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Self-bounded controlled invariant subspaces in model following by output feedback: minimal-order solution for nonminimum-phase systems

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Abstract— **Self-bounded controlled invariant subspaces play a key role in the synthesis of minimal-order dynamic regulators attaining model following by output feedback with stability. The approach, completely embedded in the geometric context, provides insight into the internal eigenstructure of the minimal self-bounded controlled invariant subspace, thus leading to effective treatment of nonminimum-phase systems.**

I. INTRODUCTION

The main contribution of this paper consists in detailing a procedure to synthesize a minimal-order regulator for model following by output feedback with stability, completely in the geometric approach context [1], [2]: in fact, after the pioneering work [3], just few papers can be found in the literature, considering model matching from the geometric point of view. In this paper, model following by dynamic feedforward is considered first and, since it can be reduced to an equivalent problem of measurable signal decoupling, connections between structural and stabilizability conditions for measurable signal decoupling and structural and stabilizability properties of the system and the model are set forth. Theorem 2 relates the structural condition for measurable signal decoupling to a relative-degree condition on the system and the model. This latter involves, along with the concept of vector relative degree, an original concept herein called vector minimum delay (they both are characterized by computational algorithms). Theorems 3, 4 relate the stabilizability condition for measurable signal decoupling to the invariant zero structure of the system and the eigenstructure of the model: they exploit the properties of self-bounded controlled invariant subspaces, herein for the first time considered within model following. Since Theorems 2, 4 state sufficient conditions for structural model following and model following with stability, respectively, they should also be regarded as guidelines to define an admissible model for a given system in a non-conventional model following problem where the designer may intervene on the model itself. The subsequent Theorems 5, 6, 7 give additional insight into the internal eigenstructure of the minimal self-bounded, through the concept of generalized frequency response, and suggest a straightforward procedure to deal with nonminimum-phase systems. In the last section, it is shown how output feedback model following can be reduced, from the structural point of view, to an equivalent feedforward problem (Theorem 8) and how the synthesis carried out according to the criteria previously considered guarantees also internal stability of the closed loop (Theorems 9, 10).

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II. SELF-BOUNDED CONTROLLED INVARIANTS IN MODEL FOLLOWING: MINIMAL-ORDER FEEDFORWARD SOLUTION

According to a procedure well-settled in the geometric approach, the model following problem is reduced to an equivalent problem of measurable signal decoupling [3]. Hence, a feedforward solution is herein considered, on the assumption that the given system is stable. This hypothesis does not cause any loss of generality with respect to that of stabilizability, usually introduced: if the system is stabilizable, it can be considered as prestabilized by an inner feedback. The discrete time-invariant linear system

$$x_s(t+1) = A_s\, x_s(t) + B_s\, u(t), \tag{1}$$
$$y_s(t) = C_s\, x_s(t), \tag{2}$$

is considered, where $x \in \mathcal{X}_s = \mathbb{R}^{n_s}$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ respectively denote the state, the control input, the controlled output. The system is assumed to be stable. The set of all admissible control input functions is defined as the set $\mathcal{U}_f$ of all bounded functions with values in $\mathbb{R}^p$. The discrete time-invariant linear model

$$x_m(t+1) = A_m\, x_m(t) + B_m\, h(t), \tag{3}$$
$$y_m(t) = C_m\, x_m(t), \tag{4}$$

is considered, where $x \in \mathcal{X}_m = \mathbb{R}^{n_m}$, $h \in \mathbb{R}^s$, $y \in \mathbb{R}^q$ respectively denote the state, the exogenous input, the measurable output. Also the model is assumed to be stable. The set of all admissible exogenous input functions is defined as the set $\mathcal{H}_f$ of all bounded functions with values in $\mathbb{R}^s$. The matrices B_s, B_m, C_s, C_m are assumed to be full-rank. The symbols $\mathcal{B}_s$, $\mathcal{B}_m$, $\mathcal{C}_s$, $\mathcal{C}_m$ are respectively used for $\operatorname{im} B_s$, $\operatorname{im} B_m$, $\ker C_s$, $\ker C_m$.

Problem 1: model following by minimal-order dynamic feedforward. Refer to Fig. 1. Let Σ_s be ruled by (1), (2), with $x_s(0) = 0$. Let Σ_m be ruled by (3), (4), with $x_m(0) = 0$. Let $\sigma(A_s) \subset \mathbb{C}^{\odot}$ and $\sigma(A_m) \subset \mathbb{C}^{\odot}$. Design a linear dynamic feedforward compensator $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$ of minimal order, such that $\sigma(A_c) \subset \mathbb{C}^{\odot}$ and, for all admissible $h(t)$ $(t \geq 0)$, $y_s(t) = y_m(t)$ for all $t \geq 0$.

Theorem 1: Problem 1 is equivalent to Problem 4 in Appendix, with $A = \mathrm{diag}\{A_s,\, A_m\}$, $B = \left[\, B_s^\top \;\, O \,\right]^\top$, $H = \left[\, O \;\, B_m^\top \,\right]^\top$, and $C = [\, C_s \; -C_m \,]$.

Proof: Refer to Fig. 1. Set $x(t) = \left[\, x_s(t)^\top \;\, x_m(t)^\top \,\right]^\top$, $y(t) = y_s(t) - y_m(t)$. The statement directly follows from the comparison of (1), (2) and (3), (4) with (10), (11). ■

In view of Theorem 1, the dynamic feedforward compensator Σ_c designed according to the procedure detailed in [4] preserves the features illustrated therein: the minimum number of internal unassignable dynamics, in particular. Moreover, Theorem 1 enables connections to be established between the necessary and sufficient condition for measurable signal decoupling with stability (Theorem 11) and the geometric properties of the original system and model. This investigation is particularly useful from a technical point of view, since it provides easy-to-check conditions to verify solvability of the considered model following problem and suggests how to modify a problem which is originally not solvable, in order to achieve a feasible and satisfactory trade-off. The following results focus on these issues.

A. Structural condition

In this section it is shown that, if the system is right-invertible and the model is reachable, a straightforward relation established between a pair of easy-to-compute vectors in the model following problem, namely the vector relative degree of the system and the vector minimum delay of the model, implies that the structural condition of the measurable signal decoupling problem, i.e. $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$, holds. The assumption of right-invertibility of the system is not usually required in model matching. However, its introduction, in the next Theorem 2, is rewarded by the statement of a particularly simple condition, which, differently from those available in the literature, focuses on the structural properties of the system and the model considered separately, instead of referring to some combination of them. The following definitions and properties are stated for a generic discrete time-invariant linear system

$$x(t+1) \;=\; A\,x(t) + B\,u(t) \qquad (5)$$

$$y(t) \;=\; C\,x(t), \qquad (6)$$

where $x \in \mathcal{X} = \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ respectively denote the state, the control input, the controlled output, and where the matrices B, C are assumed to be full-rank. The symbols $\mathcal{B}$, $\mathcal{C}$ stand for $\mathrm{im}\, B$, $\ker C$. The symbol $\mathcal{U}_f$ denotes the set of all admissible control input functions, defined as the set

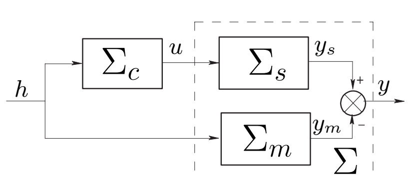

Fig. 1. Block diagram for feedforward model following.

of all bounded functions with values in $\mathbb{R}^p$. The symbol $\mathcal{I}_q$ stands for the set $\{\, i \in \mathbb{Z}^+ \colon\; 1 \le i \le q \,\}$.

Definition 1: Consider the system (5), (6) with $x(0) = 0$. Let (A, B, C) be right-invertible. The vector relative degree is the vector $\rho = [\, \rho_1 \ldots \rho_q \,]^\top$, where

$$\rho_i = \min_{u(\cdot)\,\in\,\mathcal{U}_f} \left\{ \bar t \in \mathbb{Z}^+ \colon\; y_i(\bar t) \neq 0,\;\; y_i(t) = 0,\; \forall\, t < \bar t, \right. \\ \left. y_j(t) = 0,\; \forall\, t \ge 0,\; j \in \mathcal{I}_q,\; j \neq i \right\},\; i \in \mathcal{I}_q.$$

Property 1: Consider the system (5), (6) with (A, B, C) right-invertible. For any $i \in \mathcal{I}_q$, let C_i denote the ith row of C, $\mathcal{C}_i = \ker C_i$, $\bar{\mathcal{C}}_i = \cap_{j\,\in\,\mathcal{I}_q,\, j \neq i}\, \mathcal{C}_j$, $\bar{\mathcal{V}}_i^* = \max \mathcal{V}(A, \mathcal{B}, \bar{\mathcal{C}}_i)$, $\bar{\mathcal{R}}_i^{(1)} = \mathcal{B} \cap \bar{\mathcal{V}}_i^*$, $\bar{\mathcal{R}}_i^{(\eta)} = (A\,(\bar{\mathcal{R}}_i^{(\eta-1)} \cap \bar{\mathcal{V}}_i^*) + \mathcal{B}) \cap \bar{\mathcal{V}}_i^*$, with $\eta = 2, \ldots, k_i$, where $k_i \le n$ is the least integer such that $\bar{\mathcal{R}}_i^{(k_i+1)} = \bar{\mathcal{R}}_i^{(k_i)}$. Then, for any $i \in \mathcal{I}_q$, ρ_i is the least integer such that $C_i\, \bar{\mathcal{R}}_i^{(\rho_i)} \neq 0$.

Proof: The statement follows from Definition 1 and the properties of the ith constrained reachable subspace on $\bar{\mathcal{V}}_i^*$, i.e. the subspace $\mathcal{R}_{\bar{\mathcal{V}}_i^*} = \bar{\mathcal{V}}_i^* \cap \min \mathcal{S}(A, \bar{\mathcal{V}}_i^*, \mathcal{B})$, which the considered sequence converges to. ■

The right-invertibility assumption guarantees that, for all $i \in \mathcal{I}_q$, $\rho_i \in \{1, \ldots, k_i\}$ exists, such that $C_i\, \bar{\mathcal{R}}_i^{(\rho_i)} \neq 0$.

Definition 2: Consider the system (5), (6) with $x(0) = 0$. Let (A, B) be reachable. The vector minimum delay is the vector $\delta = [\, \delta_1 \ldots \delta_q \,]^\top$, where

$$\delta_i = \min_{u(\cdot)\,\in\,\mathcal{U}_f} \left\{\, \bar t \in \mathbb{Z}^+ : y_i(\bar t) \neq 0,\;\; y_i(t) = 0,\; \forall\, t < \bar t \,\right\},\; i \in \mathcal{I}_q.$$

Property 2: Consider the system (5), (6). Let (A, B) be reachable. For any $i \in \mathcal{I}_q$, let C_i denote the ith row of C. Let $\mathcal{R}^{(1)} = \mathcal{B}$, $\mathcal{R}^{(\eta)} = A\,\mathcal{R}^{(\eta-1)} + \mathcal{B}$, $\eta = 2, \ldots, k$, where $k \le n$ is the least integer such that $\mathcal{R}^{(k+1)} = \mathcal{R}^{(k)}$. Then, δ_i is the least integer such that $C_i\, \mathcal{R}^{(\delta_i)} \neq 0$.

Proof: The property follows from Definition 2 and the properties of the subspace $\mathcal{R} = \min \mathcal{J}(A, \mathcal{B})$, which the considered sequence converges to. ■

The reachability assumption guarantees that, for all $i \in \mathcal{I}_q$, $\delta_i \in \{1, \ldots, k\}$ exists, such that $C_i\, \mathcal{R}^{(\delta_i)} \neq 0$.

Theorem 2: Consider the system (1), (2) and the model (3), (4). Let (A_s, B_s, C_s) be right-invertible and (A_m, B_m) be reachable. Consider the system (10), (11), defined according to Theorem 1. Let δ_m denote the vector minimum delay of the model and ρ_s the vector relative degree of the system. Then, $\delta_m \ge \rho_s \implies \mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$.

Proof: For any $i \in \mathcal{I}_q$, let $\delta_{m,i} \ge \rho_{s,i}$. For any $h(\cdot) \in \mathcal{H}_f$, consider the corresponding effect, with the initial condition $x_m(0) = 0$, at the ith component of the output y_m. By Definition 2, for any $i \in \mathcal{I}_q$, $\bar t_i \ge \delta_{m,i}$ exists, such that $y_{m,i}(\bar t_i) \neq 0$ and $y_{m,i}(t) = 0$ for all $t < \bar t_i$. Due to functional controllability of (A_s, B_s, C_s), if $\delta_{m,i} \ge \rho_{m,i}$, then $u_i(\cdot) \in \mathcal{U}_f$ exists, such that, with the initial condition $x_s(0) = 0$, $y_{s,i}(t) = y_{m,i}(t)$ for all $t \ge \bar t_i$, $y_{s,i}(t) = 0$, for all $t < \bar t_i$, and $y_{s,j}(t) = 0$, for all $t \ge 0$, with $j \in \mathcal{I}_q$, $j \neq i$. Consequently, by superposition, for any input function $h(\cdot) \in \mathcal{H}_f$, which, with $x_m(0) = 0$, produces a certain output $y_m(t)$, $t \ge 0$, a control function $u(\cdot) \in \mathcal{U}_f$ exists, such that $y_s(t) = y_m(t)$, for all $t \ge 0$. In the equivalent measurable signal decoupling problem, this means that for any $h(\cdot)\in\mathcal{H}_f$, $u(\cdot)\in\mathcal{U}_f$ exists, such that $y(t)=0$, for all $t\geq 0$. In other words, for any $h(\cdot)\in\mathcal{H}_f$, $u(\cdot)\in\mathcal{U}_f$ exists, such that the corresponding state trajectory, $x(t)$, $t\geq 0$, starting from $x(0)=0$, is steered on an $(A,\mathcal{B})$-controlled invariant, say $\mathcal{V}$, such that $\mathcal{V}\subseteq\mathcal{C}$ and $\mathcal{H}\subseteq\mathcal{V}+\mathcal{B}$. Finally, since $\mathcal{V}\subseteq\mathcal{V}^*$, the latter inclusion implies $\mathcal{H}\subseteq\mathcal{V}^*+\mathcal{B}$. ■

B. Stabilizability condition

In this section it is shown that, on the assumption that the structural condition of the equivalent measurable signal decoupling problem holds, the stabilizability condition, namely internal stabilizability of the subspace $\mathcal{V}_m$, the minimal $(A,\mathcal{B}+\mathcal{H})$-controlled invariant self-bounded with respect to $\mathcal{C}$ such that $\mathcal{H}\subseteq\mathcal{V}_m+\mathcal{B}$, is implied by a straightforward condition involving the invariant zeros of the plant and the poles of the model. In Theorem 3, as well as in Theorem 2, the given system is assumed to be right-invertible. Introducing this requirement, which, as already pointed out, is non-standard in model following, returns a direct and easy-to-check condition which provides insight into the way poles and zeros of the system and the model act in the extended system and suggests how to modify the model when the equivalent measurable signal decoupling problem is not solvable with stability.

Property 3: Consider the systems (1), (2), (3), (4), (10), (11), where (10), (11) is defined as in Theorem 1. Let $\mathcal{S}_s^*=\min\mathcal{S}(A_s,\mathcal{C}_s,\mathcal{B}_s)$, $\mathcal{S}^*=\min\mathcal{S}(A,\mathcal{C},\mathcal{B})$. Let S_s^* be a basis matrix of $\mathcal{S}_s^*$. Then, $\mathcal{S}^*=\operatorname{im}[S_s^{*\top}\ O]^\top$.

Proof: First, recall that $\mathcal{S}^*$ is the last term of the sequence $\mathcal{S}^{(1)}=\mathcal{B}$, $\mathcal{S}^{(i)}=A(\mathcal{S}^{(i-1)}\cap\mathcal{C})+\mathcal{B}$, $i=2,\ldots,k$, where k is the least integer such that $\mathcal{S}^{(k+1)}=\mathcal{S}^{(k)}$. Note that, due to the particular structure of C, $\mathcal{C}=\operatorname{im}\begin{bmatrix}\bar{C}_s & C_1\\ O & C_2\end{bmatrix}$, where $\bar{C}_s$ is a basis matrix of $\mathcal{C}_s$, and both C_1 and C_2 are non-zero matrices. Hence, due to the structures of A and B, the last n_m rows of the basis matrices of the subspaces subsequently generated by the algorithm of $\mathcal{S}^*$ are zero. Moreover, the first n_s rows are the basis matrices of the subspaces generated by the same algorithm applied to the triple (A_s,B_s,C_s), the algorithm of $\mathcal{S}_s^*$. ■

Property 4: Consider the systems (1), (2), (3), (4), (10), (11), where (10), (11) is defined as in Theorem 1. Let (A_s,B_s,C_s) be right-invertible. Then, (A,B,C) is right-invertible.

Proof: Note that, since C_s and C_m are full-rank, $\dim\mathcal{C}_s=n_s-q$ and $\dim\mathcal{C}=n_s+n_m-q$. Hence, $\mathcal{C}=\operatorname{im}\begin{bmatrix}\bar{C}_s & C_1\\ O & C_2\end{bmatrix}$, where $\bar{C}_s$ is a basis matrix of $\mathcal{C}_s$ and $\operatorname{rank}[C_1^\top\ C_2^\top]^\top=n_m$. Since $\mathcal{S}_s^*+\mathcal{C}_s=\mathbb{R}^{n_s}$, due to right-invertibility of (A_s,C_s,B_s), and since $\mathcal{S}^*=\operatorname{im}[S_s^{*\top}\ O]^\top$, due to Property 3, the particular structure of $\mathcal{C}$ implies $\mathcal{S}^*+\mathcal{C}=\mathbb{R}^{n_s+n_m}$, i.e. right-invertibility of (A,B,C). ■

Property 5: Consider the systems (1), (2), (3), (4), (10), (11), where (10), (11) is defined as in Theorem 1. Let $\mathcal{R}_{\mathcal{V}_s^*}=\max\mathcal{V}(A_s,\mathcal{B}_s,\mathcal{C}_s)\cap\mathcal{S}_s^*$, $\mathcal{R}_{\mathcal{V}^*}=\max\mathcal{V}(A,\mathcal{B},\mathcal{C})\cap\mathcal{S}^*$. Let $R_{\mathcal{V}_s^*}$ be a basis matrix of $\mathcal{R}_{\mathcal{V}_s^*}$. Then, $\mathcal{R}_{\mathcal{V}^*}=\operatorname{im}[R_{\mathcal{V}_s^*}^\top\ O]^\top$.

Proof: First, recall that $\mathcal{R}_{\mathcal{V}^*}$ is the last term of the sequence $\mathcal{R}_{\mathcal{V}^*}^{(1)}=\mathcal{B}\cap\mathcal{V}^*$, $\mathcal{R}_{\mathcal{V}^*}^{(i)}=(A_F\mathcal{R}_{\mathcal{V}^*}^{(i-1)}+\mathcal{B})\cap\mathcal{V}^*$, $i=2,\ldots,k$, where $A_F=A+BF$, with F being any real matrix such that $(A+BF)\,\mathcal{V}^*\subseteq\mathcal{V}^*$, and k is the least integer such that $\mathcal{R}_{\mathcal{V}^*}^{(k+1)}=\mathcal{R}_{\mathcal{V}^*}^{(k)}$. Note that, due to the particular structure of A, B, and C, $\mathcal{V}^*=\operatorname{im}\begin{bmatrix}V_s^* & V_1\\ O & V_2\end{bmatrix}$, where V_s^* is a basis matrix of $\mathcal{V}_s^*=\max\mathcal{V}(A_s,\mathcal{B}_s,\mathcal{C}_s)$, and both V_1 and V_2 are non-zero matrices. Hence, the last n_m rows of the basis matrices of the subspaces subsequently generated by the algorithm of $\mathcal{R}_{\mathcal{V}^*}$ are zero. Moreover, the first n_s rows are the basis matrices of the subspaces generated by the same algorithm applied to the triple (A_s,B_s,C_s), i.e. the algorithm of $\mathcal{R}_{\mathcal{V}_s^*}$. ■

Property 6: Consider the systems (1), (2), (3), (4), (10), (11), where (10), (11) is defined as in Theorem 1. Let (A_s,B_s,C_s) be right-invertible. Let $\mathcal{V}_s^*=\max\mathcal{V}(A_s,\mathcal{B}_s,\mathcal{C}_s)$, $\mathcal{V}^*=\max\mathcal{V}(A,\mathcal{B},\mathcal{C})$. Let V_s^* be a basis matrix of $\mathcal{V}_s^*$. Then, $\mathcal{V}^*=\operatorname{im}\begin{bmatrix}V_s^* & V_1\\ O & V_2\end{bmatrix}$, where both V_1 and V_2 are non-zero matrices and $\operatorname{rank}[V_1^\top\ V_2^\top]^\top=n_m$.

Proof: Since $\mathcal{V}_s^*$ is an $(A_s,\mathcal{B}_s)$-controlled invariant contained in $\mathcal{C}_s$, matrices X_s and U_s of appropriate dimensions exist, such that $A_sV_s=V_sX_s+B_sU_s$ and $C_sV_s=0$. As a consequence, due to the particular structures of A, B, and C, $\bar{V}_s=[V_s^\top O]^\top$ satisfies $A\bar{V}_s=\bar{V}_sX_s+BU_s$ and $C\bar{V}_s=0$, which implies that $\bar{\mathcal{V}}_s=\operatorname{im}\bar{V}_s$ is an $(A,\mathcal{B})$-controlled invariant contained in $\mathcal{C}$, hence contained in $\mathcal{V}^*$. On the other hand, by virtue of Property 4, (A,B,C) is right-invertible, or, equivalently, $\mathcal{S}^*+\mathcal{V}^*=\mathbb{R}^{n_s+n_m}$. This completes the proof, since $\mathcal{S}^*=\operatorname{im}[S_s^{*\top}\ \ O]^\top$ by virtue of Property 3 and $\mathcal{S}_s^*+\mathcal{V}_s^*=\mathbb{R}^{n_s}$ by virtue of right-invertibility of (A_s,B_s,C_s). ■

Theorem 3: Consider the systems (1), (2), (3), (4), (10), (11), where (10), (11) is defined as in Theorem 1. Let (A_s,B_s,C_s) be right-invertible. Then, $\mathcal{Z}(A,B,C)=\mathcal{Z}(A_s,B_s,C_s)\uplus\sigma(A_m)$.

Proof: Let V^* denote a basis matrix of $\mathcal{V}^*$ and let F be any real matrix such that $(A+BF)\,\mathcal{V}^*\subseteq\mathcal{V}^*$. Then, a matrix X of appropriate dimension exists, such that $(A+BF)\,V^*=V^*X$. According to Property 6, $V^*=\begin{bmatrix}V_s^* & V_1\\ O & V_2\end{bmatrix}$, where V_s^* is a basis matrix of $\mathcal{V}_s^*$ and $\operatorname{rank}[V_1^\top\ V_2^\top]^\top=n_m$. Thus, the previous equation may also be written as

$$\begin{bmatrix}A_s+B_sF_1 & B_sF_2\\ O & A_m\end{bmatrix}\begin{bmatrix}V_s & V_1\\ O & V_2\end{bmatrix}=\begin{bmatrix}V_s & V_1\\ O & V_2\end{bmatrix}\begin{bmatrix}X_1 & X_2\\ X_3 & X_4\end{bmatrix},\qquad(7)$$

where the structures A and B have been taken into account and where F and X have been partitioned according to V^*. The upper block-triangular structure of $A+BF$ and the particular structure of V^* in (7) imply $\sigma((A+BF)|_{\mathcal{V}^*}) = \sigma((A_s+B_sF_1)|_{\mathcal{V}_s^*}) \uplus \sigma(A_m)$. Finally, the thesis follows by virtue of Property 5. ■

Theorem 4: Consider the systems (1), (2), (3), (4), (10), (11), where (10), (11) is defined as in Theorem 1. Let $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$. Let (A_s, B_s, C_s) be right-invertible, $\mathcal{Z}(A_s, B_s, C_s) \subset \mathbb{C}^{\odot}$, and $\sigma(A_m) \subset \mathbb{C}^{\odot}$. Then, $\mathcal{V}_m$, i.e. the minimal $(A, \mathcal{B}+\mathcal{H})$-controlled invariant such that $\mathcal{V}_m \subseteq \mathcal{C}$ and $\mathcal{H} \subseteq \mathcal{V}_m + \mathcal{B}$, is internally stabilizable.

Proof: Recall that $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$ implies $\max \mathcal{V}(A, \mathcal{B}+\mathcal{H}, \mathcal{C}) = \max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$. Hence, $\mathcal{V}_m$ satisfies $\mathcal{V}_m = \max \mathcal{V}(A, \mathcal{B}, \mathcal{C}) \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B}+\mathcal{H})$, $\mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{V}_m \subseteq \mathcal{V}^*$, and $(A+BF)\mathcal{V}_m \subseteq \mathcal{V}_m$ for any real matrix F such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$. Therefore, $\sigma((A+BF)|_{\mathcal{V}_m/\mathcal{R}_{\mathcal{V}^*}}) \subseteq \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}})$. Moreover, $\sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \mathcal{Z}(A_s, B_s, C_s) \uplus \sigma(A_m)$, due to Theorem 3, being (A_s, B_s, C_s) right-invertible. In conclusion, $\mathcal{Z}(A_s, B_s, C_s) \subset \mathbb{C}^{\odot}$ and $\sigma(A_m) \subset \mathbb{C}^{\odot}$ imply $\sigma((A+BF)|_{\mathcal{V}_m/\mathcal{R}_{\mathcal{V}^*}}) \subset \mathbb{C}^{\odot}$. ■

In the light of Theorems 3, 4, a nonminimum-phase system seems to prevent the synthesis of an internally stable compensator. In fact, an invariant zero of the system outside the open unit disc results into an unstable internal unassignable eigenvalue of the subspace $\mathcal{V}_m$, thus violating the stabilizability condition of the equivalent measurable signal decoupling problem. However, also nonminimum-phase systems may be handled, at the cost of modifying the model so as to include the same unstable invariant zeros of the system, with some further constraints as specified below.

Theorem 5: Consider the systems (1), (2), (3), (4), (10), (11), where (10), (11) is defined as in Theorem 1. Let $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$. Then, the invariant zero structure of $(A, [B\, H], C)$ is part of the external eigenstructure of $\mathcal{V}_m$.

Proof: The invariant zero structure of $(A, [B\, H], C)$ is the internal unassignable eigenstructure of $\max \mathcal{V}(A, \mathcal{B}+\mathcal{H}, \mathcal{C})$. Hence, it is part of the external eigenstructure of the constrained reachability subspace $\max \mathcal{V}(A, \mathcal{B}+\mathcal{H}, \mathcal{C}) \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B}+\mathcal{H})$. Moreover, if $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$, then $\max \mathcal{V}(A, \mathcal{B}+\mathcal{H}, \mathcal{C}) = \max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$. This implies $\max \mathcal{V}(A, \mathcal{B}+\mathcal{H}, \mathcal{C}) \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B}+\mathcal{H}) = \mathcal{V}_m$. ■

Theorem 6: Consider the system (5), (6) and its dual, defined by the triple $(A^\top, C^\top, B^\top)$. Then, (A, B, C) and $(A^\top, C^\top, B^\top)$ have the same invariant zero structure.

Proof: Consider the system (5), (6) and perform the similarity transformations $T = [T_1\; T_2\; T_3\; T_4]$, with $\operatorname{im} T_1 = \mathcal{R}_{\mathcal{V}^*}$, $\operatorname{im}\,[T_1\; T_2] = \mathcal{V}^*$, $\operatorname{im}\,[T_1\; T_3] = \mathcal{S}^*$, and $U = [U_1\; U_2]$, with $\operatorname{im} U_1 = B^{-1}\mathcal{V}^*$, $\operatorname{im} U_2 = \left(B^{-1}\mathcal{V}^*\right)^\perp$. The matrices A', B', C', respectively corresponding to A, B, C in the new bases, partitioned according to T and U, have the structures

$$A' = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} & A'_{14} \\ O & A'_{22} & A'_{23} & A'_{24} \\ A'_{31} & A'_{32} & A'_{33} & A'_{34} \\ O & O & A'_{43} & A'_{44} \end{bmatrix}, \quad B' = \begin{bmatrix} B'_{11} & B'_{12} \\ O & O \\ O & B'_{32} \\ O & O \end{bmatrix},$$

$$C' = \begin{bmatrix} O & O & C'_{13} & C'_{14} \end{bmatrix}.$$

Consider the dual triple in the new bases, i.e. $(A'^\top, C'^\top, B'^\top)$. By simple inspection one gets

$$\mathcal{V}_d^* = \max \mathcal{V}(A^\top, \mathcal{C}^\perp, \mathcal{B}^\perp) = \operatorname{im} V_d'^* = \operatorname{im} \begin{bmatrix} O & O \\ I & O \\ O & O \\ O & I \end{bmatrix}.$$

Let $G^\top$ be any real matrix such that $(A^\top + C^\top G^\top)\mathcal{V}_d^* \subseteq \mathcal{V}_d^*$. In the new bases, let $G'^\top = [G_{11}^\top\; G_{21}^\top\; G_{31}^\top\; G_{41}^\top]$. Then, $A_G'^\top = A'^\top + C'^\top G'^\top$ has the structure

$$A_G'^\top = \begin{bmatrix} A_{11}'^\top & O & A_{31}'^\top & O \\ A_{12}'^\top & A_{22}'^\top & A_{32}'^\top & O \\ A_{G13}'^\top & O & A_{G33}'^\top & O \\ A_{G14}'^\top & A_{G24}'^\top & A_{G34}'^\top & A_{G44}'^\top \end{bmatrix},$$

where $A_{Gj4}'^\top = A_{j4}'^\top + C_{14}'^\top G_{j1}'^\top$, with $j = 1, 2, 3, 4$, $A_{Gj3}'^\top = A_{j3}'^\top + C_{13}'^\top G_{j1}'^\top$, with $j = 1, 3$, and where $A_{Gj3}'^\top = A_{j3}'^\top + C_{13}'^\top G_{j1}'^\top$, with $j = 2, 4$, are set to zero by imposing $G_{j1}'^\top = -(C_{13}'^\top)^+ A_{j3}'^\top$, with $j = 2, 4$, respectively. Then, it is trivial to verify that $A_G'^\top V_d'^* = V_d'^* X$ holds, with

$$X = \begin{bmatrix} A_{22}'^\top & O \\ A_{G24}'^\top & A_{G44}'^\top \end{bmatrix}.$$

Since $X = (A^\top + C^\top G^\top)|_{\mathcal{V}_d^*}$ is lower block-triangular, $\sigma((A^\top + C^\top G^\top)|_{\mathcal{V}_d^*}) = \sigma(A_{22}'^\top) \uplus \sigma(A_{G44}'^\top)$. Hence, the set of the internal unassignable eigenvalues of $\mathcal{V}_d^*$, i.e. $\sigma(A_{22}'^\top)$, matches that of $\mathcal{V}^*$. ■

Theorem 7: Consider the systems (1), (2), (3), (4), (10), (11), where (10), (11) is defined as in Theorem 1. Let $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$. Let (A_s, B_s, C_s) and (A_m, B_m, C_m) be right- and left-invertible. Let X be a real Jordan block, part of the invariant zero structure of both (A_s, B_s, C_s) and (A_m, B_m, C_m). If matrices V_s, V_m, and L of appropriate dimensions exist, such that $A_s^\top V_s - V_s X = -C_s^\top L$, $B_s^\top V_s = O$, $A_m^\top V_m - V_m X = C_m^\top L$, and $B_m^\top V_m = O$, then X is part of the eigenstructure external to $\mathcal{V}_m$.

Proof: Let X be part of the invariant zero structure of both (A_s, B_s, C_s) and (A_m, B_m, C_m). Then, by virtue of Theorem 6, it is also part of the invariant zero structure of $(A_s^\top, C_s^\top, B_s^\top)$ and $(A_m^\top, C_m^\top, B_m^\top)$. Since (A_s, B_s, C_s) and (A_m, B_m, C_m) are right- and left-invertible by assumption, $(A_s^\top, C_s^\top, B_s^\top)$ and $(A_m^\top, C_m^\top, B_m^\top)$ are right- and left-invertible, too. Hence, matrices V_s, V_m, L_s, and L_m of appropriate dimensions exist, such that $A_s^\top V_s - V_s X = -C_s^\top L_s$, $B_s^\top V_s = O$, $A_m^\top V_m - V_m X = C_m^\top L_m$, and $B_m^\top V_m = O$. In particular, if $L_s = L_m = L$, then the above equations can be written as

$$\begin{bmatrix} A_s^\top & O \\ O & A_m^\top \end{bmatrix} \begin{bmatrix} V_s \\ V_m \end{bmatrix} - \begin{bmatrix} V_s \\ V_m \end{bmatrix} X = -\begin{bmatrix} C_s^\top \\ -C_m^\top \end{bmatrix} L, \quad (8)$$

$$\begin{bmatrix} B_s^\top & O \\ O & B_m^\top \end{bmatrix} \begin{bmatrix} V_s \\ V_m \end{bmatrix} = \begin{bmatrix} O \\ O \end{bmatrix}. \quad (9)$$

Since the triple $(A^\top, C^\top, [B\, H]^\top)$ is left-invertible (as a consequence of Property 3 and duality), (8),(9) imply that X

is part of the invariant zero structure of $(A^\top, C^\top, [B\,H]^\top)$. Hence, by virtue of Theorem 6, X is part of the invariant zero structure of $(A, [B\,H], C)$, which implies that it is part of the eigenstructure external to $\mathcal{V}_m$, due to Theorem 5. ■

Remark 1: In view of the previous results, a real Jordan block X corresponding to an unstable invariant zero of (A_s, B_s, C_s) does not necessarily imply violation of the stabilizability condition. In fact, it may be removed from the eigenstructure internal to $\mathcal{V}_m$, by replicating it as part of the invariant zero structure of the model, with a further constraint on the so-called input distribution matrix L according to Theorem 7. Non-left-invertible systems can be handled by resorting to the techniques detailed in [4].

III. SELF-BOUNDED CONTROLLED INVARIANTS IN MODEL FOLLOWING: MINIMAL-ORDER OUTPUT FEEDBACK SOLUTION

Throughout this section, the system (1), (2) and the model (3), (4) are considered, with the further assumption that the model is square.

Problem 2: model following by minimal-order dynamic output feedback. Refer to Fig. 2. Let Σ_s be ruled by (1), (2), with $x_s(0)=0$. Let Σ_m be ruled by (3), (4), with $x_m(0)=0$. Let $\sigma(A_s) \subset \mathbb{C}^{\odot}$ and $\sigma(A_m) \subset \mathbb{C}^{\odot}$. Design a linear dynamic regulator $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$ of minimal order, such that the loop is internally and externally stable and, for all admissible $h(t)$ $(t \geq 0)$, $y_s(t) = y_m(t)$ for all $t \geq 0$.

A. Structural condition

The next Theorem 8 shows that, from the structural point of view, the output feedback model following problem is equivalent to a feedforward model following problem which refers to a suitably modified model.

Theorem 8: Refer to Fig. 2. Let Σ_s be ruled by (1), (2), with $x_s(0)=0$. Let Σ_m be ruled by (3), (4), with $x_m(0)=0$. Then, $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$ is a minimal-order regulator solving the structural output feedback model following problem, i.e. such that for all admissible $h(t)$ $(t \geq 0)$, $y_s(t) = y_m(t)$ for all $t \geq 0$, if and only if Σ_c is a minimal-order compensator solving the structural feedforward model following problem for the modified model $\Sigma'_m \equiv (A_m + B_m C_m, B_m, C_m)$.

Proof: From the structural point of view, the block diagram in Fig. 3 is equivalent to that shown in Fig. 2. In fact, it is obtained by adding the same signal $y_m(t)$ both to the input of the loop and to the input of the model and taking into account that, on the assumption that Σ_c guarantees that, for all admissible $h(t)$, $(t \geq 0)$, $y(t) = 0$ for all $t \geq 0$, it is $y_m(t) = y_s(t)$ for all $t \geq 0$. ■

Thus, the dynamic output feedback model following problem is reduced to an equivalent feedforward model following problem, as far as the structural aspects are concerned. Stability of the output feedback loop is treated as specified in the next section.

B. Stabilizability condition

The next Theorems 9, 10 concern internal and external stability of the loop when the plant is minimum-phase and nonminimum-phase, respectively. The minimal-order regulator Σ_c is designed in order to solve the feedforward model matching problem for the modified plant from the structural point of view. This is achieved by following the procedure detailed in [4], but leaving apart the question of internal stabilizability of $\mathcal{V}_m$.

Theorem 9: Consider the system (1), (2) and the model (3), (4). Let (A_s, B_s, C_s) be right-invertible, $\sigma(A_s) \subset \mathbb{C}^{\odot}$, $\sigma(A_m) \subset \mathbb{C}^{\odot}$, and $\mathcal{Z}(A_s, B_s, C_s) \subset \mathbb{C}^{\odot}$. Let $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$ be a minimal-order regulator solving the structural output feedback model following problem according to Theorem 8. Then, the loop is internally and externally stable.

Proof: Since the structural property, i.e. $y(t) = 0$, $\forall t \geq 0$, for any admissible $h(t)$ $(t \geq 0)$, is preserved in the equivalence between the block diagrams in Figs. 2, 3, stability of the original model implies external stability of the loop. As to internal stability, according to Theorem 3, the poles of Σ_c are a subset of $\mathcal{Z}(A_s, B_s, C_s) \uplus \sigma(A_m + B_m C_m)$, where $\sigma(A_m + B_m C_m)$ is not necessarily contained in the open unit disc. Hence, Σ_c is not necessarily stable. Nevertheless, the loop is internally stable since cancellations outside the open unit disc are prevented by the assumption that Σ_s is minimum-phase. ■

Theorem 10: Consider the system (1), (2) and the model (3), (4). Let (A_s, B_s, C_s) and (A_m, B_m, C_m) be right- and left-invertible. Let $\sigma(A_s) \subset \mathbb{C}^{\odot}$, $\sigma(A_m) \subset \mathbb{C}^{\odot}$, and $\mathcal{Z}(A_s, B_s, C_s) \cap \sigma(A_m + B_m C_m) = \emptyset$. Let the unstable part of the invariant zero structure of (A_s, B_s, C_s) be replicated as part of the invariant zero structure of (A_m, B_m, C_m) according to Theorem 7. Let $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$ be a minimal-order regulator solving the structural output feedback model following problem according to Theorem 8. Then, the loop is internally and

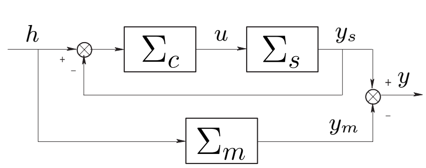

Fig. 2. Block diagram for dynamic output feedback model following.

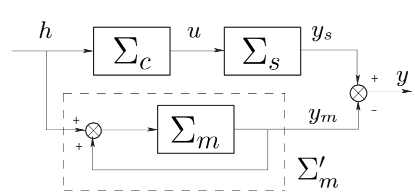

Fig. 3. Block diagram for equivalent feedforward model following.

externally stable.

Proof: External stability is guaranteed by stability of the model and preservation of the structural property ($y(t)=0$ for all $t \geq 0$, for any admissible $h(t)$, $t \geq 0$) in the equivalence between the block diagrams in Figs. 2, 3. As to internal stability, since output feedback does not modify the invariant zero structure of the model, the unstable part of the invariant zero structure of (A_s, B_s, C_s), reproduced in (A_m, B_m, C_m) according to Theorem 7, is also part of the invariant zero structure of $(A_m + B_m C_m, B_m, C_m)$. Hence, due to Theorem 7, it is not part of the internal unassignable eigenstructure of $\mathcal{V}_m$, or, equivalently, it is not part of the eigenstructure of Σ_c. Thus, cancellations outside the open unit disc are avoided for nonminimum-phase plants. ■

IV. CONCLUSIONS

The design of a dynamic regulator with the minimum number of internal unassignable dynamics achieving model following by output feedback has been thoroughly accomplished in the geometric context. The structural properties of self-bounded controlled invariant subspaces have been shown to be fundamental to both the minimization of the regulator complexity and the stabilization of the closed loop, particularly in the presence of nonminimum-phase systems.

APPENDIX
GEOMETRIC APPROACH BACKGROUND

The discrete time-invariant linear system

$$x(t+1) = A\,x(t) + B\,u(t) + H\,h(t), \qquad (10)$$

$$y(t) = C\,x(t), \qquad (11)$$

is considered, where $x \in \mathcal{X} = \mathbb{R}^n$, $u \in \mathbb{R}^p$, $h \in \mathbb{R}^s$, $y \in \mathbb{R}^q$ respectively denote the state, the control input, the exogenous input, the controlled output. The set of all admissible control input functions is the set $\mathcal{U}_f$ of all bounded functions with values in $\mathbb{R}^p$. The set of all admissible exogenous input functions is the set $\mathcal{H}_f$ of all bounded functions with values in $\mathbb{R}^s$. The matrices B, H, C are full-rank. The symbols $\mathcal{B}$, $\mathcal{H}$, $\mathcal{C}$ are used for $\operatorname{im} B$, $\operatorname{im} H$, $\ker C$, respectively. The notation $\min \mathcal{J}(A, \mathcal{B})$ is used for the minimal A-invariant containing $\mathcal{B}$. The notations $\mathcal{V}^*$ and $\max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$ are used for the maximal $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$, $\mathcal{S}^*$ and $\min \mathcal{S}(A, \mathcal{C}, \mathcal{B})$ are used for the minimal $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{B}$, and $\mathcal{R}_{\mathcal{V}^*}$ is used for the subspace reachable from the origin on $\mathcal{V}^*$. Let $\mathcal{V} \subseteq \mathcal{X}$ be an $(A, \mathcal{B})$-controlled invariant, F be any real matrix such that $(A+BF)\,\mathcal{V} \subseteq \mathcal{V}$, and $\mathcal{R}_{\mathcal{V}} = \mathcal{V} \cap \min \mathcal{S}(A, \mathcal{V}, \mathcal{B})$. The assignable and the unassignable internal eigenvalues of $\mathcal{V}$ respectively are $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}}})$ and $\sigma((A+BF)|_{\mathcal{V}/\mathcal{R}_{\mathcal{V}}})$. Let $\mathcal{R} = \min \mathcal{J}(A, \mathcal{B})$. The assignable and the unassignable external eigenvalues of $\mathcal{V}$ respectively are $\sigma((A+BF)|_{(\mathcal{V}+\mathcal{R})/\mathcal{V}})$ and $\sigma((A+BF)|_{\mathcal{X}/(\mathcal{V}+\mathcal{R})})$. Hence, $\mathcal{V}$ is internally stabilizable if and only if at least one real matrix F exists, such that $(A+BF)\,\mathcal{V} \subseteq \mathcal{V}$ and $\sigma((A+BF)|_{\mathcal{V}}) \subset \mathbb{C}^{\odot}$. Likewise, $\mathcal{V}$ is externally stabilizable if and only if at least one real matrix F exists, such that $(A+BF)\,\mathcal{V} \subseteq \mathcal{V}$ and $\sigma((A+BF)|_{\mathcal{X}/\mathcal{V}}) \subset \mathbb{C}^{\odot}$. The unassignable internal eigenvalues of $\mathcal{V}^*$ are the invariant zeros of (A, B, C), denoted by $\mathcal{Z}(A, B, C)$. The notion of invariant zero structure generalizes that of invariant zero, since it carries complete information on the complex or real Jordan form of $(A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}$. Let (A, B, C) be left-invertible, a real Jordan block X is part of the invariant zero structure of (A, B, C) if and only if matrices V, L exist, such that $AV - VX = -BL$, $CV = O$. Let $\mathcal{V} \subseteq \mathcal{X}$ be an $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$, $\mathcal{V}$ is said to be self-bounded with respect to $\mathcal{C}$ if $\mathcal{V} \supseteq \mathcal{V}^* \cap \mathcal{B}$. The set of all $(A, \mathcal{B})$-controlled invariants self-bounded with respect to $\mathcal{C}$ is a non-distributive lattice with respect to $\subseteq, +, \cap$, denoted by $\Phi(\mathcal{B}, \mathcal{C})$. Its supremum is $\mathcal{V}^*$. Its infimum is $\mathcal{R}_{\mathcal{V}^*}$.

Lemma 1: [5], [6] Let $\mathcal{H} \subseteq \mathcal{V}^*$ $(\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B})$. If the minimal $(A, \mathcal{B}+\mathcal{H})$-controlled invariant self-bounded with respect to $\mathcal{C}$, i.e. $\mathcal{V}_m = \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B}+\mathcal{H})$, is not internally stabilizable, no internally stabilizable $(A, \mathcal{B})$-controlled invariant $\mathcal{V}$ exists, which satisfies both $\mathcal{V} \subseteq \mathcal{C}$ and $\mathcal{H} \subseteq \mathcal{V}$ $(\mathcal{H} \subseteq \mathcal{V} + \mathcal{B})$.

Problem 3: measurable signal decoupling with stability. Consider the system (10), (11) with $x(0)=0$. Design a linear algebraic state feedback F and a linear algebraic feedforward S of the measurable exogenous input h on the control input u such that $\sigma(A+BF) \subset \mathbb{C}^{\odot}$ and, for all admissible $h(t)$ $(t \geq 0)$, $y(t)=0$ for all $t \geq 0$.

Theorem 11: [2] Consider the system (10), (11). Let (A, B) be stabilizable. Problem 3 is solvable if and only if *i)* $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$; *ii)* $\mathcal{V}_m$ is internally stabilizable.
If Σ is stable, the action that, starting from the zero state, is performed by the linear algebraic feedback-feedforward regulator previously considered can also be obtained by means of a linear dynamic feedforward regulator $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$, initially assumed in the zero state.

Problem 4: measurable signal decoupling with stability by minimal-order linear dynamic feedforward. Consider the system (10), (11) with $x(0)=0$. Let $\sigma(A) \subset \mathbb{C}^{\odot}$. Design a linear dynamic feedforward compensator $\Sigma_c \equiv (A_c, B_c, C_c, D_c)$ of minimal order, such that $\sigma(A_c) \subset \mathbb{C}^{\odot}$ and, for all admissible $h(t)$ $(t \geq 0)$, $y(t)=0$ for all $t \geq 0$.

The solution of Problem 4 is detailed in [4].

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