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Spectral conditions for positive realness of single-input single-output LTI systems

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Abstract— **In this note we derive necessary and sufficient conditions for a SISO system to be (strictly) positive real.**

This paper is dedicated to the memory of Professor John T. Lewis

I. Introduction

In this note we consider the problem of determining whether the transfer function $H(j\omega)$ associated with the linear time invariant (LTI) system

$$\Sigma : \dot{x} = Ax + bu \quad (1)$$
$$y = c^T x + du \quad (2)$$

is positive real, where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$, $c \in \mathbb{R}^{n\times 1}$, $d \in \mathbb{R}$, where $x \in \mathbb{R}^{n\times 1}$, $u, y \in \mathbb{R}$, and where $H(j\omega)$ is given by

$$H(j\omega) = d + c^T(j\omega I - A)^{-1}b. \quad (3)$$

Recently, several papers have appeared that give compact conditions to test whether a given transfer function is (strictly) positive real [1], [2], [3], [4]. In this paper we show that (strict) positive realness can be easily determined from: (i) the spectrum of the matrix $(A - \frac{1}{d}bc^T)A$ when $d \neq 0$; and (ii) the spectrum of the matrix $A(I - \frac{1}{c^T A^p b}A^p bc^T)A$ for some odd integer p when $d = 0$.

II. Definitions

Let A be a real $n \times n$ matrix, and suppose the transfer function $H(s) = d + c^T(sI - A)^{-1}b$ has poles and zeros that lie in the closed left half of the complex plane. Any poles on the imaginary axis are assumed to be simple. It follows that $H(s)$ is real for all real s, and that $H(s)$ is analytic in $Re(s) > 0$. Then $H(s)$ is said to be positive real (strictly positive real) if the following conditions are satisfied [5], [6].

Definition 2.1: (i) $Re(H(j\omega)) \geq 0$ for all $\omega \in \mathbb{R}$ (excluding any poles on the imaginary axis); and (ii) all residues of $H(s)$ at poles on the imaginary axis are positive.

Definition 2.2: Define $H_\epsilon(s) = H(s - \epsilon)$. Then $H(s)$ is strictly positive real (SPR) if $H_\epsilon(s)$ is PR for some $\epsilon > 0$.

Comment: Note that Definition 2.1 implies that if $H(s)$ is PR then $Re(H(s)) > 0$ whenever $Re(s) > 0$. Also Definition 2.2 implies that if $H(s)$ is SPR then $H(s)$ is a stable transfer function (the matrix A has all of its eigenvalues in the open left half of the complex plane and is said to be stable). Furthermore $Re(H(j\omega))$ cannot decay more rapidly than ω^{-2} as $|\omega| \to \infty$ [5].

III. Main results

We state results separately for the cases $d > 0$ and $d = 0$ as they require different conditions.

Theorem 3.1: Consider the transfer function $H(s) = d + c^T(sI - A)^{-1}b$ with $d > 0$. $H(s)$ is strictly positive real (SPR) if and only if (i) A is stable and (ii) the matrix $(A - \frac{1}{d}bc^T)A$ has no eigenvalues on the closed negative real axis $(-\infty, 0]$. $H(s)$ is positive real (PR) if and only if (i) the matrix $(A - \frac{1}{d}bc^T)A$ has no eigenvalue of odd (algebraic) multiplicity on the open negative real axis $(-\infty, 0)$, and (ii) all residues of $H(s)$ at poles on the imaginary axis are positive.

Theorem 3.2: Consider the transfer function $H(s) = c^T(sI - A)^{-1}b$. $H(s)$ is SPR if and only if: (i) $c^T Ab < 0$; (ii) $c^T A^{-1}b < 0$; (iii) A is stable; and (iv) $A(I - \frac{1}{c^T Ab}Abc^T)A$ has no eigenvalues on the open negative real axis $(-\infty, 0)$.

Let p be the smallest *odd* integer such that $c^T A^p b \neq 0$. Then $H(s)$ is PR if and only if: (i) $(-1)^{(p+1)/2}c^T A^p b > 0$; (ii) the matrix $A(I - \frac{1}{c^T A^p b}A^p bc^T)A$ has no eigenvalue of odd (algebraic) multiplicity on the open negative real axis $(-\infty, 0)$; and (iii) all residues of $H(s)$ at poles on the imaginary axis are positive.

Comment : The definition of strict positive realness given in [5], [6], [3] (Definition 2.2) is motivated by the desire that any proper system which is SPR should also satisfy the Kalman-Yacubovic-Popov lemma. Some authors relax this requirement and use the following definition of a strict positive real transfer function:

$H(s)$ is strictly positive real (SPR) if $Re(H(j\omega)) > 0$ for all $\omega \in \mathbb{R}$.

We briefly note that our conditions can be easily extended to account for this definition of strict positive realness. In particular, this definition allows A to have eigenvalues on the imaginary axis and $H(s)$ is SPR if and only if

[when $d > 0$:] (i) $H(s)$ is positive real, (ii) $d - c^T A^{-1}b > 0$, and (iii) if $(A - \frac{1}{d}bc^T)A$ has a nonzero eigenvalue $-\omega^2$ with (algebraic) multiplicity m, then $m = 2$ and $-\omega^2$ is also an eigenvalue of A^2 with (algebraic) multiplicity 2,

[when $d = 0$:] (i) $H(s)$ is positive real, (ii) $c^T A^{-1}b < 0$, and (iii) if $A(I - \frac{1}{c^T A^p b}A^p bc^T)A$ has a non-zero eigenvalue $-\omega^2$ with (algebraic) multiplicity m, then $m = 2$ and $-\omega^2$ is also an eigenvalue of A^2 with (algebraic) multiplicity 2.

Proof of Theorem 3.1:

The proof is based on the determinant representation of the transfer function that was first derived in [7]. Assume that $j\omega$ is not in the spectrum of A so that $(j\omega I - A)^{-1}$ is well-defined. Using the identity

$$(j\omega I - A)^{-1} + (-j\omega I - A)^{-1} = -2A(\omega^2 I + A^2)^{-1} \quad (4)$$

we can rewrite $Re(H)$ as

$$\begin{aligned} Re\{H(j\omega)\} &= d - c^T A(\omega^2 I + A^2)^{-1} b = \\ & d \left[1 - \frac{1}{d} c^T A(\omega^2 I + A^2)^{-1} b\right] \end{aligned} \quad (5)$$

We use the following observation: for any pair of vectors u and v in $\mathbf{R}^n$,

$$det[I_n + uv^T] = 1 + v^T u \quad (6)$$

Applying (6) with $v^T = -\frac{1}{d} c^T A$ and $u = (\omega^2 I + A^2)^{-1} b$ gives

$$\begin{aligned} Re\{H(j\omega)\} &= d \; det[I_n - \frac{1}{d}(\omega^2 I + A^2)^{-1} bc^T A] \\ &= d \; \frac{det[\omega^2 I_n + A^2 - \frac{1}{d} bc^T A]}{det[\omega^2 I_n + A^2]} \end{aligned} \quad (7)$$

Conditions for SPR: Suppose first that $H(s)$ is SPR. Then for some $\epsilon > 0$ the matrix $A + \epsilon I$ has spectrum in the left half of the complex plane. Hence A has no spectrum on the imaginary axis, and therefore $det[\omega^2 I_n + A^2] = |det[j\omega I_n + A]|^2 > 0$ for all real ω. Also the comment after Definition 2.1 implies that $Re(H(j\omega)) = Re(H_\epsilon(j\omega + \epsilon)) > 0$ since $H_\epsilon(s)$ is PR. It follows from (7) that $det[\omega^2 I_n + A^2 - \frac{1}{d} bc^T A] > 0$ for all real ω, and this establishes conditions (i) and (ii).

Conversely, suppose that $det[\omega^2 I_n + A^2] > 0$ and $det[\omega^2 I_n + A^2 - \frac{1}{d} bc^T A] > 0$ for all real ω. Then by continuity the same is true if A is replaced by $A + \epsilon I$ for sufficiently small $\epsilon > 0$, which implies that $H_\epsilon(s)$ is PR. Hence $H(s)$ is SPR.

Conditions for PR: Suppose first that H is PR. Since $det[\omega^2 I_n + A^2] = |det[j\omega I_n + A]|^2 \geq 0$, (7) implies that $Re\{H(j\omega)\}$ can change sign if and only if $det[\omega^2 I_n + A^2 - \frac{1}{d} bc^T A]$ has a zero of odd multiplicity for some $\omega \neq 0$. In this case $-\omega^2$ would be an eigenvalue of $A^2 - \frac{1}{d} bc^T A$ with odd (algebraic) multiplicity. This establishes condition (i). Condition (ii) is required by the definition of PR.

Conversely, suppose that conditions (i) and (ii) are satisfied. Then (7) implies that $Re\{H(j\omega)\}$ does not change sign along the imaginary axis, and since it is positive for large ω this establishes (1) in Definition 2.1. Hence $H(s)$ is PR.

Proof of Theorem 3.2:

The proof is based on a representation similar to (7). First, by considering its behavior for large ω, and using the definition of p, we see that the leading part of $Re\{H\}$ is

$$Re\{H(j\omega)\} \sim (-1)^{(p+1)/2} \frac{1}{\omega^{p+1}} c^T A^p b \quad (8)$$

Define

$$\nu_p = (-1)^{(p+1)/2} \frac{1}{\omega^{p+1}} \quad (9)$$

Following (8) we write

$$\begin{aligned} Re\{H(j\omega)\} &= \nu_p c^T A^p b - c^T A \left((\omega^2 + A^2)^{-1} + \nu_p A^{p-1}\right) b \\ = \nu_p c^T A^p b & \Big[1 - (\nu_p c^T A^p b)^{-1} c^T A \left((\omega^2 + A^2)^{-1} + \nu_p A^{p-1}\right) b\Big] \end{aligned}$$

We now repeat the argument that leads from (5) to (7), with d replaced by $\nu_p c^T A^p b$ and $(\omega^2 + A^2)^{-1}$ replaced by $(\omega^2 + A^2)^{-1} + \nu_p A^{p-1}$. The result is $Re\{H(j\omega)\}$

$$= \nu_p c^T A^p b \, \frac{det[\omega^2 I_n + A^2 - \frac{1}{c^T A^p b} A^{p+1} bc^T A - R_p]}{det[\omega^2 I_n + A^2]} \quad (10)$$

where

$$R_p = \frac{1}{\nu_p c^T A^p b} \left(I_n + \nu_p \omega^2 A^{p-1}\right) bc^T A \quad (11)$$

Notice that $R_1 = 0$, since $1 + \nu_1 \omega^2 = 0$.

We claim that for $p \geq 3$

$$\begin{aligned} & det[\omega^2 I_n + A^2 - \frac{1}{c^T A^p b} A^{p+1} bc^T A - R_p] = \\ & \qquad det[\omega^2 I_n + A^2 - \frac{1}{c^T A^p b} A^{p+1} bc^T A] \end{aligned} \quad (12)$$

To see this, first define

$$M = \omega^2 I_n + A^2 - \frac{1}{c^T A^p b} A^{p+1} bc^T A \quad (13)$$

and note that for $p \geq 3$

$$R_p = M\left(\frac{1}{\nu_p c^T A^p b} \sum_{j=0}^{(p-3)/2} \frac{(-1)^j}{\omega^{2j+2}} A^{2j} bc^T A\right) \quad (14)$$

Since R_p is rank 1, it follows from (6), (14) and the definition of p that $det[M - R_p]$

$$\begin{aligned} &= det[M] \, det[I - M^{-1} R_p] \\ &= det[M] \left(1 - \frac{1}{\nu_p c^T A^p b} \sum_{j=0}^{(p-3)/2} \frac{(-1)^j}{\omega^{2j+2}} c^T A^{2j+1} b\right) \\ &= det[M] \end{aligned} \quad (15)$$

which is just (12). Using (9) we get the that $Re\{H(j\omega)\}$ is given by

$$(-1)^{(p+1)/2} k(\omega) \, \frac{det[\omega^2 I_n + A^2 - \frac{1}{c^T A^p b} A^{p+1} bc^T A]}{det[\omega^2 I_n + A^2]} \quad (16)$$

for all $\omega \neq 0$ where $k(\omega) = \frac{1}{\omega^{p+1}} \, c^T A^p b$.

Conditions for SPR: Suppose first that H is SPR. Then as noted after Definition 2.2, $Re\{H(j\omega)\}$ can decay no faster than ω^{-2} as $\omega^2 \to \infty$, which implies that $p = 1$. From (8) it then follows that $c^T A b < 0$. The SPR condition requires that A is stable, and that $Re(H(j\omega)) > 0$ for all real ω. For $\omega \neq 0$, (16) implies that $det[\omega^2 I_n + A^2 - \frac{1}{c^T A b} A^2 bc^T A] > 0$. For $\omega = 0$, direct calculation yields the condition $Re(H(0)) = -c^T A^{-1} b > 0$.

Conversely suppose that conditions (i) – (iv) are satisfied. These conditions remain true if A is replaced by $A + \epsilon I$ for ϵ sufficiently small. Therefore H_ϵ is PR for some $\epsilon > 0$, and hence H is SPR.

Conditions for PR: Comparing (16) and (7), the only difference is that the overall multiplicative factor d is replaced by $(-1)^{(p+1)/2} \frac{1}{\omega^{p+1}} \, c^T A^p b$, and $A^2 - \frac{1}{d} bc^T A$ is replaced by $A^2 - \frac{1}{c^T A^p b} A^{p+1} bc^T A$. Positivity requires that $(-1)^{(p+1)/2} \, c^T A^p b > 0$. The other conditions for PR follow by repeating the arguments from the proof of Theorem 3.1, except at the point $\omega = 0$ where (16) does not apply. If $H(s)$ has a pole at $\omega = 0$, then positivity is guaranteed by the positivity of the residue. Otherwise the continuity of $Re(H(j\omega))$ guarantees non-negativity at $\omega = 0$.

IV. CONCLUSIONS

In this paper we have presented simple algebraic conditions for checking (strict) positive realness of a single-input single-output transfer function.

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