2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Exact decoupling with preview in the geometric context

Giovanni Marro and Elena Zattoni

Abstract— **The problem of making the output insensitive to an exogenous input signal known with preview is tackled in the geometric approach context. Necessary and sufficient constructive conditions for decoupling with minimal preview are proved by means of simple geometric arguments. The structural and the stabilizability conditions are considered separately. The use of self-bounded controlled invariant subspaces enables the minimal order solution to be straightforwardly derived. A steering along zeros technique is devised to solve decoupling in the presence of unstable unassignable dynamics of the minimal self-bounded controlled invariant subspace satisfying the structural constraint. The procedure is illustrated by an example often considered in the literature.**

I. INTRODUCTION

In control problems, references and disturbances are usually assumed to be unknown. Hence, a feedback structure of the control system guarantees the best performance. However, in many cases, signals to be tracked or rejected may be accessible for measurement or known with finite, or even infinite, preview. For instance, the route followed by an aircraft is normally planned in advance, the profile tracked by a machine-tool is typically preprogrammed, a constant-speed wind blowing on a plane is commonly forecast, or at least measured. In all these cases, better performance is achieved by exploiting information on the future of reference signals and/or disturbances by means of feedforward actions.

As to the theoretical background of preview control, many articles dealing with the problem by means of a wide variety of techniques exist, see e.g. [1], [2], [3], [4] and references therein. In this paper, we solve decoupling with preview in linear multivariable systems by using pure geometric arguments. According to a procedure well-settled in the geometric approach context, the structural and the stabilizability conditions for decoupling are considered separately. On the assumption that the structural condition holds, two different situations must be considered depending on the stabilizability properties of the system: i) decoupling can be achieved exactly, provided that a 'short' preview of the exogenous signal is available, *minimal preview*; ii) decoupling can be achieved exactly, provided that a 'theoretically infinite' preview of the exogenous signal is given, *infinite preview*. To be more specific, while the structural condition for decoupling with preview ($\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}^*$ [5], [6]) is the natural extension of the structural condition for measurable signal decoupling ($\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$ [7]), which extends, in turn, that for unaccessible signal decoupling ($\mathcal{H} \subseteq \mathcal{V}^*$ [8], [9]), as far as the stabilizability condition is concerned, we refer to that based on the use of the minimal self-bounded controlled invariant subspace satisfying the structural constraint, i.e. $\mathcal{V}_m = \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H})$ internally stabilizable, and we show that this is valid not only in the case of unaccessible or measurable exogenous signals [10], [11], [12], [13], [14], but also in the case where the signals to be localized are known in advance. Hence, if both the structural and the stabilizability conditions hold, then, exact decoupling can be achieved by means of the sole minimal preview, whose length is connected to the number of steps of the algorithm for $\mathcal{S}^*$, the minimal $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{B}$. Otherwise, if the structural condition holds but the stabilizability condition does not, it is herein shown that it is nonetheless possible to achieve decoupling of the exogenous signals with internal stability, provided that these are known in advance with infinite preview. Indeed, infinite preview is not strictly necessary: a preview sufficiently longer than the longest time-constant associated to the internal unassignable eigenvalues of $\mathcal{V}_m$ enables the problem to be solved with practically acceptable accuracy. In all cases where either minimal or infinite preview of the signals to be decoupled (or, by extension, to be tracked) is available, we face a noncausal problem.

In the above-described context, it is worth pointing out the substantial technical differences between our approach and others dealing with signal decoupling. The first feature of our work is of a theoretical nature and concerns the use of $\mathcal{V}_m$ for checking stabilizability. In fact, in [6], [8], [9], the controlled invariant considered for stability is $\mathcal{V}_g^*$, the maximal internally stabilizable $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$. According to [10], [11], here we consider $\mathcal{V}_m$, the minimal internally stabilizable $(A, \mathcal{B})$-controlled invariant self-bounded with respect to $\mathcal{C}$ (hence, satisfying either $\mathcal{H} \subseteq \mathcal{V}_m$, or $\mathcal{H} \subseteq \mathcal{V}_m + \mathcal{B}$, or $\mathcal{H} \subseteq \mathcal{V}_m + \mathcal{S}^*$). Since an internally stabilizable $\mathcal{V}_m$ is contained in $\mathcal{V}_g^*$, assuming $\mathcal{V}_m$ in place of $\mathcal{V}_g^*$ has the advantage of yielding a control system with the minimum number of internal unassignable dynamics. The second relevant feature of our work is connected with implementation: the control laws steering the states of the controlled system along trajectories defined by the unstable internal unassignable eigenvalues of $\mathcal{V}_m$ are produced by precompensators including non-conventional control devices like finite impulse response (FIR) systems.

II. DECOUPLING WITH MINIMAL PREVIEW

The discrete time-invariant linear system

$$x(k+1) = A\,x(k) + B\,u(k) + H\,h(k), \tag{1}$$
$$y(k) = C\,x(k), \tag{2}$$

G. Marro and E. Zattoni are with the Department of Electronics, Computer Science, and Systems, University of Bologna, I 40136 Bologna, Italy {gmarro, ezattoni}@deis.unibo.it

is considered, with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^p$, controlled output $y \in \mathbb{R}^q$ and exogenous input $h \in \mathbb{R}^s$ (either unaccessible, or measurable, or known with preview). Matrices B, H, and C are assumed to be full rank.

With respect to system (1),(2), the symbol $\mathcal{B}$ stands for $\operatorname{im} B$, $\mathcal{H}$ for $\operatorname{im} H$, $\mathcal{C}$ for $\ker C$, $\mathcal{V}^*$ or $\max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$ for the maximal $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$, $\mathcal{S}^*$ or $\min \mathcal{S}(A, \mathcal{C}, \mathcal{B})$ for the minimal $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{B}$, and $\mathcal{R}_{\mathcal{V}^*}$ for the constrained reachability subspace on $\mathcal{V}^*$, i.e. $\mathcal{R}_{\mathcal{V}^*} = \min \mathcal{J}\,(A + BF, \mathcal{V}^* \cap \mathcal{B})$, where F is such that $(A + BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$. Recall that $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$. The triple (A, B, C) is left-invertible if and only if $B^{-1}\mathcal{V}^* = \{0\}$, right-invertible if and only if $C\mathcal{S}^* = \mathbb{R}^q$, both right- and left-invertible if and only if $\mathcal{S}^* \oplus \mathcal{V}^* = \mathbb{R}^n$. The invariant zeros of (A, B, C) are the internal unassignable eigenvalues of $\mathcal{V}^*$, i.e. $\mathcal{Z} = \sigma((A + BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}})$, where $\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}$ is the quotient space of $\mathcal{V}^*$ with respect to $\mathcal{R}_{\mathcal{V}^*}$. If (A, B, C) is right-invertible, its relative degree is the least integer ρ such that $C\,\mathcal{S}_\rho = \mathbb{R}^q$, where $\mathcal{S}_i$, $i = 1, 2, \ldots, \rho_M$, is given by the algorithm $\mathcal{S}_1 = \mathcal{B}$, $\mathcal{S}_i = A\,(\mathcal{S}_{i-1} \cap \mathcal{C}) + \mathcal{B}$, with $i = 2, \ldots, \rho_M$ and ρ_M such that $\mathcal{S}_{\rho_M+1} = \mathcal{S}_{\rho_M}$. If (A, B, C) is both right- and left-invertible, its relative degree is the number of steps for evaluating $\mathcal{S}^*$, i.e. $\rho = \rho_M$ and $\mathcal{S}_\rho = \mathcal{S}^*$.

Theorem 1 (Unaccessible Signal Decoupling): If the input to be decoupled $h(k)$ in system (1),(2) is unaccessible, a state-feedback control law $u(k) = F\,x(k)$, decoupling the signal $h(k)$ and stabilizing the system, exists if and only if: i) $\mathcal{H} \subseteq \mathcal{V}^*$; ii) $\mathcal{V}_m = \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H})$ is internally stabilizable.

Proof: See [10], [11], [12], [13]. ■

Although the cases of $h(\cdot)$ measurable and previewed have been extensively studied from the structural point of view, a complete theory of the problem with stability is not yet available. In fact, while it is well-known that condition i) in Theorem 1 modifies into $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$ in measurable signal decoupling [7] and into $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}^*$ in previewed signal decoupling [5], [6], it is shown herein that the same stabilizability condition holds not only for unaccessible and measurable signal decoupling [12], [13], but also if the signal is previewed.

The block diagram for measurable and previewed signal decoupling is shown in Fig. 1. Σ stands for the system (1),(2), Σ_c for the feedforward compensator, and the block 'k_p-delay' for a cascade of k_p unit delays inserted on the input h signal flow to model its preview of k_p steps.

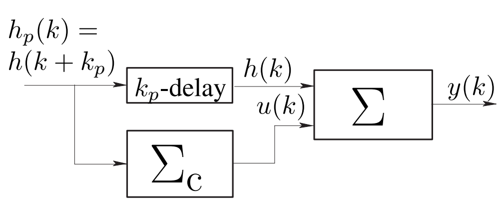

Fig. 1. Block diagram for previewed signal decoupling.

If $h(k)$ is measurable, k_p is equal to zero and $h(k)$ can be directly used as the input of the precompensator, which is a dynamic unit reproducing the eigenstructure of the stable/stabilized resolvent [14]. If $h(k)$ is previewed, k_p is equal to the number of preview samples and $h(k + k_p)$ is processed by a precompensator which, in the most general case, consists of a dynamic unit reproducing the stable dynamics of the resolvent and an FIR system reproducing both the unstable dynamics and the minimal-preview deadbeat. Details on the design of the precompensator in this general case will be given in Section III. Instead, in the remainder of this section we will consider the case where decoupling can be achieved by means of the sole minimal preview.

Problem 1 (Signal Decoupling with Minimal Preview): Refer to Fig. 1. Let Σ be ruled by (1),(2). Let A be stable. Let $x(0) = 0$. Let $h(k)$ be known with a preview of k_p steps, with $\rho_M \le k_p \le \infty$. Design a stable feedforward compensator $\Sigma_\mathrm{c} \equiv (A_c, B_c, C_c, D_c)$, having $h_p(k) = h(k + k_p)$ as input and $u(k)$ as output, such that $y(k)$ is identically zero.

Lemma 1: For any $\mathcal{Q} \subseteq \mathbb{R}^n$,

$$\min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{Q}) = \min \mathcal{S}(A, \mathcal{C}, \mathcal{S}^* + \mathcal{Q}).$$

Proof: By construction, the subspaces generated by the standard algorithms for the minimal $(A, \mathcal{C})$-conditioned invariants respectively containing $\mathcal{B} + \mathcal{Q}$ and $\mathcal{B}$ satisfy the inclusions:

$$\mathcal{S}'_1 = \mathcal{B} + \mathcal{Q} \supseteq \mathcal{S}_1 = \mathcal{B}$$

and

$$\mathcal{S}'_i = A(\mathcal{S}'_{i-1} \cap \mathcal{C}) + \mathcal{B} + \mathcal{Q} \supseteq \mathcal{S}_i = A(\mathcal{S}_{i-1} \cap \mathcal{C}) + \mathcal{B},$$

for $i = 2, 3, \ldots, \rho_M$, where ρ_M is the number of steps for evaluating $\mathcal{S}^*$. These algorithms do not necessarily converge within the same number of steps, but the last inclusion implies $\min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{Q}) \supseteq \mathcal{S}^*$. Hence, it implies $\min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{Q}) \supseteq \mathcal{S}^* + \mathcal{B} + \mathcal{Q} \supseteq \mathcal{S}^* + \mathcal{Q}$. The latter inclusion means that $\min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{Q})$ is an $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{S}^* + \mathcal{Q}$, therefore $\min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{Q}) \supseteq \min \mathcal{S}(A, \mathcal{C}, \mathcal{S}^* + \mathcal{Q})$. On the other hand, $\mathcal{B} + \mathcal{Q} \subseteq \mathcal{S}^* + \mathcal{Q}$ implies $\min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{Q}) \subseteq \min \mathcal{S}(A, \mathcal{C}, \mathcal{S}^* + \mathcal{Q})$, which completes the proof. ■

Theorem 2 (Signal Decoupling with Minimal Preview): Problem 1 is solvable if and only if: i) $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{S}^*$; ii) $\mathcal{V}_m = \mathcal{V}^* \cap \min \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H})$ is internally stabilizable.

Proof: Since condition i) is well settled in the literature, this proof will focus on condition ii).

If. First note that, owing to condition i), subspaces $\mathcal{H}_{\mathcal{S}^*} \subseteq \mathcal{S}^*$ and $\mathcal{H}_{\mathcal{V}^*} \subseteq \mathcal{V}^*$ exist such that $\mathcal{H} = \mathcal{H}_{\mathcal{S}^*} + \mathcal{H}_{\mathcal{V}^*}$. By superposition, assuming $h(k) = e_i\,\delta(k - \rho_M)$, with $k = 0, 1, \ldots$ and e_i $(i = 0, 1, \ldots, s)$ denoting the generic i-th vector of the main basis of $\mathbb{R}^s$, does not cause any loss of generality. The input $h(k)$ is assumed to be previewed of ρ_M time instants. Let τ be defined as $\tau = H\,e_i\,\delta(k - \rho_M)$ with $k = \rho_M$. Then, τ can be expressed as $\tau = \tau_{\mathcal{S}^*} + \tau_{\mathcal{V}^*}$

with $\tau_{\mathcal{S}^*} \in \mathcal{H}_{\mathcal{S}^*}$ and $\tau_{\mathcal{V}^*} \in \mathcal{H}_{\mathcal{V}^*}$. The decomposition of τ as $\tau_{\mathcal{S}^*}$ and $\tau_{\mathcal{V}^*}$ is not unique if $\mathcal{H}_{\mathcal{S}^*} \cap \mathcal{H}_{\mathcal{V}^*} \neq \{0\}$, which may occur if the system is not left-invertible, but the arguments herein presented hold for any decomposition considered. By definition of $\mathcal{S}^*$, any state belonging to $\mathcal{H}_{\mathcal{S}^*}$ can be reached from the origin in ρ_M steps at most, along a trajectory belonging to $\mathcal{C}$, therefore invisible at the output, until the last step but one. Hence, the component $\tau_{\mathcal{S}^*}$ can be nulled by applying the control input sequence driving the state from the origin to its opposite, $-\tau_{\mathcal{S}^*}$. On the other hand, the component $\tau_{\mathcal{V}^*}$ can be localized on $\mathcal{V}^*$, since both the conditions of Theorem 1 are satisfied. In fact, $\mathcal{H}_{\mathcal{V}^*} \subseteq \mathcal{V}^*$ by construction, and $\mathcal{V}^* \cap \min\, \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_{\mathcal{V}^*})$ is internally stabilizable since, by Lemma 1,

$$\begin{aligned}
&\mathcal{V}^* \cap \min\, \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}_{\mathcal{V}^*}) \\
&\quad = \mathcal{V}^* \cap \min\, \mathcal{S}(A, \mathcal{C}, \mathcal{S}^* + \mathcal{H}_{\mathcal{V}^*}) \\
&\quad = \mathcal{V}^* \cap \min\, \mathcal{S}(A, \mathcal{C}, \mathcal{S}^* + \mathcal{H}) \\
&\quad = \mathcal{V}^* \cap \min\, \mathcal{S}(A, \mathcal{C}, \mathcal{B} + \mathcal{H}) \\
&\quad = \mathcal{V}_m,
\end{aligned}$$

and $\mathcal{V}_m$ is internally stabilizable by assumption.

Only if. If $\mathcal{H} \not\subseteq \mathcal{V}^* + \mathcal{S}^*$, then the effect of the input $h(k)$ cannot be made invisible at the output because of the maximality of the respective subspaces $\mathcal{V}^*$ and $\mathcal{S}^*$. In fact, $\mathcal{V}^*$ is the maximal set of initial states in $\mathcal{C}$ corresponding to trajectories indefinitely controllable on $\mathcal{C}$, while $\mathcal{S}^*$ is the maximal set of states that can be reached from the origin in a finite number of steps with all the intermediate states in $\mathcal{C}$ except the last. On the other hand, if the structural condition holds, but $\mathcal{V}_m$ is not internally stabilizable, since $\mathcal{V}_m$ is the minimal $(A, \mathcal{B} + \mathcal{H})$-controlled invariant self-bounded with respect to $\mathcal{C}$, no internally stabilizable $(A, \mathcal{B})$-controlled invariant $\mathcal{V}$ exists satisfying both $\mathcal{V} \subseteq \mathcal{C}$ and $\mathcal{H} \subseteq \mathcal{V} + \mathcal{S}^*$. ■

Remark 1: The assumption that A is stable is not restrictive with respect to the assumptions of stabilizability of (A, B) and detectability of (A, C) usually considered. In fact, on these hypotheses, a stable system can be obtained by dynamic output feedback according to the scheme shown in Fig. 2. It can be shown that in the extended state space of the stabilized system, the internal unassignable eigenvalues of the minimal self-bounded controlled invariant satisfying the structural condition are the same as those in the state space of the original system. Hence, the minimal order of the dynamic precompensator is preserved [15].

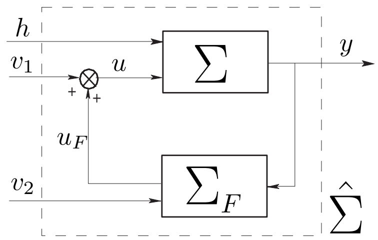

Fig. 2. Block diagram for prestabilization.

Remark 2: The assumption of zero initial state causes no loss of generality due to linearity, hence to superposition. Remarks 3–6 below focus on consequences of Theorem 2.

Remark 3: If $\mathcal{H} = \mathbb{R}^n$, then Problem 1 is solvable if and only if (A, B, C) is right-invertible.

Remark 4: If conditions i) and ii) are satisfied with $\mathcal{H} \subseteq \mathcal{V}^* + \mathcal{B}$, Problem 1 reduces to measurable signal decoupling. Hence, the algorithmic setting considered in Section III can also be used for continuous-time systems with no need for differentiators.

Remark 5: Since the internal unassignable eigenvalues of $\mathcal{V}_m$ are part of the invariant zeros of (A, B, C), condition ii) is satisfied if (A, B, C) is minimum-phase.

Remark 6: If condition ii) is not satisfied, infinite preview is required to solve Problem 1 and the compensator should include an FIR system with an infinitely large window. Although infinite preview may be available, the FIR system window must be finite. Hence, the solution is approximate, due to truncation error.

III. DECOUPLING WITH INFINITE PREVIEW: AN ALGORITHMIC SETTING

In this section, an algorithmic solution to a relaxed version of Problem 1 is devised, where the preview available is not necessarily finite and the precompensator Σ_c is not necessarily a standard dynamic system defined by a quadruple (A_c, B_c, C_c, D_c). However, the algorithmic procedure herein presented encompasses also the case of decoupling with minimal preview as a special case. The structural condition of Theorem 2 is assumed to be satisfied. Two different strategies are outlined according to whether the stabilizability condition is satisfied or not: in the former case, the minimal preview is required to obtain exact decoupling, in the latter an infinite preview is theoretically demanded.

The algorithmic setting presented in this section is built on the following basic concepts of the geometric approach. Recall that $\mathcal{V}_m$ is a locus of initial states in $\mathcal{C}$ corresponding to trajectories indefinitely controllable in $\mathcal{C}$ and that $\mathcal{S}^*$ is the maximal set of states that can be reached from the origin in ρ_M steps along trajectories with all the intermediate states in $\mathcal{C}$. Then, suppose that an impulse is applied to the input h at the time ρ_M, thus producing a component of the state $x_h \in \mathcal{H}$, which is decomposable as $x_h = x_{h,S} + x_{h,V}$, with $x_{h,S} \in \mathcal{S}^*$ and $x_{h,V} \in \mathcal{V}_m$ (note that $\mathcal{H} \subseteq \mathcal{V}_m + \mathcal{S}^*$ is implied by the structural condition [10], [11], [13]). The component $x_{h,S}$ can be nulled by applying the control sequence that drives the state from the origin to $-x_{h,S}$ along a trajectory in $\mathcal{S}^*$. The component $x_{h,V}$ can be maintained on $\mathcal{V}_m$ by a suitable control action in the time interval $\rho_M \leq k < \infty$ while avoiding state divergence, if all the internal unassignable modes of $\mathcal{V}_m$ are stable (or stabilizable). Otherwise, $x_{h,V}$ must be further decomposed as $x_{h,V} = x_{h,V_S} + x_{h,V_U}$, with x_{h,V_S} belonging to the subspace of the stable (or stabilizable) internal modes of $\mathcal{V}_m$

and $x_{h,\mathcal{V}_U}$ belonging to that of the unstable modes. The former component can be maintained on $\mathcal{V}_m$, avoiding state divergence, by a suitable control action in the time interval $\rho_M \le k < \infty$, while the latter can be nulled by reaching $-x_{h,\mathcal{V}_U}$ with a control action, applied in the time interval $-\infty < k \le \rho_M - 1$, corresponding to a trajectory in $\mathcal{V}_m$ from the origin.

The hypothesis that $\mathcal{V}_m$ does not have internal unassignable eigenvalues on the unit circle is implicit in order to discriminate between stable and unstable modes, when $\mathcal{V}_m$ is not internally stabilizable. Moreover, Algorithms 1 and 2 require that system (1),(2) is left-invertible with respect to the control input: Algorithm 3 provides a means to deal with non left-invertible systems. Algorithms 1 and 2 provide the control and state sequences for motions on $\mathcal{S}^*$ and $\mathcal{V}_m$, respectively, assuming $h(k) = I\,\delta(k-\rho_M)$. This particular choice of the input h directly yields the FIR system convolution profiles and the matrices of the dynamic unit.

Matrix H must be decomposed as $H = V H_1' + S H_2'$, where V and S denote basis matrices of $\mathcal{V}_m$ and $\mathcal{S}^*$, respectively. Let F be such that $(A+BF)\,\mathcal{V}_m \subseteq \mathcal{V}_m$ and let $T = [V\ S\ T_1]$ be a state space basis transformation. The system matrices in the new basis have the structures

$$A' = \begin{bmatrix} A_{11}' & A_{12}' & A_{13}' \\ 0 & A_{22}' & A_{23}' \\ 0 & A_{32}' & A_{33}' \end{bmatrix}, \tag{3}$$

$$B' = \begin{bmatrix} 0 \\ B_2' \\ 0 \end{bmatrix}, \qquad H' = \begin{bmatrix} H_1' \\ H_2' \\ 0 \end{bmatrix}, \tag{4}$$

$$C' = \begin{bmatrix} 0 & C_2' & C_3' \end{bmatrix}, \ F' = \begin{bmatrix} F_1' & F_2' & F_3' \end{bmatrix}. \tag{5}$$

Algorithm 1 (Motion on $\mathcal{S}^*$): The controls $U_1(k)$, $k = 0, \dots, \rho_M - 1$, and the corresponding states $X_1(k)$, $k = 1, \dots, \rho_M$, are derived through the following steps.

1. Compute basis matrices M_i of the subspace $\mathcal{S}_i \cap \mathcal{C}$ for $i = 1, \dots, \rho_M - 1$.
2. Compute the sequences $\beta(i)$ and $U_1(i)$, $i = 1, \dots, \rho_M - 1$, as
$$\begin{bmatrix} \beta(\rho_M - j) \\ U_1(\rho_M - j) \end{bmatrix} = \begin{bmatrix} A\,M_{\rho_M - j} & B \end{bmatrix}^{\#} M_{\rho_M - j + 1}\beta(\rho_M - j + 1),$$
for $j = 1, \dots, \rho_M - 1$, with $M_{\rho_M} = S$ and $\beta(\rho_M) = -\,H_2'$.
3. Compute $U_1(0)$ driving the states from the origin to $M_1\beta(1)$ as
$$U_1(0) = B^{\#} M_1\,\beta(1).$$
4. Compute the states $X_1(i)$, $i = 1, \dots, \rho_M$, as
$$X_1(i) = M_i\,\beta(i), \quad i = 1, \dots, \rho_M.$$

Algorithm 2 (Motion on $\mathcal{V}_m$): Two different strategies must be implemented depending on whether $\mathcal{V}_m$ is internally stabilizable or not.

1. If $\mathcal{V}_m$ is internally stabilizable, the motion on $\mathcal{V}_m$ is provided by the pair (A_{11}', H_1') in (3),(4), i.e. the states restricted to $\mathbb{R}^{n_\mathcal{V}}$, $n_\mathcal{V} = \dim(\mathcal{V}_m)$, are $X_2(\rho_M + i) = (A_{11}')^i\,H_1'$, $i = 0, 1, \dots$, and the controls are $U_2(\rho_M + i) = F_1'\,(A_{11}')^i\,H_1'$, $i = 0, 1, \dots$.
2. If $\mathcal{V}_m$ is not internally stabilizable, a second state space basis transformation T', whose aim is to separate the stable and unstable modes of $\mathcal{V}_m$, is required. The matrices A_{11}'', H_1'' and F_1'', respectively corresponding to A_{11}', H_1' and F_1' in the new basis, have the structures
$$A_{11}'' = \begin{bmatrix} A_S & 0 \\ 0 & A_U \end{bmatrix}, \quad H_1'' = \begin{bmatrix} H_S \\ H_U \end{bmatrix},$$
$$F_1'' = \begin{bmatrix} F_S & F_U \end{bmatrix}.$$
A preaction, nulling the unstable component of the state H_U at the time instant ρ_M must be computed backwards through the matrix A_U. The states restricted to $\mathbb{R}^{n_u}$, $n_u = \dim(\mathcal{V}_m^U)$, are $X_3(\rho_M - j) = -\,A_U^{-j}\,H_U$, $j = 0, 1, \dots$, and the controls are $U_3(\rho_M - j) = -\,F_U A_U^{-j} H_U$, $j = 1, \dots$. The stable component of the state H_S is managed as in the case of $\mathcal{V}_m$ stabilizable.

Algorithms 1 and 2 directly yield the compensator. If all the internal modes of $\mathcal{V}_m$ are stable, decoupling is achieved by means of the minimal preaction (dead-beat, motion on $\mathcal{S}^*$) and postaction (motion on $\mathcal{V}_m$ along the stable zeros). The first can be obtained as the output of a ρ_M-step FIR system with suitable convolution profiles, the latter can be realized as the output of a stable dynamic unit. Hence, the compensator turns out to be the parallel of a ρ_M-step FIR system and a dynamic unit. The input/output equation of the FIR system is

$$u_\mathrm{F}(k) = \sum_{\ell=0}^{\rho_M - 1} \varPhi(\ell)\,h(k-\ell), \quad k = 0, 1, \dots, \tag{6}$$

with $\varPhi(\ell) = U_1(\ell)$, $\ell = 0, \dots, \rho_M - 1$. The equations of the dynamic unit are

$$w(k+1) = N\,w(k) + L\,h(k-\rho_M), \quad k = 0, 1, \dots, \tag{7}$$

$$u_\mathrm{D}(k) = M\,w(k), \tag{8}$$

where $N = A_{11}'$, $L = H_1'$, $M = F_1'$. Hence, the control input is $u(k) = u_\mathrm{F}(k) + u_\mathrm{D}(k)$, $k = 0, 1, \dots$.

Otherwise, if unstable modes are also present in $\mathcal{V}_m$, infinite preaction is required. Since the evolution of the state along the unstable modes of $\mathcal{V}_m$ can only be computed backwards in time and reproduced through an FIR system, the FIR system window is enlarged to include the preaction time k_a, where this latter should be large enough to make the truncation error negligible. In this case, the compensator is the parallel of a $(k_a + \rho_M)$-step FIR system and a

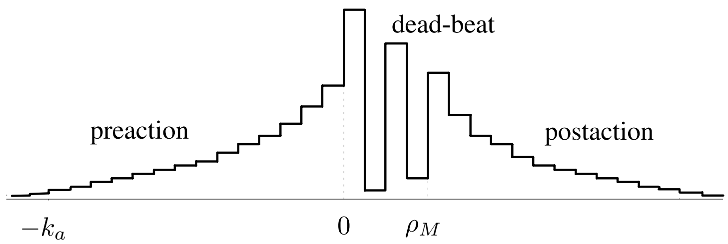

Fig. 3. A scalar input sequence for decoupling an impulse applied at time ρ_M.

dynamic unit. Equation (6) is modified into

$$u_{\mathrm{F}}(k) = \sum_{\ell=-k_a}^{\rho_M-1} \Phi(\ell)\, h(k-\ell), \quad k=0,1,\ldots, \qquad (9)$$

with $\Phi(\ell)=U_1(\ell)+U_3(\ell)$, $\ell=-k_a,\ldots,\rho_M-1$, (with a slight abuse of notation the control sequences are assumed to be zero wherever they are not explicitly defined). The dynamic unit is described by (7),(8) with $N=A_S$, $L=H_S$, $M=F_S$.

Fig. 3 shows a typical scalar control input sequence for decoupling an impulse applied to one entry of input h at the time ρ_M in the most general case where both infinite and minimal preview are present.

If the triple (A,B,C) is not left-invertible, the previous procedure can be applied anyhow, provided that a preliminary manipulation is performed to obtain a left-invertible triple and the results thus obtained are adapted to fit the original system. The proofs of the results exploited by the following algorithm can be found in [15], [16].

Algorithm 3 (Extension to Non Left-Invertible Systems): If the triple (A,B,C) is not left-invertible, the previous procedure should be applied to (A^*,B^*,C), with

1. $A^*=A+BF^*$, where F^* is a state feedback matrix such that $(A+BF^*)\,\mathcal{V}^*\subseteq\mathcal{V}^*$ and all the elements of $\sigma(A+BF^*)|_{\mathcal{R}_{\mathcal{V}^*}}$ are stable;
2. $B^*=B\,U^*$, where U^* is a basis matrix of the subspace $\mathcal{U}^*=(B^{-1}\,\mathcal{V}^*)^{\perp}$, the orthogonal complement of the inverse image of $\mathcal{V}^*$ with respect to B.

Let $\bar{U}_i(k)$ and $\bar{X}_i(k)$, with $i=1,2,3$ and k consistently defined, be the sequences of controls and states provided by Algorithms 1 and 2 applied to (A^*,B^*,C). The corresponding control sequences for (A,B,C) must be computed as $U_i(k)=U^*\bar{U}_i(k)+F^*\bar{X}_i(k)$, $i=1,2,3$.

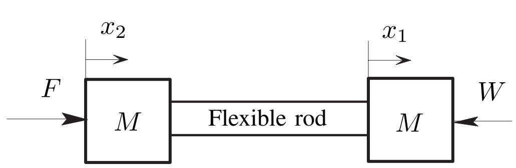

Fig. 4. Example system.

IV. AN ILLUSTRATIVE EXAMPLE

The proposed method is illustrated by an example often considered in the literature (see e.g. [17] and references therein). The system consists of two masses connected by a flexible rod (Fig. 4). The manipulable input is a force F applied to the mass with displacement x_2. In addition, we consider a disturbance W, which is a force acting on the mass with displacement x_1. Assuming $x_3=\dot{x}_1$ and $x_4=\dot{x}_2$, the state equations turn out to be

$$\begin{aligned} \dot{x}(t) &= A\,x(t)+B\,u(t)+H\,h(t), \\ y(t) &= C\,x(t), \end{aligned}$$

with

$$A=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.0909 & 0.0909 & -0.0091 & 0.0091 \\ 0.0909 & -0.0909 & 0.0091 & -0.0091 \end{bmatrix},$$

$$B=\begin{bmatrix} 0 \\ 0 \\ -0.0070 \\ 0.0839 \end{bmatrix},\ H=\begin{bmatrix} 0 \\ 0 \\ -0.0839 \\ 0.0070 \end{bmatrix},$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

The system is stabilized by feedback of the displacement x_2 on the manipulable variable, i.e. we consider a state feedback matrix $K=[0\ \ 20\ \ 0\ \ 0]$, so that the new system matrix is $A_s=A-BK$. The poles of the stabilized system are $\sigma(A_s)=\{-0.0055\pm1.2653\,j,\ -0.0036\pm0.2775\,j\}$. A discrete-time model is derived by ZOH-sampling with $T=0.1$ s. The invariant zeros of the sampled data system are $\mathcal{Z}=\{1.1205,\ 0.9014,\ -0.9916\}$. Hence, the system has nonminimum-phase dynamics. Standard computations provide $\mathcal{S}^*=\mathcal{S}_1=\mathcal{B}$ and $\mathcal{V}^*=\mathcal{V}_m=\mathrm{im}\,[\,e_2\ e_3\ e_4\,]$, where e_j, with $j=2,3,4$, denotes the j-th vector of the main basis of $\mathbb{R}^4$. The relative degree is $\rho_M=1$. Hence, the minimal preaction (or dead-beat control) consists of one single step. Exact decoupling requires infinite preaction, due to the unstable internal unassignable eigenvalue $z=1.1205$ of $\mathcal{V}_m$. Stable dynamics are managed through infinite postaction. Software developed by means of the standard geometric routines [13] allows an output error amplitude of about 10^{-5} to be achieved with a 60-sample preview. The control sequence decomposed into its different contributions is shown in Fig. 5.

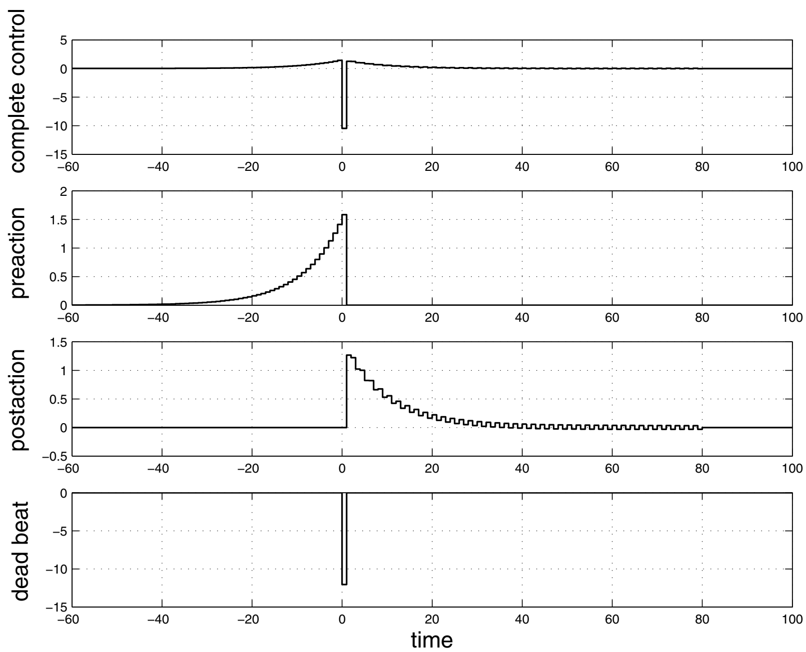

Fig. 5. Decomposition of the control sequence with a preview of 60 samples.

V. CONCLUSIONS

The problem of making the output totally insensitive to an exogenous input signal known a certain amount of time in advance has been solved in the geometric context, by exploiting the properties of the minimal self-bounded controlled invariant subspace satisfying the structural constraint. An algorithmic procedure was detailed for designing the compensator both in the case where the stabilizability condition is satisfied, and in the case where unstable internal unassignable eigenvalues of the minimal self-bounded controlled invariant are present. In the former case only the minimal preview is required, while in the latter case a theoretically infinite preview is necessary. Indeed, from the practical point of view, a preview amounting to about three times the greatest time constant associated to the unstable internal unassignable eigenvalues of $\mathcal{V}_m$ is sufficient to guarantee a negligible truncation error. The precompensator devised includes an FIR system working in connection with a standard dynamic unit.

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