Fredholm equation of the first kind. One way to obtain the analytical expression for $\kappa(\sigma)$ is to explore successive approximations, but for a realizable compensator design, a numerical approach using discretized $\kappa(\sigma)$ would be more desirable. We divide the dipole region into n segments of equal width $\Delta\eta$, as shown in Fig.11. We designate by $\kappa_j\Delta\eta$, the total dipole strength within the j segment, whose center is at a distance η_j from the origin; κ_j is taken as constant, equal to the average of the exact distribution within the segment. κ_j will of course vary from one segment to another. The dipoles within the j segment will contribute phase $\Delta\Psi_j$ to the phase function at a given point P in the field. This contribution may be written as

$$\Delta\Psi_j=-\frac{\kappa_j\Delta\eta\omega_p}{(\sigma_p-\eta_j)^2+\omega_p^2} \tag{31}$$

Hence, the approximate formula corresponding to the exact form of Eq.(15) is

$$\frac{\pi}{\omega_p}=-\sum_{j=1}^{n}\frac{\kappa_j\Delta\eta}{(\sigma_p-\eta_j)^2+\omega_p^2} \tag{32}$$

We now apply this formula to n points on the desired **CRL** to obtain a set of n simultaneous linear algebraic equations, the solution of which yields the dipole strength $X=[\kappa_1\ \ \kappa_2\ \ \cdots\ \ \kappa_n]^T$, i.e.,

$$AX=B \tag{33}$$

where $A=[a_{ij}]$ is a $n\times n$ matrix with its elements a_{ij} given by

$$a_{ij}=-\frac{\Delta\eta}{(\sigma_j-\eta_j)^2+\omega_i^2}$$

and $B=[b_i]$ is a n-dimensional vector with its element b_i given by $b_i=\pi/y_i$. As n approaches infinity the numerical result for the dipole strength distribution κ_j approaches the exact solution. By utilizing a program for solving simultaneous linear equations, the solution for a reasonably large number of segments can readily be found on a digital computer. Once κ_j is known, the phase function at any point can be computed by

$$\Psi(\sigma,\omega)=-\sum_{j=1}^{n}\frac{\kappa_j\Delta\eta\omega}{(\sigma-\eta_j)^2+\omega^2} \tag{34}$$

Finally, the open-loop transfer function for this phase function is obtained as

$$G(s)=C(s)P(s)=\prod_{j=1}^{n}\left(\frac{s-\eta_j-\Delta\eta_j}{s-\eta_j}\right)^{n_j} \tag{35}$$

where $\Delta\eta_j$ is the distance between the pole and zero of the j th dipole, which, in the limiting, must be an infinitesimal quantity; however, to obtain a realizable compensator, it is approximated by a finitely small value satisfying $n_j\Delta\eta_j=\kappa_j\Delta\eta_j$.

V. Equations of Motion for Phasers

(A) Phaser Fluid

In our experience of plotting root loci, the phenomena has been learned that adding zeros in the left-half s-plane to open-loop transfer function has the effect of moving the root loci toward the left half of the s-plane and that adding poles in the left-plane s-plane has the effect of pushing the original loci toward the right-half plane. These phenomena can be well explained by the concept of phase flow. From the observation of **GRL** for poles we have known that a pole can be conceived of as a point from which phasers emanate along all radial directions. When added to the original loci, these phasers emanating from the pole create a pressure on the loci and cause it to move away from the pole; while from the observation of **GRL** for zeros, we have known that a zero is a point into which phasers flows radially from all directions. Thus, the phasers on the original loci in the neighborhood of the zero will be sucked into the zero and cause the loci to move toward the zero. In the following paragraph, we will give a quantitative description of above phenomena.

(B) The Continuity Equations

Consider the two-dimensional phase flow in the complex plane, and cut out a control volume of infinitesimal dimensions (see Fig.12). Consider the total phase rate across the control surface, and the rate of change of phase storage within, for a unit depth normal to the complex plane. In terms of density ρ and phase flux rate (velocity of phaser) **q** of phaser fluid, the mass flux rate **Q** can be defined accordingly as

$$Q = \left[Q_\sigma \; Q_\omega\right] = \rho q = \left[\rho q_\sigma \; \rho q_\omega\right] \tag{36}$$

Application of the principle of conservation of phase, yields

$$\frac{\partial Q_\sigma}{\partial \sigma} + \frac{\partial Q_\omega}{\partial \omega} + \frac{\partial \rho}{\partial t} = 0 \tag{37}$$

or, in vector form

$$\nabla \cdot \mathrm{Q} + \frac{\partial \rho}{\partial \mathrm{t}} = 0 \tag{38}$$

This continuity equation must be satisfied for any control systems including nonlinear and time-varying systems. If we assume that phaser fluid is "incompressible", i.e., ρ =constant, we have

$$\nabla \cdot \mathrm{q} = 0 \tag{39}$$

(C) The Momentum Equations

Cut out an infinitesimal stationary control volume of unit depth, and consider the phase pressure p acting on this control volume in the σ direction and the σ momentum fluxes across the control surface (see Fig.13). Applying the momentum theorem [1], in the σ direction, we have

Outflow of momentum

$$= \left[Q_\sigma q_\sigma + \frac{\partial}{\partial \sigma}\left(Q_\sigma q_\sigma\right)\delta\sigma\right]\delta\omega + \left[Q_\omega q_\sigma + \frac{\partial}{\partial \omega}\left(Q_\omega q_\sigma\right)\delta\omega\right]\delta\sigma$$

Inflow of momentum $= \left(Q_\sigma \delta\omega\right) q_x + \left(Q_\omega \delta\sigma\right) q_x$

Increase of momentum storage $= a\partial Q_\sigma / \partial t$

External force $= p_\sigma \delta\omega - \left(p_\sigma + \frac{\partial p_\sigma}{\partial \sigma}\delta\sigma\right)\delta\omega$

Combining together and simplifying, we have

$$\frac{\partial q_\sigma}{\partial t} + \frac{\partial}{\partial \sigma}\left(Q_\sigma q_x\right) + \frac{\partial}{\partial \omega}\left(Q_\omega q_x\right) = -\frac{\partial p_\sigma}{\partial \sigma} \tag{40}$$

Expanding the left-hand terms and using Eq.(36) and Eq.(37), yields

$$\frac{\partial q_\sigma}{\partial t} + q_\sigma \frac{\partial q_\sigma}{\partial \sigma} + q_\omega \frac{\partial p_\sigma}{\partial \omega} = -\frac{1}{\rho}\frac{\partial p_\sigma}{\partial \sigma} \tag{41a}$$

Similarly, applying the momentum theorem in the ω direction, we have

$$\frac{\partial q_\omega}{\partial t} + q_\sigma \frac{\partial q_\omega}{\partial \sigma} + q_\omega \frac{\partial p_\omega}{\partial \omega} = -\frac{1}{\rho}\frac{\partial p_\omega}{\partial \omega} \tag{41b}$$

Eqs.(41) can be put into a compact vector form as

$$\frac{\partial q}{\partial t} + \left(q \cdot \nabla\right) q + \frac{\nabla p}{\rho} = 0 \tag{42}$$

This can be simplified further by noting from vector analysis that

$$\left(q \cdot \nabla\right) q = \frac{1}{2}\nabla q^2 - q \times \left(\nabla \times q\right) \tag{43}$$

Substituting Eq.(43) into Eq.(42), we obtain for incompressible phaser fluid

$$\frac{\partial q}{\partial t} - q \times \left(\nabla \times q\right) = -\nabla\left(\frac{p}{\rho} + \frac{q^2}{2}\right) \tag{44}$$

VI. Force Action of Root Loci

Let $L_1 \in GRL(\Psi_1)$ is a segment of the root locus with phase Ψ_1. We now want to find the force exerted by the remaining loci of **GRL** on L_1. The phase pressure force acting on the surface element ds (which has a unit thickness perpendicular to the complex plane) is pds and is perpendicular to ds. The two components of the phase pressure force dF_σ and dF_ω in the positive $\sigma -$ and $\omega -$ directions are

$$dF_\sigma = -p\sin\theta ds = pd\omega, \quad dF_\omega = p\cos\theta ds = pd\sigma$$

Where θ is the angle that ds makes with the positive $\sigma -$axis. These two differential force components may be written in complex form as

$$dF_\sigma - jdF_\omega = -p\left(d\omega + jd\omega\right) = -jpd\overline{s} \tag{45}$$

in which $d\overline{s}$ $\left(= d\sigma - jd\omega\right)$ is the conjugate of ds. For an irrotational time-invariant system, the pressure p on the surface element ds can be determined as

$$p = H - \rho q^2 / 2 \tag{46}$$

where the constant H may be evaluated from the known conditions at a reference root on L_1 and q is the magnitude of phase flux rate at ds. From Eq.(5) and Eq.(12), we can express q^2 in terms of complex potential function $\Omega(s)$ as

$$q^2 = \left(q_\sigma - jq_\omega\right)\left(q_\sigma + jq_\omega\right) = \frac{d\Omega(s)}{ds}\overline{\left(\frac{d\Omega(s)}{ds}\right)} \tag{47}$$

Substituting Eq.(47) into Eq.(46), yields

$$p = H - \frac{1}{2}\rho \frac{d\Omega(s)}{ds}\overline{\left(\frac{d\Omega(s)}{ds}\right)} \tag{48}$$

so that Eq.(45) becomes

$$dF_{\sigma} - jdF_{\omega} = -jHd\overline{s} + j\frac{1}{2}\rho\frac{d\Omega(s)}{ds}\overline{\left(\frac{d\Omega(s)}{ds}\right)}d\overline{s} \quad (49)$$

Noting that along L_1, $d\Psi = 0$ and $d\Omega = d\overline{\Omega} = d\Phi$, therefore, Eq.(49) can also be written as

$$dF_{\sigma} - jdF_{\omega} = -jHds + j\frac{1}{2}\rho\left(\frac{d\Omega(s)}{ds}\right)^2 ds \quad (50)$$

which, upon integrating along L_1, becomes

$$F_{\sigma} - jF_{\omega} = -jH\Delta s + j\frac{1}{2}\rho\int_L\left(\frac{d\Omega(s)}{ds}\right)^2 ds \quad (51)$$

where Δs is the difference between the starting point and the end point of L_1. When L_1 is a closed path, Eq.(51) reduces to the Blasius formula [3, 5].

$$F_{\sigma} - jF_{\omega} = j\frac{1}{2}\rho\oint_L\left(\frac{d\Omega(s)}{ds}\right)^2 ds \quad (52)$$

which was originally used as a method for determining the force exerted by the fluid on a cylinder of any cross-section shape in a steady two-dimensional potential flow. Given the transfer function $G(s)$ of an irrotational system, we can evaluate the complex potential function $\Omega(s)$ from Eq.(4) and after the substitution of $\Omega(s)$ into Eq.(51) or Eq.(52), we obtain the force action upon any portion of root loci, immediately.

VIII. Conclusions

This paper presents a potential flow formulation of root locus for irrotational time-varying systems and gives a vivid description of root locus in terms of the phenomena of potential flows existing in the real world. Dynamic model and governing equations for root locus are developed and used in the derivation of force action among root loci and used in relating root sensitivity to root robustness. The introduced concept of generalized root locus broadens the view of root locus and gives a clue to the inverse root-locus problem. It is shown that the superposition of elementary generalized root loci can synthesize root locus in control system design and to extend the above results to rotational systems whose transfer functions do not exist.

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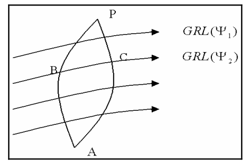

Fig.1. Phase across ABP equals phase across ACP

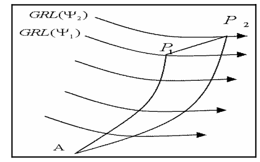

Fig.2. Phase across P_1P_2 is $\Psi_1 - \Psi_2$

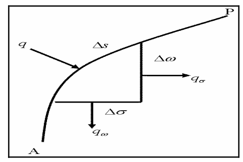

Fig.3 σ and ω components of phase flux rate.

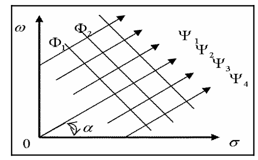

Fig.4 Time-delay GRL are parallel straight lines.

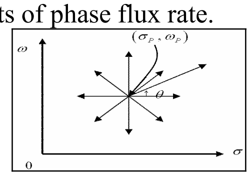

Fig.5 GRL for poles are radial lines.

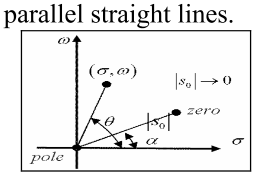

Fig.6 A pole-zero pair.

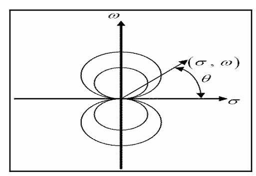

Fig.7. GRL for dipole is a family of circles passing through origin.

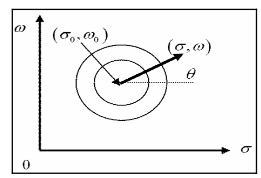

Fig.8.GRL for vortex are circles

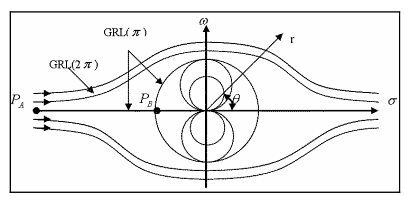

Fig.9. GRL for a delay system with a dipole at origin

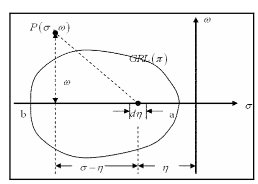

Fig.10 Continuous dipole distribution.

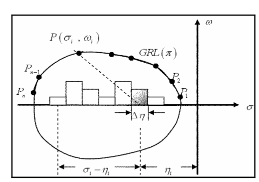

Fig.11 Discrete dipole distribution.

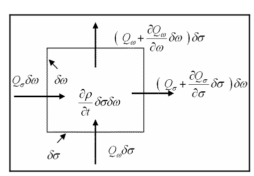

Fig.12 Conservation of phase.

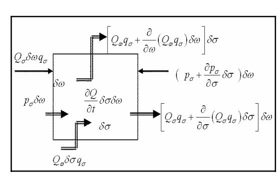

Fig.13 Conservation of phase momentum.