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Multiobjective Controllability Assessment by Finite Dimensional Approximation

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Abstract— **This paper deals with the calculation of the optimal control performance for linear time invariant plant models under various types of constraints on performance channels in the time as well as in the frequency domain. The approach presented here is based on finite dimensional Q-parametrisation of all stable closed-loop systems. The objective function and the constraints are evaluated on a grid in the time domain and in the frequency domain which facilitates the specification of problem specific goals and constraints. The focus of this paper is on the integration of $\mathcal{H}_\infty$-constraints in this design concept. To render the computation numerically tractable, a sequence of quadratic optimisations instead of a linear matrix inequality (LMI) formulation is used. The method is benchmarked against the standard LMI formulation for multiobjective performance calculation which introduces conservatism due to requiring common Lyapunov matrices for each criterion. As expected, the degree of conservatism is reduced showing the efficiency of the new approach.**

I. Introduction

A key issue in assessing potential control structures is flexibility: it is seldom sufficient to synthesise feedback controllers based upon just one mathematical design criterion. To meet typical design specifications such as minimising the control error while keeping bounds on actuator saturation and robust stability, it is more convenient to combine different mathematical problem formulations. This is called the multiobjective controllability assessment problem in the sequel. One well-known approach to this problem is to map different criteria into LMIs (see [14] for an extensive overview), and then to force the Lyapunov matrices of each criterion to be equal in order to render the overall controller design amenable to synthesis. This introduces a certain degree of conservatism which is in general hard to quantify. Our goal here is not to synthesize controllers directly but to quantify the attainable performance of a control structure. Low order controllers can be obtained by applying frequency response approximation in a second step ([3], [4], [5], [10]): this technique has been successfully used in many examples. In the context of assessing the best achievable performance we follow the lines of [1], [15], [13], [7] using a finite dimensional Q-parametrisation, where reduced conservatism is bought at the expense of computational complexity. In our approach however, for the sake of increased flexibility, the solution is not obtained in the state space but by evaluating the criteria on a grid in the time and in the frequency domain as in ([18], [17]). This requires some engineering judgement in choosing the gridding parameters, but it can handle many different performance objectives such as e. g. actuator saturation, overshoot constraints, steady-state accuracy and enables the consideration of time delays [16] and arbitrary trajectories of the external inputs etc. [18]. It is amenable to intrinsically point-wise constraint descriptions such as e. g. uncertainty bounds resulting from the asymptotic theory of identification (see e. g. [20]) which renders it attractive from a practical point of view. The focus of this contribution is on recent results how to incorporate $\mathcal{H}_\infty$ constraints in this approach. These constraints are not transformed into a linear matrix inequality (LMI) but satisfied by solving a sequence of convex quadratic problems with linear constraints.

II. Preliminaries

All possibly multivariate variables are printed in bold font. We use the Matlab like notation $\mathbf{X}(:,j)$ to denote subparts of matrices, in this case the subpart is the jth column of the matrix $\mathbf{X}$. Frequent use is made of the following operators, which can be applied to frequency- or time-dependent matrices $(\mathbf{X}(j\omega), \mathbf{X}(t))$ as well as to transfer matrices $(\mathbf{X}(s))$:

- Transpose ^{T},
- complex conjugate transpose ^{H},
- stacking operator col, where

$$col(\mathbf{X}) := [X_{11}, X_{21}, \ldots, X_{m1}, X_{12}, \ldots, X_{1n}, \ldots, X_{mn}]^T, \quad (1)$$

- Kronecker operator $\otimes$, where

$$\mathbf{X} \otimes \mathbf{Y} := \begin{bmatrix} X_{11} \cdot \mathbf{Y} & X_{12} \cdot \mathbf{Y} & \ldots & X_{1n} \cdot \mathbf{Y} \\ X_{21} \cdot \mathbf{Y} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ X_{m1} \cdot \mathbf{Y} & & & X_{mn} \cdot \mathbf{Y} \end{bmatrix}. \quad (2)$$

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III. Computation of the Optimal Control Performance

A. Generalised plant and multiobjective specifications

The interaction of a linear time invariant plant with a controller $\mathbf{K}(s)$ and the relation of the external inputs $\mathbf{w} := [\mathbf{w}_\infty^T, \mathbf{w}_l^T, \mathbf{w}_2^T]^T \in \mathcal{R}^{(n_{w_\infty}+n_{w_l}+n_{w_2})}$ to the external outputs $\mathbf{z} := [\mathbf{z}_\infty^T, \mathbf{z}_l^T, \mathbf{z}_2^T]^T \in \mathcal{R}^{(n_{z_\infty}+n_{z_l}+n_{z_2})}$ are described by introducing a generalised plant $\mathbf{P}(s)$ which is shown in Fig. 1. $\mathbf{u} \in \mathcal{R}^{n_u}$ denotes the outputs of the controller and $\mathbf{v} \in \mathcal{R}^{n_v}$ denotes the inputs of the controller. The relationship between $\mathbf{w}$ and $\mathbf{z}$ including the controller

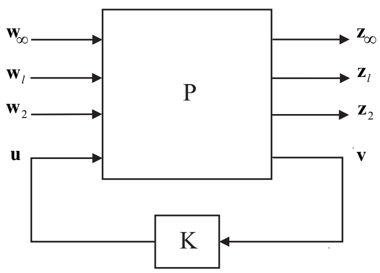


Fig. 1. Generalised plant setup for multiobjective performance calculation.

is then given by the standard linear fractional transformation $\mathbf{T}_{zw} := \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(\mathbf{I} - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}$. The performance specifications are described in the sequel.

- $\mathbf{w}_2, \mathbf{z}_2$ This performance channel is used as the objective function. Several different objectives can be combined. We here use time domain signals to formulate a finite horizon integral squared error problem

$$\|\mathbf{T}_{z_2 w_2}\|_{2,\mathbf{t}} := \int_0^{t_{n_t}} foh\left\{\mathbf{z}_2^T(t)\mathbf{z}_2(t)\right\} dt, \quad (3)$$

  where *foh* denotes first order hold approximation ([12]) and $\mathbf{w}_2$ is specified a priori.
- $\mathbf{w}_l, \mathbf{z}_l$: Arbitrarily many constraints in the time domain of the form:

$$\max |\mathbf{z}_l(t)| < \gamma_l \ \forall\ t \in \mathbf{t}_{spec}, \mathbf{w}_l \in \mathbb{W}_{l,spec} \quad (4)$$

  can be imposed, where $\mathbb{W}_{spec}$ is a set of predefined signals and $\mathbf{t}_{spec}$ is an arbitrary time window. These constraints model steady-state accuracy, actuator saturation avoidance (e. g. in an impulse-to-peak sense), constraints on the maximum overshoot, just to mention a few (see [1], [18] for more details).
- $\mathbf{w}_\infty, \mathbf{z}_\infty$: Constraints of the form

$$\max_{\omega \in \mathbf{\Omega}} \|\mathbf{T}_{\mathbf{z}_\infty \mathbf{w}_\infty}(j\omega)\|_{i2} < \gamma_\infty, \quad (5)$$

  where $\mathbf{\Omega}$ is a set of frequency points and $_{i2}$ denotes the induced 2-norm, are imposed to specify robustness.

B. Finite Dimensional Q-Parametrisation

The standard Youla parametrisation ([19]) expresses any relationship between the external signals $\mathbf{w}(s)$ and the external outputs $\mathbf{z}(s)$ which is attainable by a stabilising controller as

$$\mathbf{T}_{zw}(s) = \mathbf{T}_{11}(s) + \mathbf{T}_{12}(s) \cdot \mathbf{Q}(s) \cdot \mathbf{T}_{21}(s), \quad (6)$$

for some $\mathbf{Q} \in \mathcal{H}_\infty$; all such $\mathbf{Q}$ yield an internally stable closed-loop system. The stable transfer matrices $\mathbf{T}_{11}(s)$, $\mathbf{T}_{12}(s)$ and $\mathbf{T}_{21}(s)$ are determined by the plant. The Youla parameter $\mathbf{Q}(s)$ is any stable transfer matrix with compatible dimensions. (6) is a complete and convex description of all possible closed-loop systems. The formulation of an optimisation problem with convex criteria and convex constraints is very attractive as it has a global, i. e. unique, solution. To compute the optimal control performance, the Youla parameter $\mathbf{Q}(s)$ must be parametrised by a finite number of parameters. The usual approach is to represent $\mathbf{Q}$ by a finite series in terms of suitable fixed transfer matrices $q_i(s)$ and variable coefficients x_i. In the general multivariate case a finite dimensional $\tilde{\mathbf{Q}}$ can be written as

$$col(\tilde{\mathbf{Q}}(s, \mathbf{x})) := \mathbf{I}_{n_v n_u} \otimes \mathbf{q}^T(s) \cdot \mathbf{x}, \quad (7)$$

with

$$\mathbf{q}^T(s) := [q_0(s) \ldots q_{n_b-1}(s)] \quad (8)$$

being a basis of $\mathcal{H}_\infty$ for $n_b \to \infty$.

C. Finite horizon and gridding

The closed-loop relationship (6) can be reformulated to avoid time consuming computations during the optimisation, [18]. The basic idea is that (6) can be written as

$$col(\mathbf{T}_{zw}(s, x)) = \underbrace{col(\mathbf{T}_{11}(s))}_{:=\mathbf{T}_A} + \underbrace{\mathbf{T}_{21}^T(s) \otimes \mathbf{T}_{12}(s) \cdot \mathbf{I}_{n_v n_u} \otimes \mathbf{q}^T(s)}_{:=\mathbf{T}_B} \cdot \mathbf{x}, \quad (9)$$

using the relationship

$$col(\mathbf{ABC}) = \mathbf{C}^T \otimes \mathbf{A} \cdot col(\mathbf{B}). \quad (10)$$

The proof of (10) can be found in [2]. By means of (9), after discretising the performance objectives in the time domain and in the frequency domain on a finite grid, the computation can be accelerated by precomputing most of the matrices ([18]). In the frequency domain, (9) is evaluated point-wise:

$$col(\mathbf{T}_{zw}(j\omega, \mathbf{x})) = \mathbf{T}_A(j\omega) + \mathbf{T}_B(j\omega) \cdot \mathbf{x}, \quad \omega \in \mathbf{\Omega} = [\omega_1 \ldots \omega_{n_\omega}]. \quad (11)$$

Note that it is possible to weight these frequency domain matrices by first computing the frequency response of $\mathbf{T}_{11}(s)$, $\mathbf{T}_{12}(s)$ and $\mathbf{T}_{21}(s)$ and then applying the Kronecker product point-wise. Hence, it is not necessary in this approach that the uncertainty is represented by a realisation of a weighting matrix, instead a point-wise uncertainty description can be used. In the time domain, the reaction of an external output $\mathbf{z}_{w_j}$ to the jth external input w_j is given by:

$$\mathbf{z}_{w_j}(t) = \mathcal{L}^{-1}\left\{\mathbf{T}_{11}(:,j)(s)\cdot w_j(s)\right\} + \\ \mathcal{L}^{-1}\left\{\mathbf{T}_{12}(s)\cdot\tilde{\mathbf{Q}}(s,\mathbf{x})\cdot\mathbf{T}_{21}(:,j)(s)\cdot w_j(s)\right\}, \quad (12)$$

where $\mathcal{L}^{-1}$ denotes inverse Laplace transformation. (12) can be reformulated and evaluated point-wise as an affine function of the optimisation vector $\mathbf{x}$:

$$\mathbf{z}_{w_j}(t,\mathbf{x}) = \tilde{\mathbf{T}}_{A:w_j}(t) + \tilde{\mathbf{T}}_{B:w_j}(t)\cdot\mathbf{x}, \\ t\in\mathbf{t} = [t_1\dots t_{n_t}]. \quad (13)$$

Eqs. (11) and (13) give a complete and affine description of all possible closed-loop systems with respect to the optimisation vector $\mathbf{x}$. It is obvious that with this formulation arbitrary trajectories of the external inputs in the time domain can be considered.

D. Formulation as a Quadratic Problem

We will here restrict ourselves to quantify control performance by the integral squared error of the outputs $\mathbf{z}_2$, since this criterion leads to a quadratic optimisation problem and is familiar to practitioners. In principle, the point-wise descriptions are amenable to a larger class of convex objective function formulations. Since the LMI formulation of the integral squared error in the point-wise approach (3) is not invariant against the size of the time vector $\mathbf{t}$ ([11]), we directly solve a quadratic optimisation problem [12]:

$$\min_{\mathbf{x}} \underbrace{\frac{1}{2}\cdot\mathbf{x}^T\cdot\mathbf{H}\cdot\mathbf{x} + \mathbf{c}^T\cdot\mathbf{x}}_{:=\mathbf{\Phi}(\mathbf{x})}, \quad (14)$$

where the dimension of the positive definite matrix $\mathbf{H}$ does only depend on the size of $\mathbf{x}$.

E. Hard modulus constraints

Hard bounds on the reaction of the outputs $\mathbf{z}_l(t)$ to the external inputs $\mathbf{w}_l(t)$, i. e. $\max|\mathbf{z}_l(t)| < \gamma_l \; for \; t\in\mathbf{t}_l$, where γ_l is a constant factor, can be mapped to a set of linear constraints by imposing

$$\pm\{\tilde{\mathbf{T}}_{A:w_j}(t) + \tilde{\mathbf{T}}_{B:w_j}(t)\cdot\mathbf{x}\} < \gamma_l \;\forall\; t\in\mathbf{t}_l. \quad (15)$$

The flexibility lies in the possibility to choose arbitrary inputs $\mathbf{w}_l$ and to pick $\mathbf{t}_l$ to differ from $\mathbf{t}$ such that time dependent hard bounds can be realised. Typical applications of these bounds could be e. g. to constrain the impulse-to-peak norm of the setpoint-to-actuator channel or generally to enforce that time-varying trajectories are kept within specified bounds.

F. $\mathcal{H}_\infty$-constraints

The control performance calculation described above does not account for $\mathcal{H}_\infty$-constraints. Such constraints are considered in the sequel. For ease of notation, define $\mathbf{M}(j\omega) := \mathbf{T}_{\mathbf{z}_\infty\mathbf{w}_\infty}(j\omega)$. The goal is to compute $\mathbf{x}$ which minimises (14) and satisfies the constraints (15) as well as:

$$\|\mathbf{M}(j\omega,\mathbf{x})\|_{i2} < \gamma_\infty \forall\omega\in\mathbf{\Omega}. \quad (16)$$

In the sequel, we will for clarity drop the dependency on ω assuming it implicitly understood that calculations are performed at all frequencies considered. The nonlinear constraint (16) can be mapped to an LMI by a Schur complement argument. However, the point-wise criteria cannot be handled efficiently within the LMI framework. To overcome this difficulty we instead solve a sequence of quadratic optimisation problems. We proceed as follows:

The singular value decomposition of $\mathbf{M}$ for some $\mathbf{x}$ yields:

$$\mathbf{\Sigma}(\mathbf{x}) \;=\; \mathbf{L}^H(\mathbf{x})\cdot\mathbf{M}(\mathbf{x})\cdot\mathbf{R}(\mathbf{x}), \quad (17)$$

with $\mathbf{L}$, $\mathbf{R}$ unitary and $\mathbf{\Sigma}$ real diagonal. Although $\mathbf{M}$ is affine in $\mathbf{x}$, $\mathbf{\Sigma}$ is not, since $\mathbf{L}$, $\mathbf{R}$ also depend on $\mathbf{x}$. The multiplication of $\mathbf{M}$ with unitary matrices independent of $\mathbf{x}$, denoted by $\mathbf{L}', \mathbf{R}'$, yields:

$$\mathbf{\Sigma}'(\mathbf{x}) \;:=\; (\mathbf{L}')^H\cdot\mathbf{M}(\mathbf{x})\cdot\mathbf{R}', \quad (18)$$

where $\mathbf{\Sigma}'$ need neither be real nor diagonal. $\mathbf{\Sigma}'$ is affine in $\mathbf{x}$, since $\mathbf{M}$ is affine in $\mathbf{x}$. For $\mathbf{\Sigma}'$ in (18) we introduce the element by element matrix norm:

$$\|\mathbf{\Sigma}'\|_{max,\Re,\Im} := \max_{i,j}\max(|\Re\{\Sigma'_{ij}\}|, |\Im\{\Sigma_{ij}\}|), \quad (19)$$

where $\Re$ and $\Im$ denote the real and the imaginary part of a complex number. It can be verified by straightforward manipulations and consulting [6] that

$$\|\mathbf{\Sigma}'\|_{max,\Re,\Im} \le \|\mathbf{\Sigma}'\|_{i2} \le \sqrt{2\cdot n_{z_\infty}\cdot n_{w_\infty}}\|\mathbf{\Sigma}'\|_{max,\Re,\Im}. \quad (20)$$

The main idea is that if $\mathbf{L}'$, $\mathbf{R}'$ become equal to $\mathbf{L}(\mathbf{x}), \mathbf{R}(\mathbf{x})$ then $\|\mathbf{M}(\mathbf{x})\|_{i2}$ becomes equal to $\|\mathbf{\Sigma}'(\mathbf{x})\|_{max,\Re,\Im}$. and we have replaced a nonlinear constraint by an affine one. In the sequel, it will be shown that this can be achieved by a sequence of quadratic optimisations. (18) can be reformulated using again (10) as

$$col(\mathbf{\Sigma}'(\mathbf{x})) = \underbrace{(\mathbf{R}')^T\otimes(\mathbf{L}')^H}_{:=\mathbf{\Upsilon}}\cdot col(\mathbf{M}(\mathbf{x})). \quad (21)$$

It is obvious from definitions (1) and (19) that

$$\|\mathbf{\Sigma}'(\mathbf{x})\|_{max,\Re,\Im} = \|col(\mathbf{\Sigma}'(\mathbf{x})\|_{max,\Re,\Im},$$

and with the help of (21) linear constraints can be formulated. The idea is to iterate over unitary matrices $\mathbf{L}'(\mathbf{x}_{i-1})$, $\mathbf{R}'(\mathbf{x}_{i-1})$, which are obtained from the singular value decomposition

$$\mathbf{M}(\mathbf{x}_{i-1}) = \mathbf{L}'(\mathbf{x}_{i-1})\mathbf{\Sigma}(\mathbf{x}_{i-1})(\mathbf{R}'(\mathbf{x}_{i-1}))^H,$$

at the previous optimisation step or at the beginning of the sequence of optimisations where no robustness constraints have been considered. In each step, the additional constraints

$$\underbrace{\|\mathbf{\Sigma}_i'(\mathbf{x})\|_{max,\Re,\Im} < 1}_{:=C_i}, \tag{22}$$

where

$$\mathbf{\Sigma}_i'(\mathbf{x}) := (\mathbf{L}'(\mathbf{x}_{i-1}))^H \cdot \mathbf{M}(\mathbf{x}) \cdot \mathbf{R}'(\mathbf{x}_{i-1}),$$

are added to (14), (15) until

$$\|\mathbf{\Sigma}_i'(\mathbf{x}_i)\|_{max,\Re,\Im} = \|\mathbf{\Sigma}_i'(\mathbf{x}_i)\|_{i2} \tag{23}$$

and then

$$\|\mathbf{M}(\mathbf{x}_i)\|_{i2} < 1. \tag{24}$$

We claim that for $i \to \infty$ the union of the constraints (22) will lead to the same solution of (14) as the constraint (24), if the latter constraint is feasible. For the proof of this claim we will need some auxiliary results.

Let $\mathbb{S} \in \mathbb{R}^{n_x}$ denote the solution space of the quadratic problem for which (23) holds, and $\mathbb{S}_{C_i}$ denote the solution space for which (22) holds, formally:

$$\mathbb{S} := (\mathbf{x} | \|\mathbf{\Sigma}'(\mathbf{x})\|_{max,\Re,\Im} = \|\mathbf{\Sigma}'(\mathbf{x})\|_{i2}), \tag{25}$$

$$\mathbb{S}_{C_i} := (\mathbf{x} | \|\mathbf{\Sigma}_i'(\mathbf{x})\|_{max,\Re,\Im} < 1). \tag{26}$$

Furthermore let

$$\mathbb{S}_i := (\mathbb{S}_{C_0} \cap \mathbb{S}_{C_1} \ldots \cap \mathbb{S}_{C_i}). \tag{27}$$

Lemma 3.1 (Reduction of solution space): If the ith iteration does not yield a solution in $\mathbb{S}$ i. e.:

$$\mathbf{x}_i \notin \mathbb{S}, \tag{28}$$

then the solution space for the next optimisation is reduced

$$\mathbb{S}_{i+1} \subset \mathbb{S}_i. \tag{29}$$

Proof: Using the definition of the solution space $\mathbb{S}$ (25) it follows from (28) that

$$\|\mathbf{\Sigma}_i'(\mathbf{x}_i)\|_{max,\Re,\Im} \neq \|\mathbf{\Sigma}_i'(\mathbf{x}_i)\|_{i2}. \tag{30}$$

In the sequel we show that (30) enforces $\mathbf{x}_{i+1} \neq \mathbf{x}_i$, which is established by contradiction. Let us therefore assume that $\mathbf{x}_{i+1} = \mathbf{x}_i$. The solution $\mathbf{x}_{i+1}$ has to fulfil:

$$\|\mathbf{\Sigma}_{i+1}'(\mathbf{x}_{i+1})\|_{max,\Re,\Im} = \\ \|(\mathbf{L}'(\mathbf{x}_i))^H \cdot \mathbf{M}(\underbrace{\mathbf{x}_{i+1}}_{=\mathbf{x}_i}) \cdot \mathbf{R}'(\mathbf{x}_i)\|_{max,\Re,\Im} = \\ \|\mathbf{\Sigma}_i'(\mathbf{x}_i)\|_{i2},$$

which contradicts (30) proving that $\mathbf{x}_{i+1} \neq \mathbf{x}_i$. For a strictly convex optimisation problem with a unique solution, as it is the case here, this can only be due to

$$\mathbb{S}_{i+1} \neq \mathbb{S}_i,$$

and therefore by construction of (27)

$$\mathbb{S}_{i+1} \subset \mathbb{S}_i.$$

□

Lemma 3.2 (Retention of $\mathbb{S}$): To show that the constraints (22) do not exclude solutions to (24), we formally prove

$$\mathbb{S} \subseteq \mathbb{S}_i \ \forall \ i. \tag{31}$$

Proof: Due to the construction of $\mathbb{S}_i$ it is sufficient to check $\mathbb{S} \subseteq \mathbb{S}_{C_i} \ \forall \ i$. By the definitions of $\mathbb{S}$ and $\mathbb{S}_{C_i}$ and by noting that due to (20) $\|\mathbf{\Sigma}_i'\|_{max,\Re,\Im} \leq \|\mathbf{\Sigma}_i'\|_{i2} \ \forall \ i$ it follows that $\mathbb{S} \subseteq \mathbb{S}_{C_i}$. □

Lemma 3.3 (Boundedness of $\mathbb{S}_i$): For $i \geq 1$ $\mathbb{S}_i$ is a bounded subspace of $\mathbb{R}^{n_x}$.

Proof: Due to the construction of $\mathbb{S}_i$ (27) it is sufficient to show that $\mathbb{S}_1$ is bounded. From the upper bound in (20) we have that

$$\|\mathbf{\Sigma}_1'(\mathbf{x}_1)\|_{i2} \leq \sqrt{2n_{z_\infty} n_{w_\infty}} \underbrace{\|\mathbf{\Sigma}_1'(\mathbf{x}_1)\|_{max,\Re,\Im}}_{<1}.$$

Substituting the left hand side by its $max, \Re, \Im$-norm where the unitary matrices are known yields

$$\|(\mathbf{L}'(\mathbf{x}_1))^H \mathbf{M}(\mathbf{x}_1) \mathbf{R}'(\mathbf{x}_1)\|_{max,\Re,\Im} = \\ \|(\mathbf{R}'(\mathbf{x}_1))^T \otimes (\mathbf{L}'(\mathbf{x}_1))^H \cdot col(\mathbf{M}(\mathbf{x}_1))\|_{max,\Re,\Im} \\ \leq \sqrt{2n_{z_\infty} n_{w_\infty}}.$$

Due to the fact that $col(\mathbf{M})$ is affine in $\mathbf{x}$ and unitary matrices have full rank, $\mathbb{S}_1$ is a bounded subspace of $\mathbb{R}^{n_x}$. □

We are now ready to state the main theorem:

Theorem 3.1: For $i \to \infty$ the solution of (14) subject to (s. t.) the union of constraints (22) is equivalent to the solution of (14) s. t. the constraint (24), if the latter constraint is feasible.

Proof: Due to lemma 3.2 an existing solution of (14) s. t. (24) will never be excluded from the search space. Furthermore we have that due to lemma 3.1 the search space is reduced in every step unless the solution is found. The fact that by lemma 3.3 the search space is bounded completes the proof. □

G. Technical considerations

Employing (21), linear constraints can be formulated. Let $\mathbf{A}_i, \mathbf{b}_i$ be the constraints in the ith iteration corresponding to $\mathbb{S}_i$, $\mathbf{A}_{r,i}, \mathbf{b}_{r,i}$ the added robustness constraints corresponding to $\mathbb{S}_{C_i}$ and $\mathbf{A}_0, \mathbf{b}_0$ the constraints corresponding to (15). Additionally let $\gamma < 1$ be a factor which realises a trade off between conservatism and computation time. E. g. $\gamma^* := \sup \gamma$ corresponds to the least conservative solution

solution while $\gamma^{**} := 1/\sqrt{2n_{y_c}n_{u_c}} - \epsilon$ corresponds to trivial convergence in one step which is most conservative. The operator $\pm\Re\Im$ is defined on a matrix $\mathbf{X}$ as

$$\pm\Re\Im\{\mathbf{X}\} := \\ \left[+\Re\{\mathbf{X}^T\} \quad +\Im\{\mathbf{X}^T\} \quad -\Re\{\mathbf{X}^T\} \quad -\Im\{\mathbf{X}^T\}\right]^T,$$

and the $\underline{\gamma}$ is the vectorised form of γ with appropriate dimensions. We propose the following algorithm to realise the sequence of quadratic optimisations with linear constraints:

Algorithm 3.1:
- i) $i := 0; \mathbf{x}_0 = \arg\min_{\mathbf{x}} \frac{1}{2} \cdot \mathbf{x}^T \cdot \mathbf{H} \cdot \mathbf{x} + \mathbf{c}^T \cdot \mathbf{x}$ $s.t.\quad \mathbf{A}_0 \cdot \mathbf{x} \leq \mathbf{b}_0$,
- ii) $i++; \|\mathbf{M}(\mathbf{x}_{i-1})\|_{i2} < 1$? true: end; false: proceed with iii),
- iii) $\mathbf{M}(\mathbf{x}_{i-1}) = \mathbf{L}(\mathbf{x}_{i-1}) \cdot \mathbf{\Sigma}(\mathbf{x}_{i-1}) \cdot \mathbf{R}(\mathbf{x}_{i-1})^H$,
- iv) $\mathbf{A}_{r,i} = \begin{pmatrix} \pm\Re\Im\{\mathbf{\Upsilon}(\mathbf{x}_{i-1}, j\omega_1) \cdot \mathbf{T}_B(j\omega_1)\} \\ \vdots \\ \pm\Re\Im\{\mathbf{\Upsilon}(\mathbf{x}_{i-1}, j\omega_{n_\omega}) \cdot \mathbf{T}_B(j\omega_{n_\omega})\} \end{pmatrix}$,
  $\mathbf{b}_{r,i} = \begin{pmatrix} \underline{\gamma} \mp \Re\Im\{\mathbf{\Upsilon}(\mathbf{x}_{i-1}, j\omega_1) \cdot \mathbf{T}_A(j\omega_1)\} \\ \vdots \\ \underline{\gamma} \mp \Re\Im\{\mathbf{\Upsilon}(\mathbf{x}_{i-1}, j\omega_{n_\omega}) \cdot \mathbf{T}_A(j\omega_{n_\omega})\} \end{pmatrix}$,
- v) $\mathbf{A}_i = \begin{pmatrix} \mathbf{A}_{i-1} \\ \mathbf{A}_{r,i} \end{pmatrix}; \quad \mathbf{b}_i = \begin{pmatrix} \mathbf{b}_{i-1} \\ \mathbf{b}_{r,i} \end{pmatrix}$,
- vi) $\mathbf{x}_i = \arg\min_{\mathbf{x}} \frac{1}{2} \cdot \mathbf{x}^T \cdot \mathbf{H} \cdot \mathbf{x} + \mathbf{c}^T \cdot \mathbf{x}$ $s.t.\quad \mathbf{A}_i \cdot \mathbf{x} \leq \mathbf{b}_i$,
- vii) back to ii).

IV. Numerical Examples

For benchmarking, we consider mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis problems where the setpoint-to-control-error ($r \rightarrow e$ in Fig. 2) 2-norm to steps in the setpoints (or its first order hold approximation) is minimized subject to robust stability constraints. The uncertainty weighting matrices $\mathbf{l}_\Delta$ ($\|\mathbf{\Delta}\|_\infty < 1$) are given in output-multiplicative form such that with $\mathbf{T}_{z_\infty w_\infty}$ being equal to the negative setpoint-to-output complimentary sensitivity ($r \rightarrow y$ Fig. 2) the condition for robust stability is given by:

$$\|\mathbf{T}_{z_\infty w_\infty}(j\omega)\mathbf{l}_\Delta(j\omega)\|_{i2} < 1 \ \forall\omega \in \mathbf{\Omega}. \tag{32}$$

We also impose steady-state-accuracy of the controlled variables, i. e. we specify

$$|\mathbf{z}_l(t)| < \gamma_l \quad \forall t \in \mathbf{t}_l, \tag{33}$$

where $\mathbf{z}_l = \mathbf{z}_2$ is the control error output (e in Fig. 2). We consider a simple SISO second order system:

$$\mathbf{G}_1(s) = \frac{2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \omega_n = 1, \zeta = 0.5,$$

with multiplicative uncertainty given by:

$$l_\Delta(s) = 0.1\frac{10s+1}{0.1s+1}, \tag{34}$$

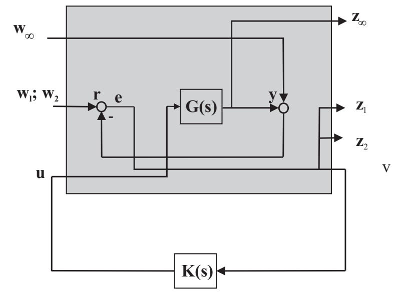


Fig. 2. Generalised plant for mixed design.

and the MIMO CDC benchmark system [8]:

$$\mathbf{G}_2(s) = \frac{1}{75s+1}\begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix},$$

where according to [9] each element of the diagonal output uncertainty can be expressed as:

$$|l_{\Delta_i}(j\omega)| = \\ \begin{cases} \sqrt{r_k^2 + 2(1+r_k)(1-cos(\theta_{max}\omega))} \mid \omega < \pi/\theta_{max} \\ 2 + r_k \mid \omega \geq \pi/\theta_{max} \end{cases}, \tag{35}$$

with $r_k = 0.2$ and $\theta_{max} = 1.0$. Note that we do not have to find a realisation as a transfer matrix for the second problem, since we are dealing with a finite set of frequency domain matrices. In the computations we used the optimisation parameters given in Tab. I. For both problems we also calculated controllers using the Matlab LMI control toolbox that implements the LMI approach [14]. To minimise the step-to-control error instead of the impulse-to-control-error in this LMI-approach we added scalar performance weights given by:

$$W_{z_2}(s) = \frac{1}{s + 10 \cdot eps}, \tag{36}$$

where eps is the Matlab machine tolerance. For the uncertainty weight in the LMI approach we used the realisation from [9] in the second example. Tab. II shows the comparison of our new approach (OCP) and the LMI approach (LMI). The infinite horizon $\mathcal{H}_\infty$ norms of the z_∞, w_∞

Example I	Example II
basis functions $\left(\mathbf{q}^T = \left[1,\ (\frac{s-a}{s+a}) \ldots (\frac{s-a}{s+a})^{n_b}\right]\right)$	
$a = 10, n_b = 15$	$a = 1, n_b = 10$
discretisation of the objective function ($\mathbf{t} = [0 \ldots t_{n_t}]$)	
$t_{n_t} = 100; n_t = 10001$	$t_{n_t} = 200; n_t = 2001$
discretisation of the steady-state constraint($\mathbf{t}_l = [0.9 \cdot t_{n_t} \ldots t_{n_t}]$)	
$\gamma_l = 10^3 \cdot eps$	$\gamma_l = 10^4 \cdot eps$
discretisation of the robustness constraint ($\mathbf{\Omega} = [0.01 \ldots \omega_{n_\omega}]$)	
$\omega_{n_\omega} = 100; n_\omega = 200$	$\omega_{n_\omega} = 10; n_\omega = 200$

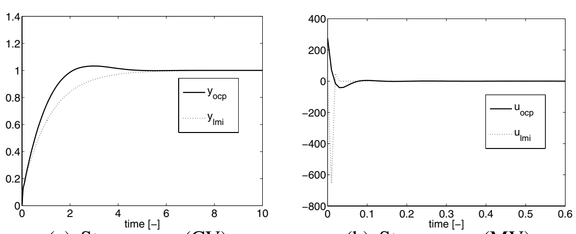
TABLE I

Parameters of performance computation.

	Example I		Example II	
	LMI	OCP	LMI	OCP
$\|\mathbf{T}_{z_\infty w_\infty} \cdot \mathbf{1}_\Delta\|_\infty$	0.88	1.00	0.97	1.00
$\|\mathbf{T}_{z_2 w_2}\|_2$	0.69	0.60	0.73	0.69
$\|\mathbf{T}_{z_2 w_2}\|_{2,\mathbf{t}}$	0.69	0.60	0.76	0.70

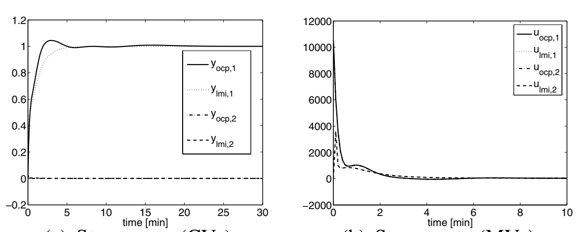
TABLE II
RESULTS OF THE COMPUTATION.

channels were calculated using the realisations of the uncertainty weights. The $\mathcal{H}_2$ norms of the performance channels were calculated by means of the quasi-integral performance weight (36). The finite horizon objective function (3), which is evaluated for independent steps in the setpoint, indicates that the infinite horizon 2-norm is approximated accurately. The $\mathcal{H}_\infty$-norm computation shows that the gridding did not lead to a loss of accuracy. Figs. 3 and 4 show the closed-loop step responses of the controlled variables (CVs) and the manipulated variables (MVs) and further illustrate that the control performances of the two examples are, as expected, improved relative to the LMI approach.



(a) Step on y_1 (CV) (b) Step on y_1 (MV)

Fig. 3. Closed-loop step responses for example 1.



(a) Step on y_1 (CVs) (b) Step on y_1 (MVs)

Fig. 4. Closed-loop step responses for example 2.

V. CONCLUSION

We have shown how to integrate $\mathcal{H}_\infty$-norm constraints into a scheme for multiobjective control performance assessment in a numerically efficient fashion. The validity of the approach was proven and demonstrated for two examples. The point-wise evaluation of the closed-loop performance yields a larger versatility than the well-known state-space approaches for finite dimensional Q-parametrisation while it requires a suitable choice of the gridding parameters. The major advantages of the presented scheme are its high flexibility to integrate many practically relevant constraints and that the conservatism is not increased by combining different types of constraints. A desirable improvement of our and other methods employing finite-dimensional Q-parametrisation is to improve the convergence of the series expansions leading to a reduction of computational complexity.

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