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Further Studies on Stabilizing Input/Output Receding-horizon Control

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Abstract— **In this paper the stability of the well-known stabilizing input/output receding horizon control (SIORHC) is further studied. The results on deadbeat stability and asymptotic stability of SIORHC are improved by using Ackermann's formula and Lyapunov's method respectively. The equivalence conclusion between SIORHC and generalized predictive control (GPC) is also improved.**

I. Introduction

GENERALIZED predictive control (GPC, see [1]) is a popular form of model predictive control (MPC) that has gained widespread acceptance. Because of using finite horizon, stability was not guaranteed in the original version of GPC. This has been overcome after 90s with new versions of GPC. One idea is that the stability of GPC could be guaranteed if in the last part of the prediction horizon the future outputs are constrained at the desired setpoint and the prediction horizon is properly selected. The well-known example for this category is the stabilizing input/output receding horizon control (SIORHC, see [2]), or constrained receding horizon predictive control (CRHPC, see [3]).

This paper further studies the stability properties of SIORHC (or CRHPC). For deadbeat property, [3] obtained some results by using Ackermann's formula. With different method, [2] also obtained some results under certain conditions. In this paper the same idea as in [3] is used to analyze the deadbeat property of SIORHC. For asymptotic stability, [2] and [3] obtained similar results by using different methods. As in most stability analysis of MPC, Lyapunov's method will be chosen to obtain more general results for asymptotic stability of SIORHC. The equivalence conclusion between SIORHC and GPC is also improved in the paper.

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II. Problem Statement

Consider SISO input/output model

$$\bar{a}(z^{-1})y(k)=\bar{b}(z^{-1})u(k-1), \tag{1}$$

where u and y are the input and output; z^{-1} is the unit delay operator; $\bar{a}(z^{-1})=1+\bar{a}_1 z^{-1}+\cdots+\bar{a}_{n_a-1}z^{-n_a+1}$, $\bar{a}_{n_a-1}\neq 0$, $\bar{b}(z^{-1})=b_1+\cdots+b_{n_b}z^{-n_b+1}$, $b_{n_b}\neq 0$. Multiplying both sides of equation (1) by $\Delta=1-z^{-1}$ obtains

$$a(z^{-1})y(k)=b(z^{-1})\Delta u(k), \tag{2}$$

where $a(z^{-1})=\bar{a}(z^{-1})\Delta=1+a_1z^{-1}+\cdots+a_{n_a}z^{-n_a}$, $a_{n_a}\neq 0$, $b(z^{-1})=z^{-1}\bar{b}(z^{-1})=b_1z^{-1}+\cdots+b_{n_b}z^{-n_b}$. For system with delay of d samples, $b_1,\cdots,b_d=0$ and $n_b>d$. Assume that $\{a(z^{-1}),b(z^{-1})\}$ is an irreducible pair. At sampling time k the objective function of SIORHC is

$$J=\sum_{i=N_0}^{N_1-1}q_i\left[y(k+i\,|\,k)-\omega(k+i)\right]^2+\sum_{j=1}^{N_u}\lambda_j\Delta u^2(k+j-1\,|\,k), \tag{3a}$$

$$\text{s.t.}\quad y(k+l\,|\,k)=\omega(k+N_1),\quad l=N_1,\cdots,N_2, \tag{3b}$$

$$\Delta u(k+l-1\,|\,k)=0,\qquad l=N_u+1,\cdots,N_2 \tag{3c}$$

where ω is the setpoint; $y(k+i\,|\,k)$ ($\Delta u(k+i\,|\,k)$) is the future output (input) at time $k+i$, predicted at time k; $q_i\geq 0$ and $\lambda_j\geq 0$ are the weighting factors; N_0, N_1 and N_1, N_2 are the starting and end points of the output optimization horizon and constraint horizon respectively; N_u is the control horizon. For the stability analysis, assume without loss of generality that $\omega=0$.

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Equation (2) can be transformed into the following state space model with minimal canonical form

$$\boldsymbol{x}(k+1)=\boldsymbol{A}\boldsymbol{x}(k)+\boldsymbol{B}\Delta u(k)\,, \tag{4a}$$

$$y(k)=\boldsymbol{C}\boldsymbol{x}(k)\,, \tag{4b}$$

where $\boldsymbol{x}\in R^n$, $n=\max\{n_a,n_b\}$, $\boldsymbol{A}=\begin{bmatrix}-\boldsymbol{\alpha}^T & -a_n\\ \boldsymbol{I}_{n-1} & \boldsymbol{0}\end{bmatrix}$, $\boldsymbol{B}=[1\ 0\ \cdots\ 0]^T$, $\boldsymbol{C}=[b_1\ b_2\ \cdots\ b_n]$, $\boldsymbol{\alpha}^T=[a_1\ a_2\ \cdots\ a_{n-1}]$.

Other *Notations*:
$\boldsymbol{W}_o=[\boldsymbol{C}^T\ \ \boldsymbol{A}^T\boldsymbol{C}^T\ \ (\boldsymbol{A}^T)^{n-1}\boldsymbol{C}^T]^T$.
For $\forall i\geq 1, \boldsymbol{W}_i=[\boldsymbol{A}^{i-1}\boldsymbol{B}\ \cdots\ \boldsymbol{A}\boldsymbol{B}\ \ \boldsymbol{B}]$,
$\Delta\boldsymbol{U}_i(k)=[\Delta u(k),\Delta u(k+1\,|\,k),\cdots,\Delta u(k+i-1\,|\,k)]^T$,
$\boldsymbol{I}_i$ is i-ordered identity matrix.

III. The Dead-beat Properties of SIORHC

In deducing the deadbeat properties of SIORHC, first express (3b) (i.e., $y(k+l\,|\,k)=0$, $l=N_1,\cdots,N_2$) by $\boldsymbol{x}(k+N_1\,|\,k)$ (and, if $N_1<N_u$, $[\Delta u(k+N_1\,|\,k),\cdots,\Delta u(k+N_u-1\,|\,k)]$), then substitute $\boldsymbol{x}(k+N_1\,|\,k)$ by $\boldsymbol{x}(k)$ and $\Delta\boldsymbol{U}_i(k)$, $i=N_u$, n_a or N_1. And then, solve $\Delta u(k)$ as the Ackermann's formula for deadbeat control.

Lemma 1: Under the following conditions the closed-loop system of SIORHC is deadbeat stable:

$$n_a<n_b,\ N_u=n_a,\ N_1\geq n_b,\ N_2-N_1\geq n_a-1\,. \tag{5}$$

Proof: Firstly, since $N_1>N_u$,

$$\boldsymbol{x}(k+N_1\,|\,k)=\boldsymbol{A}^{N_1}\boldsymbol{x}(k)+\boldsymbol{A}^{N_1-N_u}\boldsymbol{W}_{N_u}\Delta\boldsymbol{U}_{N_u}(k)\,. \tag{6}$$

Take a nonsingular transformation to (4), $\bar{\boldsymbol{x}}(k+1)=\bar{\boldsymbol{A}}\bar{\boldsymbol{x}}(k)+\bar{\boldsymbol{B}}\Delta u(k)$, $y(k)=\bar{\boldsymbol{C}}\bar{\boldsymbol{x}}(k)$, where $\bar{\boldsymbol{x}}=[\boldsymbol{x}_0^T,\boldsymbol{x}_1^T]^T$, $\bar{\boldsymbol{A}}=block\ diag\{\boldsymbol{A}_0,\boldsymbol{A}_1\}$, $\bar{\boldsymbol{B}}=[\boldsymbol{B}_0^T\ \boldsymbol{B}_1^T]^T$ and $\bar{\boldsymbol{C}}=[\boldsymbol{C}_0\ \boldsymbol{C}_1]$, with $\boldsymbol{A}_0\in R^{n_a\times n_a}$ nonsingular, all the eigenvalues of $\boldsymbol{A}_1$ zero. Denote $n_b=n_a+p$, then $\boldsymbol{A}_1\in R^{p\times p}$. Since $N_1\geq n_b$ and $N_u=n_a$, $\boldsymbol{A}_1^h=\boldsymbol{0}$, $\forall h\geq N_1-N_u$. Then (6) becomes

$$\begin{bmatrix}\boldsymbol{x}_0(k+N_1\,|\,k)\\ \boldsymbol{x}_1(k+N_1\,|\,k)\end{bmatrix}=\begin{bmatrix}\boldsymbol{A}_0^{N_1} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_0(k)\\ \boldsymbol{x}_1(k)\end{bmatrix}+\begin{bmatrix}\boldsymbol{A}_0^{N_1-1}\boldsymbol{B}_0 & \cdots & \boldsymbol{A}_0^{N_1-N_u}\boldsymbol{B}_0\\ \boldsymbol{0} & \cdots & \boldsymbol{0}\end{bmatrix}\Delta\boldsymbol{U}_{N_u}(k)\,. \tag{7}$$

According to (7), $\boldsymbol{x}_1(k+N_1\,|\,k)=\boldsymbol{0}$ is automatically satisfied, therefore, considering deadbeat control of (4) is equivalent to considering deadbeat control of its subsystem $\{\boldsymbol{A}_0,\boldsymbol{B}_0,\boldsymbol{C}_0\}$. Further, consider $N_2-N_1=n_a-1$, then (3b) becomes

$$\begin{bmatrix}\boldsymbol{C}_0 & \boldsymbol{C}_1\\ \boldsymbol{C}_0\boldsymbol{A}_0 & \boldsymbol{C}_1\boldsymbol{A}_1\\ \vdots & \vdots\\ \boldsymbol{C}_0\boldsymbol{A}_0^{n_a-1} & \boldsymbol{C}_1\boldsymbol{A}_1^{n_a-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_0(k+N_1\,|\,k)\\ \boldsymbol{0}\end{bmatrix}=\boldsymbol{0}\,. \tag{8}$$

Since $(\boldsymbol{A}_0,\boldsymbol{C}_0)$ is observable, imposing (8) is equivalent to letting $\boldsymbol{x}_0(k+N_1\,|\,k)=\boldsymbol{0}$. Then (7) becomes

$$\boldsymbol{0}=\boldsymbol{A}_0^{N_u}\boldsymbol{x}_0(k)+\boldsymbol{W}_{0,N_u}\Delta\boldsymbol{U}_{N_u}(k) \tag{9}$$

where $\boldsymbol{W}_{0,j}=[\boldsymbol{A}_0^{j-1}\boldsymbol{B}_0\ \cdots\ \boldsymbol{A}_0\boldsymbol{B}_0\ \ \boldsymbol{B}_0]$, $\forall j\geq 1$. Similar to [3] (proof of Proposition 3), the optimal control law is given by

$$\Delta u(k)=-\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}\times\begin{bmatrix}\boldsymbol{B}_0 & \boldsymbol{A}_0\boldsymbol{B}_0\cdots & \boldsymbol{A}_0^{N_u-1}\boldsymbol{B}_0\end{bmatrix}^{-1}\boldsymbol{A}_0^{N_u}\boldsymbol{x}_0(k) \tag{10}$$

which, since $N_u=n_a$, is the Ackermann's formula for deadbeat control of $\{\boldsymbol{A}_0,\boldsymbol{B}_0,\boldsymbol{C}_0\}$. □

Lemma 2: Under the following conditions the closed-loop system of SIORHC is deadbeat stable:

$$n_a<n_b,\ N_u\geq n_a,\ N_1=n_b,\ N_2-N_u\geq n_b-1\,. \tag{11}$$

Proof: (a) $N_1\geq N_u$. For $N_u=n_a$, the conclusion follows from Lemma 1. For $N_u>n_a$, take a nonsingular transformation the same as in Lemma 1, then

$$\begin{bmatrix}\boldsymbol{x}_0(k+N_1\,|\,k)\\ \boldsymbol{x}_1(k+N_1\,|\,k)\end{bmatrix}=\begin{bmatrix}\boldsymbol{A}_0^{N_1} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_0(k)\\ \boldsymbol{x}_1(k)\end{bmatrix}+\begin{bmatrix}\boldsymbol{A}_0^{N_1-1}\boldsymbol{B}_0\cdots\boldsymbol{A}_0^{p}\boldsymbol{B}_0 & \boldsymbol{A}_0^{p-1}\boldsymbol{B}_0\cdots\boldsymbol{A}_0^{N_1-N_u}\boldsymbol{B}_0\\ \boldsymbol{0}\ \ \cdots\ \ \boldsymbol{0} & \boldsymbol{A}_1^{p-1}\boldsymbol{B}_1\cdots\boldsymbol{A}_1^{N_1-N_u}\boldsymbol{B}_1\end{bmatrix}\Delta\boldsymbol{U}_{N_u}(k)\,. \tag{12}$$

Assume that $\boldsymbol{A}_1=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}_{p-1}\\ 0 & \boldsymbol{0}\end{bmatrix}$ and $\boldsymbol{B}_1=[0\cdots 0\,1]^T$, then $\begin{bmatrix}\boldsymbol{A}_1^{p-1}\boldsymbol{B}_1\cdots\boldsymbol{A}_1^{N_1-N_u}\boldsymbol{B}_1\end{bmatrix}=\begin{bmatrix}\boldsymbol{I}_{N_u-n_a}\\ \boldsymbol{0}\end{bmatrix}$. Denote $\boldsymbol{x}_1=[\boldsymbol{x}_2^T,\boldsymbol{x}_3^T]^T$ where $\dim\boldsymbol{x}_2=N_u-n_a$ and $\dim\boldsymbol{x}_3=N_1-N_u$. According to (12), $\boldsymbol{x}_3(k+N_1\,|\,k)=\boldsymbol{0}$ is automatically satisfied, therefore, considering deadbeat control of (4) is equivalent to considering deadbeat control of its partial states $[\boldsymbol{x}_0^T, \boldsymbol{x}_2^T]$. Further, assume that $\boldsymbol{C}_1 = [c_{11}\ c_{12} \cdots c_{1p}]$. Consider $N_2 - N_1 = N_u - 1$ (i.e., $N_1 = n_b$ and $N_2 - N_u = n_b - 1$), then (3b) becomes

$$\begin{bmatrix} \boldsymbol{C}_0 & c_{11} & c_{12} & \cdots & c_{1N_u-n_a} & \cdots & \cdots & c_{1p} \\ \boldsymbol{C}_0\boldsymbol{A}_0 & 0 & c_{11} & \cdots & c_{1N_u-n_a-1} & \cdots & \cdots & c_{1p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{C}_0\boldsymbol{A}_0^{N_u-n_a-1} & 0 & 0 & \cdots & c_{11} & \cdots & \cdots & c_{1p-N_u+n_a+1} \\ \boldsymbol{C}_0\boldsymbol{A}_0^{N_u-n_a} & 0 & 0 & \cdots & 0 & c_{11} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}_0\boldsymbol{A}_0^{N_u-1} & 0 & 0 & \cdots & 0 & 0 & \cdots & * \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_0(k+N_1 \mid k) \\ \boldsymbol{x}_2(k+N_1 \mid k) \\ \boldsymbol{0} \end{bmatrix} = \boldsymbol{0}. \quad (13)$$

Since $(\boldsymbol{A}_0, \boldsymbol{C}_0)$ is observable and $c_{11} \neq 0$, $\begin{bmatrix} \boldsymbol{C}_0 & c_{11} & \cdots & c_{1N_u-n_a} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}_0\boldsymbol{A}_0^{N_u-n_a-1} & 0 & \cdots & c_{11} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{C}_0\boldsymbol{A}_0^{N_u-1} & 0 & \cdots & 0 \end{bmatrix}$ is nonsingular. Therefore, imposing (13) is equivalent to letting $[\boldsymbol{x}_0^T(k+N_1 \mid k), \boldsymbol{x}_2^T(k+N_1 \mid k)] = \boldsymbol{0}$. According to (12), $\left[\Delta u(k+n_a \mid k), \cdots, \Delta u(k+N_u-1 \mid k)\right]^T = \boldsymbol{x}_2(k+N_1 \mid k) = \boldsymbol{0}$. Therefore, (12) becomes

$$\boldsymbol{0} = \boldsymbol{A}_0^{n_a}\boldsymbol{x}_0(k) + \boldsymbol{W}_{0,n_a}\Delta\boldsymbol{U}_{n_a}(k). \quad (14)$$

The optimal control law is given by

$$\Delta u(k) = -\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{B}_0 & \boldsymbol{A}_0\boldsymbol{B}_0 & \cdots & \boldsymbol{A}_0^{n_a-1}\boldsymbol{B}_0 \end{bmatrix}^{-1} \boldsymbol{A}_0^{n_a}\boldsymbol{x}_0(k) \quad (15)$$

which is the Ackermann's formula for deadbeat control of $\{\boldsymbol{A}_0, \boldsymbol{B}_0, \boldsymbol{C}_0\}$ and induces deadbeat control of (4).

(b) $N_1 < N_u$. Firstly,

$$\boldsymbol{x}(k+N_1 \mid k) = \boldsymbol{A}^{N_1}\boldsymbol{x}(k) + \boldsymbol{W}_{N_1}\Delta\boldsymbol{U}_{N_1}(k). \quad (16)$$

Since $N_1 = n$ and $N_2 - N_u \geq n-1$, $N_2 - N_1 \geq n + N_u - N_1 - 1$. Consider $N_2 - N_1 = n + N_u - N_1 - 1$, then (3b) becomes

$$\begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \vdots \\ \boldsymbol{CA}^{N_u-N_1} \\ \vdots \\ \boldsymbol{CA}^{n+N_u-N_1-1} \end{bmatrix} \boldsymbol{x}(k+N_1 \mid k) + \begin{bmatrix} 0 & \cdots & 0 \\ \boldsymbol{CB} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \boldsymbol{CA}^{N_u-N_1-1}\boldsymbol{B} & \cdots & \boldsymbol{CB} \\ \vdots & \ddots & \vdots \\ \boldsymbol{CA}^{n+N_u-N_1-2}\boldsymbol{B} & \cdots & \boldsymbol{CA}^{n-1}\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \Delta u(k+N_1 \mid k) \\ \Delta u(k+N_1+1 \mid k) \\ \vdots \\ \Delta u(k+N_u-1 \mid k) \end{bmatrix} = \boldsymbol{0}. \quad (17)$$

Substituting (16) into (17) obtains

$$\begin{bmatrix} \boldsymbol{W}_o \\ \boldsymbol{CA}^{N_1} \\ \vdots \\ \boldsymbol{CA}^{N_u-1} \end{bmatrix} \boldsymbol{A}^{N_1}\boldsymbol{x}(k) + \begin{bmatrix} \boldsymbol{W}_o\boldsymbol{W}_{N_1} & \boldsymbol{G}_1 \\ \begin{bmatrix} \boldsymbol{CA}^{N_1} \\ \vdots \\ \boldsymbol{CA}^{N_u-1} \end{bmatrix} \boldsymbol{W}_{N_1} & \boldsymbol{G}_2 \end{bmatrix} \Delta\boldsymbol{U}_{N_u}(k) = \boldsymbol{0} \quad (18)$$

where $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are matrices of the corresponding parts in (17). Denote J as

$$\begin{aligned} J &= \sum_{i=N_0}^{N_1-1} q_i y(k+i \mid k)^2 + \sum_{j=1}^{N_1} \lambda_j \Delta u^2(k+j-1 \mid k) \\ &\quad + \sum_{j=N_1+1}^{N_u} \lambda_j \Delta u^2(k+j-1 \mid k) \\ &= J_1 + \sum_{j=N_1+1}^{N_u} \lambda_j \Delta u^2(k+j-1 \mid k) = J_1 + J_2. \end{aligned} \quad (19)$$

By optimality principle, $\min J \geq \min J_1 + \min J_2 \geq \min J_1$, $\left[\Delta u(k+N_1 \mid k), \cdots, \Delta u(k+N_u-1 \mid k)\right] = \boldsymbol{0}$ is the best choice for minimizing (19). By this choice, on the other hand, (18) is simplified as $\boldsymbol{W}_o\boldsymbol{A}^{N_1}\boldsymbol{x}(k) + \boldsymbol{W}_o\boldsymbol{W}_{N_1}\Delta\boldsymbol{U}_{N_1}(k) = \boldsymbol{0}$. Hence, the optimal control law is given by

$$\Delta u(k) = -\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{N_1-1}\boldsymbol{B} \end{bmatrix}^{-1} \boldsymbol{A}^{N_1}\boldsymbol{x}(k) \quad (20)$$

which, since $N_1 = n_b = n$, is the Ackermann's formula for deadbeat control of (4). □

Lemma 3: Under the following conditions the closed-loop system of SIORHC is deadbeat stable:

$$n_a > n_b,\ N_u \ge n_a,\ N_1 = n_b,\ N_2 - N_u \ge n_b - 1. \quad (21)$$

Proof: (a) $N_u = n_a$. Firstly, since $N_1 < N_u$,

$$\boldsymbol{x}(k+N_1 \mid k) = \boldsymbol{A}^{N_1}\boldsymbol{x}(k) + \boldsymbol{W}_{N_1}\Delta\boldsymbol{U}_{N_1}(k). \quad (22)$$

Since $N_1 = n_b$ and $N_2 - N_u \ge n_b - 1$, $N_2 - N_1 \ge N_u - 1 = n - 1$. Consider $N_2 - N_1 = N_u - 1$, then (3b) becomes

$$\begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \vdots \\ \boldsymbol{CA}^{N_u - N_1} \\ \vdots \\ \boldsymbol{CA}^{N_u - 1} \end{bmatrix} \boldsymbol{x}(k+N_1 \mid k) + \begin{bmatrix} 0 & \cdots & 0 \\ \boldsymbol{CB} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \boldsymbol{CA}^{N_u - N_1 - 1}\boldsymbol{B} & \cdots & \boldsymbol{CB} \\ \vdots & \ddots & \vdots \\ \boldsymbol{CA}^{N_u - 2}\boldsymbol{B} & \cdots & \boldsymbol{CA}^{N_1 - 1}\boldsymbol{B} \end{bmatrix} \times \begin{bmatrix} \Delta u(k+N_1 \mid k) \\ \Delta u(k+N_1+1 \mid k) \\ \vdots \\ \Delta u(k+N_u-1 \mid k) \end{bmatrix} = \boldsymbol{0}. \quad (23)$$

Denote $q = n_a - n_b$. Since the last q elements of $\boldsymbol{C}$ are zeros, by the special forms of $\boldsymbol{A}$ and $\boldsymbol{B}$, it is easy to conclude that $\boldsymbol{CA}^{-h}\boldsymbol{B} = 0$, $\forall h = 1, 2, \cdots, q$. Therefore, (23) can be re-expressed as

$$\begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \vdots \\ \boldsymbol{CA}^{N_u - 1} \end{bmatrix} \boldsymbol{x}(k+N_1 \mid k) + \begin{bmatrix} \boldsymbol{CA}^{-1}\boldsymbol{B} & \cdots & \boldsymbol{CA}^{-N_u + N_1}\boldsymbol{B} \\ \boldsymbol{CB} & \cdots & \boldsymbol{CA}^{-N_u + N_1 + 1}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ \boldsymbol{CA}^{N_u - 2}\boldsymbol{B} & \cdots & \boldsymbol{CA}^{N_1 - 1}\boldsymbol{B} \end{bmatrix} \times \begin{bmatrix} \Delta u(k+N_1 \mid k) \\ \Delta u(k+N_1+1 \mid k) \\ \vdots \\ \Delta u(k+N_u-1 \mid k) \end{bmatrix} = \boldsymbol{0}. \quad (24)$$

Considering Cayley-Hamilton's Theorem [4], since $\boldsymbol{A}$ is nonsingular, for any integer j, $\begin{bmatrix} \boldsymbol{CA}^{N_u - 1 + j} & \boldsymbol{CA}^{N_u - 2 + j}\boldsymbol{B} & \cdots & \boldsymbol{CA}^{N_1 - 1 + j}\boldsymbol{B} \end{bmatrix}$ can be represented as a linear combination of the rows in $\begin{bmatrix} \boldsymbol{C} & \boldsymbol{CA}^{-1}\boldsymbol{B} & \cdots & \boldsymbol{CA}^{-N_u + N_1}\boldsymbol{B} \\ \boldsymbol{CA} & \boldsymbol{CB} & \cdots & \boldsymbol{CA}^{-N_u + N_1 + 1}\boldsymbol{B} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{CA}^{N_u - 1} & \boldsymbol{CA}^{N_u - 2}\boldsymbol{B} & \cdots & \boldsymbol{CA}^{N_1 - 1}\boldsymbol{B} \end{bmatrix}$. Therefore, (24) is equivalent to

$$\begin{bmatrix} \boldsymbol{CA}^{N_u - N_1} \\ \boldsymbol{CA}^{N_u - N_1 + 1} \\ \vdots \\ \boldsymbol{CA}^{n + N_u - N_1 - 1} \end{bmatrix} \boldsymbol{x}(k+N_1 \mid k) + \begin{bmatrix} \boldsymbol{CA}^{N_u - N_1 - 1}\boldsymbol{B} & \cdots & \boldsymbol{CAB} & \boldsymbol{CB} \\ \boldsymbol{CA}^{N_u - N_1}\boldsymbol{B} & \cdots & \boldsymbol{CA}^2\boldsymbol{B} & \boldsymbol{CAB} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{CA}^{n + N_u - N_1 - 2}\boldsymbol{B} & \cdots & \boldsymbol{CA}^n\boldsymbol{B} & \boldsymbol{CA}^{n-1}\boldsymbol{B} \end{bmatrix} \times \begin{bmatrix} \Delta u(k+N_1 \mid k) \\ \Delta u(k+N_1+1 \mid k) \\ \vdots \\ \Delta u(k+N_u-1 \mid k) \end{bmatrix} = \boldsymbol{0}. \quad (25)$$

Substituting (22) into (25) obtains

$$\boldsymbol{W}_o\boldsymbol{A}^{N_u}\boldsymbol{x}(k) + \boldsymbol{W}_o\boldsymbol{W}_{N_u}\Delta\boldsymbol{U}_{N_u}(k) = \boldsymbol{0}. \quad (26)$$

Hence, the optimal control law is given by

$$\Delta u(k) = -\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} \cdots & \boldsymbol{A}^{N_u - 1}\boldsymbol{B} \end{bmatrix}^{-1} \boldsymbol{A}^{N_u}\boldsymbol{x}(k) \quad (27)$$

which, since $N_u = n_a = n$, is the Ackermann's formula for deadbeat control of (4).

(b) $N_u > n_a$. By the reason same as in Lemma 2 (b), it is best that $[\Delta u(k+n_a \mid k), \cdots, \Delta u(k+N_u-1 \mid k)] = \boldsymbol{0}$. Hence, the same conclusion can be obtained. □

Lemma 4: Under the following conditions the closed-loop system of SIORHC is deadbeat stable:

$$n_a > n_b,\ N_u = n_a,\ N_1 \ge n_b,\ N_2 - N_1 \ge n_a - 1. \quad (28)$$

Proof: (a) $N_1 \ge N_u$. Firstly,

$$\boldsymbol{x}(k+N_1 \mid k) = \boldsymbol{A}^{N_1}\boldsymbol{x}(k) + \boldsymbol{A}^{N_1 - N_u}\boldsymbol{W}_{N_u}\Delta\boldsymbol{U}_{N_u}(k). \quad (29)$$

Similar to Lemma 1, since $(\boldsymbol{A}, \boldsymbol{C})$ is observable, choosing $N_2 - N_1 \ge n_a - 1 = n - 1$ is equivalent to letting $\boldsymbol{x}(k+N_1 \mid k) = \boldsymbol{0}$. Then, because $\boldsymbol{A}$ is nonsingular, the optimal control law is given by

$$\Delta u(k) = -\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{N_u - 1}\boldsymbol{B} \end{bmatrix}^{-1} \boldsymbol{A}^{N_u}\boldsymbol{x}(k) \quad (30)$$

which, since $N_u = n_a = n$, is the Ackermann's formula for deadbeat control of (4).

(b) $N_1 < N_u$. For $N_1 = n_b$, the conclusion follows from Lemma 3 (a). For $N_1 > n_b$, by similar reason and deduction, (3b) is equivalent to (25) which induces the Ackermann's formula for deadbeat control of (4). □

Considering Lemma 1-Lemma 4 and the conclusion in [3], the following result is obtained.

Theorem 1: Under either of the following two conditions the closed-loop system of SIORHC is deadbeat stable:

$$\text{(i) } N_u = n_a,\ N_1 \ge n_b,\ N_2 - N_1 \ge n_a - 1; \tag{31a}$$

$$\text{(ii) } N_u \ge n_a,\ N_1 = n_b,\ N_2 - N_u \ge n_b - 1. \tag{31b}$$

Remark 1: However, for simplest computation, the parameters for deadbeat control should be chosen as:

$$N_u = n_a,\ N_1 = n_b,\ N_2 = n_a + n_b - 1. \tag{32}$$

This result generalizes the conclusion of [3], where only the case of $n_a = n_b$ was discussed. For systems with d samples delay, denote $\tilde{n} = \max\{n_a, n_b - d\}$ and $\hat{n} = \tilde{n} + d$, then the deadbeat conditions in [2] can be expressed as:

$$N_u = \tilde{n},\ N_1 = \hat{n},\ N_2 = \tilde{n} + \hat{n} - 1. \tag{33}$$

For the special case $n_a = n_b - d$, (32) is equivalent to (33).

Remark 2: For the objective function of GPC [1]

$$J = \sum_{i=N_1}^{N_2} \left[y(k+i \mid k) - \omega(k+i) \right]^2 + \sum_{j=1}^{N_u} \lambda \Delta u^2 (k+j-1 \mid k), \tag{34a}$$

$$\text{s.t. } \Delta u(k+l-1 \mid k) = 0,\ \ l = N_u + 1, \cdots, N_2, \tag{34b}$$

the deadbeat condition with $\lambda = 0$ has been achieved and is same as (31) [5]. With deadbeat control, the output of system (1) will reach the setpoint in n_b samples by changing input n_a times. This is the quickest response that system (1) can achieve. Also, at this speed it is the unique response. Therefore, it can be readily stated that, for $\lambda = 0$, SIORHC and GPC are equivalent under (31). This equivalence conclusion also improves that in [2], i.e., condition (33).

IV. Further Stability Properties of SIORHC

In this section, the asymptotic stability property of SIORHC is mainly studied.

Theorem 2: Under the following conditions the closed-loop system of SIORHC is stable, irrespective of the choices for N_0, $\lambda_i \ge 0$ and $q_i \ge 0$:

$$N_u \ge n_a,\ N_1 \ge n_b,\ N_2 - N_u \ge n_b - 1,\ N_2 - N_1 \ge n_a - 1. \tag{35}$$

Proof: By Theorem 1, since $N_u \ge n_a$, $N_1 \ge n_b$, there exists feasible solution in the optimization. At time k, let $\Delta \boldsymbol{U}_{N_u}^*(k) = [\Delta u^*(k), \Delta u^*(k+1 \mid k), \cdots, \Delta u^*(k+N_u-1 \mid k)]^T$ be the optimal solution, resulting in optimal predictions $y^*(k+1 \mid k), \cdots, y^*(k+N_2 \mid k)$ and an optimal cost $J^*(k)$. Since $N_2 - N_1 \ge n_a - 1$, $N_2 - N_u \ge n_b - 1$ and $y^*(k+l \mid k) = 0$, $\forall l = N_1, \cdots, N_2$,

$$\begin{aligned} & y^*(k+N_2 \mid k) \\ & = -\bar{a}_1 y^*(k+N_2-1 \mid k) - \cdots - \bar{a}_{n_a-1} y^*(k+N_2-n_a+1 \mid k) \\ & \quad + b_1 u^*(k+N_2-1 \mid k) + \cdots + b_{n_b} u^*(k+N_2-n_b \mid k) \\ & = \left(b_1 + b_2 + \cdots + b_{n_b} \right) u^*(k+N_u-1 \mid k) \\ & = 0. \end{aligned}$$

Since $b_1 + b_2 + \cdots + b_{n_b} \ne 0$, $u^*(k+N_u-1 \mid k) = 0$. Further, $u^*(k+l-1 \mid k) = 0$, $\forall l = N_u, \cdots, N_2$. At time $k+1$, if the control profile $\Delta \boldsymbol{U}_{N_u}(k+1) = [\Delta u^*(k+1 \mid k), \cdots, \Delta u^*(k+N_u-1 \mid k), 0]^T$ is applied and $\Delta u(k+l \mid k+1) = 0$, $\forall l = N_u+1, \cdots, N_2$, then $u(k+N_2 \mid k+1) = 0$ and $y(k+l+1 \mid k+1) = 0$, $\forall l = N_1, \cdots, N_2 - 1$. Hence,

$$\begin{aligned} & y(k+N_2+1 \mid k+1) \\ & = -\bar{a}_1 y(k+N_2 \mid k+1) - \cdots - \bar{a}_{n_a-1} y(k+N_2-n_a+2 \mid k+1) \\ & \quad + b_1 u(k+N_2 \mid k+1) + \cdots + b_{n_b} u(k+N_2-n_b+1 \mid k+1) \\ & = b_1 u(k+N_2 \mid k+1) + b_2 u^*(k+N_2-1 \mid k) \\ & \quad + \cdots + b_{n_b} u^*(k+N_2-n_b+1 \mid k) \\ & = 0. \end{aligned}$$

This means that $\Delta \boldsymbol{U}_{N_u}(k+1)$ is feasible at time $k+1$. Denote $J(k+1)$ the cost corresponding to $\Delta \boldsymbol{U}_{N_u}(k+1)$, then the successive proof is similar to [6] (section 5.1). □

Remark 3: Apparently the deadbeat condition (31) is contained in (35). The asymptotic stability conditions can be obtained by eliminating (31) from (35):

$$N_u \ge n_a + 1,\ N_1 \ge n_b + 1,\ N_2 - N_u \ge n_b - 1,\ N_2 - N_1 \ge n_a - 1. \tag{36}$$

For the special case of $d = p$, and with special cost weightings, [3] have actually obtained the following asymptotic stability conditions:

$$N_u \geq n_a + 2\,,\ N_1 \geq n_b + 2\,,\ N_2 - N_u = n_b - 1\,,\ N_2 - N_1 = n_a - 1\,. \tag{37}$$

Remark 4: With certain special conditions firstly satisfied, [2] have obtained the following stability conditions:

$$N_u \geq \tilde{n}\,,\ N_1 \geq \hat{n}\,,\ N_2 - N_u = \hat{n} - 1\,,\ N_2 - N_1 = \tilde{n} - 1\,. \tag{38}$$

With deadbeat conditions removed, the asymptotic stability conditions in (38) are close to that in [3].

V. Conclusion

This paper considers predictive control of linear systems under terminal equality constraints. Specially, the equivalence with deadbeat control is studied, with the minimum feasible output optimization horizon, constraint horizon and control horizon. The stability results are suitable to systems with real constraints such as input, output constraint etc., in the sense that "the closed-loop system of SIORHC is asymptotically stable if and only if the optimization problem is feasible at the initial time" [7].

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