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Robust H_∞ Control and Quadratic Stabilization of Uncertain Discrete-time Switched Linear Systems

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***Abstract*— We focus on robust H_∞ control analysis and synthesis for discrete-time switched systems with norm-bounded time-varying uncertainties. Sufficient conditions are derived to guarantee quadratic stability of switched systems with a prescribed H_∞-norm bound γ. Each of these conditions can be dealt with as a linear matrix inequality (LMI) which can be easily tested with efficient algorithms. All the switching rules adopted are constructively designed and do not rely on any uncertainties.**

I. INTRODUCTION

Switched systems have gained much attention during the last decade, which deserve investigation for theoretical development as well as for practical applications. Many real-world systems can be modelled as switched systems and they also have lots of applications in control of many other fields, see for instance [1]-[19] for examples.

Although there have been many results on switched systems (e.g., [1]-[16] and the references therein), there has been relatively little work on study of uncertain switched systems. But this study is important since uncertainty is ubiquitous. One of the problems associated with this study is how to design switching rules which not only don't rely on uncertainties but also can guarantee system stability or other performances. Here, we will cope with this problem. A method is proposed to constructively design a state-dependent switching rule that is not dependent on any uncertainties. By employing this switching rule, the uncertain switched system is quadratically stable with a prescribed H_∞-norm bound γ.

As to performance analysis of switched systems, [14] presented a method to compute slow switching RMS gain for switched linear systems. [15] investigated the disturbance attenuation properties of time-controlled switched systems. In these two papers, it is assumed that at least one subsystem must be Hurwitz-stable. Here, we do not take this assumption and focus on the following problem:

Is it possible for us to obtain a prescribed disturbance attenuation level γ via a properly designed switching rule which do not rely on any uncertainties when all subsystems are not Schur-stable ?

We will show that the answer to this question is *YES*. Moreover, the H_∞ synthesis problem via switched state feedback and switched static output feedback is also studied.

Notations: $L_2[0,\infty)$ denotes the space of square integrable functions on $[0,\infty)$ and $\|\cdot\|_2$ stands for the usual $L_2[0,\infty)$-norm. The symbol $*$ is used to denote a symmetric structure in a matrix, i.e.

$$\begin{bmatrix} L & N \\ * & R \end{bmatrix} = \begin{bmatrix} L & N \\ N^T & R \end{bmatrix}$$

II. QUADRATIC STABILIZATION WITH DISTURBANCE ATTENUATION VIA SWITCHING

Consider the following uncertain discrete-time switched linear systems:

$$\begin{cases} x(t+1) = (A_{r(x,t)} + \triangle A_{r(x,t)})x(t) + B_{1r(x,t)}w(t) \\ \qquad\qquad +(B_{2r(x,t)} + \triangle B_{2r(x,t)})u(t) \\ z(t) = C_{r(x,t)}x(t) + D_{r(x,t)}u(t) \\ y(t) = H_{r(x,t)}x(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the control input, $w(t) \in \mathbb{R}^h$ is the exogenous input which belongs to $L_2[0,\infty)$, $z(t) \in \mathbb{R}^q$ is the controlled output, $y(t) \in \mathbb{R}^s$ is the measurement output. The right continuous function $r(x,t) : \mathbb{R}^n \times \mathbb{R}^+ \to \{1, 2, \cdots, l\}$ (denoted as $\underline{l}$) is the switching rule to be designed. Moreover, $r(x,t) = i$ implies that the i-th subsystem is activated.

$$[\Delta A_i,\ \Delta B_{2i}] = E_i\Gamma[F_{1i},\ F_{2i}],\ \ \forall i \in \underline{l}. \tag{2}$$

$A_i, B_{1i}, B_{2i}, C_i,\ D_i$ and H_i are constant matrices of appropriate dimensions that describe the nominal systems, E_i, F_{1i}, F_{2i} are given matrices which characterize the structure of uncertainty. Γ is the norm-bounded time-varying uncertainty, i.e.,

$$\Gamma = \Gamma(t) \in \{\Gamma(t) : \Gamma(t)^T\Gamma(t) \le I, \Gamma(t) \in \mathbb{R}^{m\times k}\}$$

In [20], it is pointed out that there are several reasons for assuming that the system uncertainty has the structure given in (2). One is that a linear interconnection of a nominal plant with the uncertainty Γ leads to the structure of the form (2). The other comes from the fact that uncertainties in many physical systems can be modelled in this manner, e.g., satisfying 'matching conditions'.

Let us first consider the following unforced switched systems simplified from (1):

$$\begin{cases} x(t+1) = (A_{r(x,t)} + \triangle A_{r(x,t)})x(t) + B_{1r(x,t)}w(t) \\ z(t) = C_{r(x,t)}x(t) \end{cases} \tag{3}$$

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To formulate the problem concerned here clearly, we need the following definitions.

Definition 1: The system (3) with $w \equiv 0$ is said to be **quadratically stabilizable via switching** if there exist a switching rule $r(x,t)$, a positive definite function $V(x) = x^T Px$ and a positive scalar ε such that, for any admissible uncertainty Γ with $\Gamma^T\Gamma \leq I$

$$V(x(t+1)) - V(x(t)) < -\varepsilon x^T(t)x(t)$$

holds for all trajectories of system (3).

Definition 2: The system (1) is said to be **quadratically stabilizable via switched state feedback** if there exist a switching rule $r(x,t)$ and an associated state feedback $u = K_{r(x,t)}x$ with $K_i (i \in \underline{l})$ not depending on uncertainty Γ, such that with $u = K_{r(x,t)}x$, the resulting closed-loop nominal system ($w \equiv 0$) is quadratically stable.

Remark 1: It should be noted that in the above two definitions, not only the state feedback gain matrices $K_i (i \in \underline{l})$ but also the switching rule $r(x,t)$ to be designed do not depend on any uncertainty Γ.

In order to study disturbance attenuation properties of system (3), we give the following definition.

Definition 3: Given a constant $\gamma > 0$, system (3) is said to be **quadratically stabilizable with H_∞ disturbance attenuation γ via switching** if there exists a switching rule $r(x,t)$ such that under this switching, it satisfies

(1) system (3) with $w \equiv 0$ is quadratically stabilizable for all admissible uncertainties Γ,

(2) with zero-initial condition $x(0) = 0$, $\|z\|_2 < \gamma \|w\|_2$ for all admissible uncertainties Γ and all nonzero $w \in L_2[0,\infty)$, where $||z||_2 = \sqrt{\sum\limits_{t=0}^{\infty} z^T(t)z(t)}$.

To develop the main result, we need the following two lemmas.

Lemma 1: Suppose A, E, F are given matrices, P is a positive definite matrix and η is a scalar such that $\eta^{-1}I - E^T PE > 0$. Then

$$\begin{aligned}&(A + E\Gamma F)^T P(A + E\Gamma F) \\ &\leq A^T(P^{-1} - \eta EE^T)^{-1}A + \eta^{-1}F^T F\end{aligned}$$

holds for arbitrary norm-bounded time-varying uncertainty Γ with $\Gamma^T\Gamma \leq I$.

Proof: Since

$$\begin{aligned}
& A^T PE(\eta^{-1}I - E^T PE)^{-1}E^T PA - A^T PE\Gamma F \\
& \quad -F^T\Gamma^T E^T PA + F^T\Gamma^T(\eta^{-1}I - E^T PE)\Gamma F \\
= \; & [A^T PE(\eta^{-1}I - E^T PE)^{-\frac{1}{2}} \\
& \quad -F^T\Gamma^T(\eta^{-1}I - E^T PE)^{\frac{1}{2}}] \\
& \times[A^T PE(\eta^{-1}I - E^T PE)^{-\frac{1}{2}} \\
& \quad -F^T\Gamma^T(\eta^{-1}I - E^T PE)^{\frac{1}{2}}]^T \\
\geq \; & 0
\end{aligned}$$

and $\Gamma^T\Gamma \leq I$, we have

$$\begin{aligned}
& A^T PE(\eta^{-1}I - E^T PE)^{-1}E^T PA + \eta^{-1}F^T F \\
& \geq A^T PE\Gamma F + F^T\Gamma^T E^T PA + F^T\Gamma^T E^T PE\Gamma F, \quad (4)
\end{aligned}$$

It follows from (4) that

$$\begin{aligned}
& (A + E\Gamma F)^T P(A + E\Gamma F) \\
& = A^T PA + A^T PE\Gamma F + F^T\Gamma^T E^T PA \\
& \quad +F^T\Gamma^T E^T PE\Gamma F \\
& \leq A^T PA + A^T PE(\eta^{-1}I - E^T PE)^{-1}E^T PA \\
& \quad +\eta^{-1}F^T F \\
& = A^T[P + PE(\eta^{-1}I - E^T PE)^{-1}E^T P]A \\
& \quad +\eta^{-1}F^T F \qquad (5)
\end{aligned}$$

On the other hand, by the Schur complement technique, it can be verified that

$$\eta^{-1}I - E^T PE > 0 \Longleftrightarrow P^{-1} - \eta EE^T > 0$$

thus $P^{-1} - \eta EE^T$ is invertible. Since

$$(P^{-1} - \eta EE^T)^{-1} = P + PE(\eta^{-1}I - E^T PE)^{-1}E^T P \quad (6)$$

we can get the result by combining (5) and (6). ■

Lemma 2: Take as given the $\alpha_1, \cdots, \alpha_l$ with $\alpha_i \geq 0$ and $\sum_{i=1}^{l} \alpha_i > 0$, then the following two statements are equivalent:

(i)There exist a symmetric matrix $P > 0$ and a scalar $\eta > 0$ such that

$$\begin{aligned}
& \sum_{i=1}^{l} \alpha_i[A_i^T(P^{-1} - \gamma^{-2}B_{1i}B_{1i}^T - \eta E_i E_i^T)^{-1}A_i \\
& \quad +\eta^{-1}F_{1i}^T F_{1i} - P + C_i^T C_i] < 0 \qquad (7)
\end{aligned}$$

with

$$P^{-1} - \gamma^{-2}B_{1i}B_{1i}^T - \eta E_i E_i^T > 0, \quad \forall i \in \underline{l} \qquad (8)$$

where γ is a given constant.

(ii) There exist a symmetric matrix $Q > 0$, a scalar $\eta > 0$ such that the following LMI

$$\begin{array}{ccccccc}
-\sum_{i=1}^{l}\alpha_i Q & \sqrt{\alpha_1}QA_1^T & \cdots & \sqrt{\alpha_l}QA_l^T & 0 & \cdots & 0 \\
* & -Q+\eta E_1E_1^T & \cdots & 0 & B_{11} & \cdots & 0 \\
 & & \ddots & \vdots & \vdots & \ddots & \vdots \\
* & * & & & & & \\
* & * & * & -Q+\eta E_lE_l^T & 0 & \cdots & B_{1l} \\
* & * & * & * & -\gamma^2 I & \cdots & 0 \\
 & & & & & \ddots & \vdots \\
* & * & * & * & * & & \\
* & * & * & * & * & * & -\gamma^2 I \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & *
\end{array}$$

$$\begin{array}{cccccc}
\sqrt{\alpha_1}QC_1^T & \cdots & \sqrt{\alpha_l}QC_l^T & \sqrt{\alpha_1}QF_{11}^T & \cdots & \sqrt{\alpha_l}QF_{1l}^T \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
-I & \cdots & 0 & 0 & \cdots & 0 \\
 & \ddots & \vdots & \vdots & \ddots & \vdots \\
* & & & & & \\
* & * & -I & 0 & \cdots & 0 \\
* & * & * & -\eta I & \cdots & 0 \\
 & & & & \ddots & \vdots \\
* & * & * & * & & \\
* & * & * & * & * & -\eta I
\end{array} < 0$$

is satisfied.

Proof: By computation, the feasibility of (7) and (8) is equivalent to

$$\overline{A}^T(\overline{P}^{-1}-\gamma^{-2}\overline{B}_1\overline{B}_1^T-\eta\overline{E}\overline{E}^T)^{-1}\overline{A}+\eta^{-1}\overline{F}_1^T\overline{F}_1 -\sum_{i=1}^{l}\alpha_i P+\overline{C}^T\overline{C}<0 \tag{9}$$

with

$$\overline{P}^{-1}-\gamma^{-2}\overline{B}_1\overline{B}_1^T-\eta\overline{E}\overline{E}^T>0 \tag{10}$$

where

$$\overline{A}=\begin{bmatrix}\sqrt{\alpha_1}A_1\\ \vdots\\ \sqrt{\alpha_l}A_l\end{bmatrix},\overline{C}=\begin{bmatrix}\sqrt{\alpha_1}C_1\\ \vdots\\ \sqrt{\alpha_l}C_l\end{bmatrix}$$

$$\overline{B}_1=diag\{B_{11},\cdots,B_{1l}\},\overline{P}=diag\underbrace{\{P,\cdots,P\}}_{l}$$

$$\overline{E}=diag\{E_1,\cdots,E_l\},\overline{F}_1=\begin{bmatrix}\sqrt{\alpha_1}F_{11}\\ \vdots\\ \sqrt{\alpha_l}F_{1l}\end{bmatrix}$$

By virtue of Schur complement formula, (9) and (10) hold if and only if

$$\begin{bmatrix}-\sum_{i=1}^{l}\alpha_i P & \overline{A}^T & 0 & \overline{C}^T & \overline{F}_1^T\\ \overline{A} & -\overline{P}^{-1}+\eta\overline{E}\overline{E}^T & \overline{B}_1 & 0 & 0\\ 0 & \overline{B}_1^T & -\gamma^2 I & 0 & 0\\ \overline{C} & 0 & 0 & -I & 0\\ \overline{F}_1 & 0 & 0 & 0 & -\eta I\end{bmatrix}<0 \tag{11}$$

Multiplying $diag\{P^{-1},I,I,I,I\}$ on both sides of the left-hand-side matrix of (11) and denote $P^{-1}=Q$, then by Schur complement formula, (11) is equivalent to (ii). This concludes the proof. ■

In what follows, we will drop the state and time dependence in $r(x,t)$, i.e., denote $r(x,t)$ as r when the switching rule $r(x,t)$ is used as a subscript to a matrix.

Theorem 1: Given a constant $\gamma>0$, system (3) is quadratically stabilizable with H_∞ disturbance attenuation γ via switching if there exist a positive definite matrix Q and a positive scalar η such that the LMI (ii) is satisfied for some nonnegative scalars $\alpha_1,\alpha_2,\cdots,\alpha_l$ with $\sum_{i=1}^{l}\alpha_i>0$. In this case, the switching rule is taken as

$$r(x,t)=\arg\min_{i\in\underline{l}}\{x(t)^T[A_i^T(Q-\gamma^{-2}B_{1i}B_{1i}^T-\eta E_iE_i^T)^{-1}A_i+\eta^{-1}F_{1i}^TF_{1i}-Q^{-1}+C_i^TC_i]x(t)\}, \tag{12}$$

Proof: We first show the quadratic stabilization of systems (3) via switching (12). By Lemma 2, the feasibility of (ii) means that

$$\sum_{i=1}^{l}\alpha_i[A_i^T(P^{-1}-\gamma^{-2}B_{1i}B_{1i}^T-\eta E_iE_i^T)^{-1}A_i +\eta^{-1}F_{1i}^TF_{1i}-P+C_i^TC_i]<0$$

with

$$P^{-1}-\gamma^{-2}B_{1i}B_{1i}^T-\eta E_iE_i^T>0,\quad i\in\underline{l} \tag{13}$$

where $P=Q^{-1}$ and Q is the positive definite matrix satisfying (ii). This implies that the following inequality always holds for some $\zeta>0$

$$\sum_{i=1}^{l}\alpha_i[A_i^T(P^{-1}-\gamma^{-2}B_{1i}B_{1i}^T-\eta E_iE_i^T)^{-1}A_i +\eta^{-1}F_{1i}^TF_{1i}-P+C_i^TC_i]<-\zeta I$$

Consequently, for any nonzero $x(t)\in\mathbb{R}^n$

$$\begin{aligned}&\sum_{i=1}^{l}\alpha_i[\min_{i\in\underline{l}}x^T(t)(A_i^T(P^{-1}-\gamma^{-2}B_{1i}B_{1i}^T -\eta E_iE_i^T)^{-1}A_i+\eta^{-1}F_{1i}^TF_{1i}-P+C_i^TC_i)x(t)]\\ &\le\sum_{i=1}^{l}\alpha_i x^T(t)[A_i^T(P^{-1}-\gamma^{-2}B_{1i}B_{1i}^T -\eta E_iE_i^T)^{-1}A_i+\eta^{-1}F_{1i}^TF_{1i}-P+C_i^TC_i]x(t)\\ &<-\zeta x^T(t)x(t)\end{aligned} \tag{14}$$

On the other hand, let's consider the following discrete-type Lyapunov function for systems (3)

$$V(x(t))=x^T(t)Px(t)$$

For $w(t)\equiv 0$, we have

$$\begin{aligned}&V(x(t+1))-V(x(t))\\ &=x^T(t)[(A_r+\Delta A_r)^TP(A_r+\Delta A_r)-P]x(t)\\ &\le x^T(t)[(A_r+\Delta A_r)^TP(A_r+\Delta A_r)-P+C_r^TC_r\\ &\quad+(A_r+\Delta A_r)^TPB_{1r}(\gamma^2I-B_{1r}^TPB_{1r})^{-1}B_{1r}^TP\\ &\quad\times(A_r+\Delta A_r)]x(t)\end{aligned} \tag{15}$$

where the inequality in (15) follows from the fact that

$$\gamma^2I-B_{1i}^TPB_{1i}>0,\quad i\in\underline{l}$$

which is due to (8) since by the Schur complement technique

$$\gamma^2I-B_{1i}^TPB_{1i}>0\Longleftrightarrow P^{-1}-\gamma^{-2}B_{1i}B_{1i}^T>0$$

Furthermore, by (6) and Lemma 1

$$\begin{aligned}&(A_r+\Delta A_r)^TP(A_r+\Delta A_r)+(A_r+\Delta A_r)^TPB_{1r}\\ &\quad\times(\gamma^2I-B_{1r}^TPB_{1r})^{-1}B_{1r}^TP(A_r+\Delta A_r)\\ &=(A_r+E_r\Gamma F_{1r})^T[P+PB_{1r}(\gamma^2I-B_{1r}^TPB_{1r})^{-1}\\ &\quad\times B_{1r}^TP](A_r+E_r\Gamma F_{1r})\\ &=(A_r+E_r\Gamma F_{1r})^T(P^{-1}-\gamma^{-2}B_{1r}B_{1r}^T)^{-1}\\ &\quad\times(A_r+E_r\Gamma F_{1r})\\ &\leqslant A_r^T(P^{-1}-\gamma^{-2}B_{1r}B_{1r}^T-\eta E_rE_r^T)^{-1}A_r\\ &\quad+\eta^{-1}F_{1r}^TF_{1r}\end{aligned} \tag{16}$$

holds. Hence, (15) and (16) gives that

$$\begin{aligned}&V(x(t+1))-V(x(t))\\&\leqslant x^T(t)[A_r^T(P^{-1}-\gamma^{-2}B_{1r}B_{1r}^T-\eta E_rE_r^T)^{-1}A_r\\&\quad+\eta^{-1}F_{1r}^TF_{1r}-P+C_r^TC_r]x(t)\end{aligned}\qquad(17)$$

Thus, combining (14), (17) and the switching rule (12), we have

$$V(x(t+1))-V(x(t))<-\varepsilon x^T(t)x(t)$$

proving the quadratic stability, where $\varepsilon=\zeta(\sum_{i=1}^{l}\alpha_i)^{-1}$.

Secondly, we will investigate the disturbance attenuation of system (3). In order to establish the upper bound $\gamma\|w\|_2$, we introduce the criterion function

$$J=\sum_{t=0}^{\infty}[z^T(t)z(t)-\gamma^2w^T(t)w(t)].$$

Since system (3) is stable, all states converge to zero and noticing that $x(t_0)=x(0)=0$, we have

$$J=\sum_{t=0}^{\infty}[\Delta V(x(t))+z^T(t)z(t)-\gamma^2w^T(t)w(t)],$$

where $\Delta V(x(t)):=V(x(t+1))-V(x(t))$. By computation, we get

$$\begin{aligned}&\Delta V(x(t))+z^T(t)z(t)-\gamma^2w^T(t)w(t)\\&=x^T(t)[(A_r+\Delta A_r)^TP(A_r+\Delta A_r)-P+C_r^TC_r]\\&\quad\times x(t)+x^T(t)(A_r+\Delta A_r)^TPB_{1r}w(t)\\&\quad+w^T(t)B_{1r}^TP(A_r+\Delta A_r)x(t)\\&\quad-w^T(t)(\gamma^2I-B_{1r}^TPB_{1r})w(t)\\&=x^T(t)[(A_r+\Delta A_r)^TP(A_r+\Delta A_r)-P+C_r^TC_r\\&\qquad+(A_r+\Delta A_r)^TPB_{1r}(\gamma^2I-B_{1r}^TPB_{1r})^{-1}\\&\qquad\times B_{1r}^TP(A_r+\Delta A_r)]x(t)-[\gamma^{-1}(I\\&\qquad-\gamma^{-2}B_{1r}^TPB_{1r})^{-1}B_{1r}^TP(A_r+\Delta A_r)x(t)-\gamma w(t)]^T\\&\qquad\times(I-\gamma^{-2}B_{1r}^TPB_{1r})[\gamma^{-1}(I-\gamma^{-2}B_{1r}^TPB_{1r})^{-1}\\&\qquad\times B_{1r}^TP(A_r+\Delta A_r)x(t)-\gamma w(t)]\\&\leq x^T(t)[(A_r+\Delta A_r)^TP(A_r+\Delta A_r)-P+C_r^TC_r\\&\qquad+(A_r+\Delta A_r)^TPB_{1r}(\gamma^2I-B_{1r}^TPB_{1r})^{-1}\\&\qquad\times B_{1r}^TP(A_r+\Delta A_r)]x(t)\end{aligned}\qquad(18)$$

where the last inequality follows from the fact that $I-\gamma^{-2}B_{1r}^TPB_{1r}>0$. Hence, (18) and (16) gives that

$$\begin{aligned}&\Delta V(x(t))+z^T(t)z(t)-\gamma^2w^T(t)w(t)\\&\leqslant x^T(t)[A_r^T(P^{-1}-\gamma^{-2}B_{1r}B_{1r}^T-\eta E_rE_r^T)^{-1}A_r\\&\quad+\eta^{-1}F_{1r}^TF_{1r}-P+C_r^TC_r]x(t)\end{aligned}\qquad(19)$$

Therefore, combing (19) with the switching rule (12), we get

$$J<0$$

holds for any nonzero $w(t)\in L_2[0,\infty)$, i.e., $\|z\|_2<\gamma\|w\|_2$ for all nonzero $w(t)\in L_2[0,\infty)$. Obviously, the designed switching rule (12) does not depend on any uncertainties. This concludes the proof. ■

III. CONTROL SYNTHESIS

A. Switched state feedback

In this section, we study H_∞ control problem for system (1) via switched state feedback.

The **switched state feedback robust H_∞ control** problem addressed in this section is as follows: *for a given constant $\gamma>0$, design a switching rule $r(x,t)$ and an associated state feedback $u=K_rx$ such that the resulting closed-loop system of (1) is quadratically stable with H_∞ disturbance attenuation γ for all admissible uncertainties.*

The resulting closed-loop system of (1) under switched state feedback $u(t)=K_rx(t)$ can be written in the form of

$$\begin{cases}x(t+1)=(\widehat{A}_r+\Delta\widehat{A}_r)x(t)+B_{1r}w(t)\\ z(t)=\widehat{C}_rx(t),\quad x(0)=0\end{cases}\qquad(20)$$

where $\widehat{A}_r:=A_r+B_{2r}K_r,\Delta\widehat{A}_r:=E_r\Gamma\widehat{F}_r,\widehat{F}_r:=F_{1r}+F_{2r}K_r,\widehat{C}_r:=C_r+D_rK_r$.

Lemma 3: Take as given the $\alpha_1,\cdots,\alpha_l$ with $\alpha_i>0(\forall i\in\underline{l})$, then the following two statements are equivalent:
(i)There exist a symmetric matrix $P>0$, a scalar $\eta>0$ and feedback gain matrices $K_1,\cdots,K_l$ such that

$$\begin{aligned}&\sum_{i=1}^{l}\alpha_i[\widehat{A}_i^T(P^{-1}-\gamma^{-2}B_{1i}B_{1i}^T-\eta E_iE_i^T)^{-1}\widehat{A}_i\\&+\eta^{-1}\widehat{F}_i^T\widehat{F}_i-P+\widehat{C}_i^T\widehat{C}_i]<0\end{aligned}\qquad(21)$$

with

$$P^{-1}-\gamma^{-2}B_{1i}B_{1i}^T-\eta E_iE_i^T>0,\quad\forall i\in\underline{l}\qquad(22)$$

where $\widehat{A}_i:=A_i+B_{2i}K_i,\widehat{F}_i:=F_{1i}+F_{2i}K_i,\widehat{C}_i:=C_i+D_iK_i$.
(ii) There exist a symmetric matrix $Q>0$, a scalar $\eta>0$ and matrices $Y_1,\cdots,Y_l$ such that the following LMI

$$\begin{array}{ccccc}-\sum_{i=1}^{l}\alpha_iQ & \widetilde{B}_{21} & \cdots & \widetilde{B}_{2l} & 0\\ * & -Q+\eta E_1E_1^T & \cdots & 0 & B_{11}\\ & & \ddots & \vdots & \vdots\\ * & * & & & \\ * & * & * & -Q+\eta E_lE_l^T & 0\\ * & * & * & * & -\gamma^2I\\ * & * & * & * & *\\ * & * & * & * & *\\ * & * & * & * & *\\ * & * & * & * & *\\ * & * & * & * & *\\ * & * & * & * & *\\ * & * & * & * & *\\ * & * & * & * & *\\ * & * & * & * & *\end{array}$$

$$\begin{array}{cccccccc}\cdots & 0 & D_1 & \cdots & D_l & F_{21} & \cdots & F_{2l}\\ \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \cdots & B_{1l} & 0 & \cdots & 0 & 0 & \cdots & 0\\ \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0\\ \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ * & -\gamma^2I & 0 & \cdots & 0 & 0 & \cdots & 0\\ * & * & -I & \cdots & 0 & 0 & \cdots & 0\\ & & & \ddots & \vdots & \vdots & \ddots & \vdots\\ * & * & * & & & & & \\ * & * & * & * & -I & 0 & \cdots & 0\\ * & * & * & * & * & -\eta I & \cdots & 0\\ & & & & & & \ddots & \vdots\\ * & * & * & * & * & * & & \\ * & * & * & * & * & * & * & -\eta I\end{array}<0\qquad(23)$$

is satisfied. Moreover, if (ii) holds, then (i) will hold for matrices $K_i=\frac{1}{\sqrt{\alpha_i}}Y_iQ^{-1}$, where

$$\widetilde{B}_{21}:=\sqrt{\alpha_1}QA_1^T+Y_1^TB_{21}^T,\widetilde{B}_{2l}:=\sqrt{\alpha_l}QA_l^T+Y_l^TB_{2l}^T$$

$$\widetilde{D}_1 := \sqrt{\alpha_1}QC_1^T + Y_1^T D_1^T, \widetilde{D}_l := \sqrt{\alpha_l}QC_l^T + Y_l^T D_l^T$$

$$\widetilde{F}_{21} := \sqrt{\alpha_1}QF_{11}^T + Y_1^T F_{21}^T, \widetilde{F}_{2l} := \sqrt{\alpha_1}QF_{1l}^T + Y_l^T F_{2l}^T$$

Proof: Since

$$\begin{aligned}
&\sum_{i=1}^{l} \alpha_i [\widehat{A}_i^T (P^{-1} - \gamma^{-2} B_{1i} B_{1i}^T - \eta E_i E_i^T)^{-1} \widehat{A}_i \\
&\quad + \eta^{-1} \widehat{F}_i^T \widehat{F}_i - P + \widehat{C}_i^T \widehat{C}_i] \\
&= \widetilde{A}^T (\overline{P}^{-1} - \gamma^{-2} \overline{B}_1 \overline{B}_1^T - \eta \overline{E}\,\overline{E}^T)^{-1} \widetilde{A} \\
&\quad + \eta^{-1} \widetilde{F}^T \widetilde{F} - \sum_{i=1}^{l} \alpha_i P + \widetilde{C}^T \widetilde{C}
\end{aligned}$$

where

$$A := \begin{matrix} \sqrt{\alpha_1}(A_1 + B_{21}K_1) \\ \vdots \\ \sqrt{\alpha_l}(A_l + B_{2l}K_l) \end{matrix}, C := \begin{matrix} \sqrt{\alpha_1}(C_1 + D_1 K_1) \\ \vdots \\ \sqrt{\alpha_l}(C_l + D_l K_l) \end{matrix}$$

$$\widetilde{F} := \begin{bmatrix} \sqrt{\alpha_1}(F_{11} + F_{21}K_1) \\ \vdots \\ \sqrt{\alpha_l}(F_{1l} + F_{2l}K_l) \end{bmatrix}$$

$\overline{P}, \overline{B}_1, \overline{E}$ are matrices defined in (9), by virtue of Schur complement formula, (21) and (22) hold if and only if

$$\begin{matrix} -\sum_{i=1}^{l} \alpha_i P & A^T & C^T & F^T \\ A & -\overline{P}^{-1} + \gamma^{-2}\overline{B}_1\overline{B}_1^T + \eta\overline{E}\,\overline{E}^T & 0 & 0 \\ C & 0 & -I & 0 \\ F & 0 & 0 & -\eta I \end{matrix} < 0 \tag{24}$$

Multiplying $diag\{P^{-1}, I, I, I\}$ on both sides of the left-hand-side matrix of (24) and denote $P^{-1} = Q, Y_i = \sqrt{\alpha_i} K_i Q$, then again, by Schur complement formula, (24) is equivalent to (23). This completes the proof. ∎

Theorem 2: Given a constant $\gamma > 0$, the switched state feedback robust H_∞ control of systems (1) is feasible if there exist a matrix $Q > 0$, matrices $Y_1, \cdots, Y_l$ and a scalar $\eta > 0$ such that the LMI (23) is satisfied for some scalars $\alpha_1, \cdots, \alpha_l > 0$, where the state feedback gain matrices are given by

$$K_i = \frac{1}{\sqrt{\alpha_i}} Y_i Q^{-1}, \quad i \in \underline{l}$$

In this case, the switching rule is taken as

$$\begin{aligned}
r(x,t) = \arg\min_{i \in \underline{l}} \{ & x(t)^T [\widehat{A}_i^T (Q - \gamma^{-2} B_{1i} B_{1i}^T \\
& - \eta E_i E_i^T)^{-1} \widehat{A}_i + \eta^{-1} \widehat{F}_i^T \widehat{F}_i - Q^{-1} + \widehat{C}_i^T \widehat{C}_i] x(t) \}
\end{aligned}$$

where $\widehat{A}_i := A_i + B_{2i}K_i, \widehat{F}_i := F_{1i} + F_{2i}K_i, \widehat{C}_i := C_i + D_i K_i$.

Proof: By Lemma 3 and following similar arguments to the proof of theorem 1, we can prove this result. ∎

B. Switched static output feedback

For switched systems (1), let's consider the synthesis problem of switched static output feedback $u(t) = K_r y(t)$ ensuring that the closed-loop system

$$\begin{cases} x(t+1) = [(A_r + B_{2r}K_r H_r) \\ \quad + E_r \Gamma (F_{1r} + F_{2r} K_r H_r)] x(t) + B_{1r} w(t) \\ z(t) = (C_r + D_r K_r H_r) x(t), \quad x(0) = 0 \end{cases}$$

is quadratically stable with a prescribed H_∞ disturbance attenuation γ for all admissible uncertainties. Without loss of generality, the system matrices $H_i (\forall i \in \underline{l})$ are assumed to be of full row rank. This assumption is reasonable since it can be achieved by discarding redundant measurement components of the output $y(t)$.

Lemma 4: Take as given the $\alpha_1, \cdots, \alpha_l$ with $\alpha_i > 0 (\forall i \in \underline{l})$, then the following condition (ii) implies condition (i):

(i)There exist a symmetric matrix $P > 0$, a scalar $\eta > 0$ and feedback gain matrices $K_1, \cdots, K_l$ such that

$$\begin{aligned}
&\sum_{i=1}^{l} \alpha_i [(A_i + B_{2i}K_iH_i)^T (P^{-1} - \gamma^{-2} B_{1i} B_{1i}^T \\
&\quad - \eta E_i E_i^T)^{-1} (A_i + B_{2i}K_iH_i) + \eta^{-1} (F_{1i} + F_{2i}K_iH_i)^T \\
&\quad \times (F_{1i} + F_{2i}K_iH_i) - P + (C_i + D_iK_iH_i)^T \\
&\quad \times (C_i + D_iK_iH_i)] < 0
\end{aligned} \tag{25}$$

with

$$P^{-1} - \gamma^{-2} B_{1i} B_{1i}^T - \eta E_i E_i^T > 0, \quad \forall i \in \underline{l} \tag{26}$$

(ii) There exist a symmetric matrix $Q > 0$, a scalar $\eta > 0$ and matrices $N_i, V_i (i = 1, \cdots, l)$ such that the following LMI

$$\begin{matrix}
-\sum_{i=1}^{l} \alpha_i Q & H_1 & \cdots & H_l & 0 \\
* & -Q + \eta E_1 E_1^T & \cdots & 0 & B_{11} \\
 & & \ddots & \vdots & \vdots \\
* & * & & & \\
* & * & * & -Q + \eta E_l E_l^T & 0 \\
* & * & * & * & -\gamma^2 I \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{matrix}$$

$$\begin{matrix}
\cdots & 0 & C_1 & \cdots & C_l & F_{11} & \cdots & F_{1l} & \\
\cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \\
\ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
\cdots & B_{1l} & 0 & \cdots & 0 & 0 & \cdots & 0 & \\
\cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \\
\ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\
* & -\gamma^2 I & 0 & \cdots & 0 & 0 & \cdots & 0 & < 0 \\
* & * & -I & \cdots & 0 & 0 & \cdots & 0 & \\
 & & & \ddots & \vdots & \vdots & \ddots & \vdots & \\
* & * & * & & & & & & \\
* & * & * & * & -I & 0 & \cdots & 0 & \\
* & * & * & * & * & -\eta I & \cdots & 0 & \\
 & & & & & & \ddots & \vdots & \\
* & * & * & * & * & * & & & \\
* & * & * & * & * & * & * & -\eta I &
\end{matrix}$$

and

$$H_i Q = V_i H_i, \quad \forall i \in \underline{l} \tag{27}$$

is satisfied. Moreover, if (ii) holds, then (i) will hold for matrices $K_i = \frac{1}{\sqrt{\alpha_i}} N_i V_i^{-1}$, where

$$H_1 := \sqrt{\alpha_1} Q A_1^T + H_1^T N_1^T B_{21}^T, H_l := \sqrt{\alpha_l} Q A_l^T + H_l^T N_l^T B_{2l}^T$$

$$C_1 := \sqrt{\alpha_1} Q C_1^T + H_1^T N_1^T D_1^T, C_l := \sqrt{\alpha_l} Q C_l^T + H_l^T N_l^T D_l^T$$

$$F_{11} := \sqrt{\alpha_1} Q F_{11}^T + H_1^T N_1^T F_{21}^T, F_{1l} := \sqrt{\alpha_l} Q F_{1l}^T + H_l^T N_l^T F_{2l}^T$$

Proof: Assume there exist $Q > 0$ and matrices N_i, V_i such that the LMI in condition (ii) and (27) are satisfied. As H_i is of full row rank and Q is positive definite, it follows from (27) that V_i is of full rank for all $i = 1, \cdots, l$ and then invertible. Again, by (27) and note that $K_i = \frac{1}{\sqrt{\alpha_i}} N_i V_i^{-1}$, we have

$$\sqrt{\alpha_i} K_i H_i Q = N_i H_i, \quad i = 1, \cdots, l$$

Replacing $N_i H_i$ in condition (ii) by $\sqrt{\alpha_i} K_i H_i Q$ and by the Schur complement formula, the result can be proved in the same way as the proof of Lemma 3. ■

Theorem 3: Given a constant $\gamma > 0$, the switched static output feedback robust H_∞ control of systems (1) is feasible if there exist a matrix $Q > 0$, matrices $N_i, V_i (i = 1, \cdots, l)$ and a scalar $\eta > 0$ such that the LMI in condition (ii) and (27) are satisfied for some scalars $\alpha_1, \cdots, \alpha_l > 0$, where the output feedback gain matrices are given by

$$K_i = \frac{1}{\sqrt{\alpha_i}} N_i V_i^{-1}$$

In this case, the switching rule is taken as

$$\begin{aligned} r(x,t) = \arg\min_{i \in \underline{l}} \{ & x(t)^T [(A_i + B_{2i} K_i H_i)^T (Q \\ & -\gamma^{-2} B_{1i} B_{1i}^T - \eta E_i E_i^T)^{-1} (A_i + B_{2i} K_i H_i) \\ & +\eta^{-1} (F_{1i} + F_{2i} K_i H_i)^T (F_{1i} + F_{2i} K_i H_i) - Q^{-1} \\ & +(C_i + D_i K_i H_i)^T (C_i + D_i K_i H_i)] x(t) \} \end{aligned}$$

Proof: By Lemma 4, the result can be proved in the same way as the proof of theorem 1. ■

Remark 2: The method adopted here to construct switching rules is named as the min-projection strategy in some papers (e.g., [13][16][17]). The direct application of min-projection strategy may result in sliding motions. We refer to Pettersson [13] and Sun [10] for discussions of how this behavior can be avoided.

IV. CONCLUSIONS

This paper has studied disturbance attenuation properties of uncertain discrete-time switched systems by employing a constructively designed state-dependent switching rule. A method is proposed to design a switching rule which is not dependent on any uncertainties to guarantee quadratic stability with a prescribed H_∞−norm bound for a switched system. The feasibility of this method is associated with the solvability of a matrix inequality which can be dealt with as a linear matrix inequality (LMI). How to develop other switching rules to cope with the H_∞ control problem for switched systems should be studied in the future work.

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REFERENCES

[1] P. Varaiya, Smart Cars on Smart Roads: Problems of Control, *IEEE Transactions on Automatic Control,* vol. 38, 1993, pp 195-207.

[2] S. Pettersson, Analysis and Design of Hybrid Systems, Ph.D. dissertation, Control Engineering Laboratory, Chalmers University of Technology, 1999.

[3] W.S. Wong, R.W. Brockett, Systems with Finite Communication Bandwidth Constraints-Part I: State Estimation Problems, *IEEE Transactions on Automatic Control,* vol. 42, 1997, pp 1294-1299.

[4] D. Liberzoin, A.S. Morse, Basic Problems in Stability and Design of Switched Systems, *IEEE Control Systems Magazine,* vol.19, 1999, pp 59-70.

[5] R.A. DeCarlo, M.S. Branicky, S. Pettersson, B. Lennartson, Perspectives and Results on the Stability and Stabilizability of Hybrid Systems, *Proceedings of the IEEE,* vol.88, 2000, pp 1069-1082.

[6] Z. Ji, L. Wang, G. Xie and F. Hao, Linear Matrix Inequality Approach to Quadratic Stabilisation of Switched Systems, *IEE Proceedings-Control Theory and Applications,* vol. 151, 2004, pp 289-294.

[7] Z. Ji, L. Wang and D. Xie, Robust H_∞ Control and Quadratic Stabilization of Uncertain Switched Linear Systems, *Proceedings of the 2004 American Control Conference,* Boston, Massachusetts June 30 - July 2, 2004, pp 4543-4548.

[8] G. Zhai, H. Lin, P.J. Antsaklis, Quadratic Stabilizability of Switched Linear Systems with Polytopic Uncertainties, *Int. J. Control,* vol. 76, 2003, pp 747-753.

[9] D. Xie, L. Wang, F. Hao and G. Xie, LMI Approach to L_2-gain Analysis and Control Synthesis of Uncertain Switched Systems, *IEE Proceedings-Control Theory and Applications,* vol. 151, 2004, pp 21-28.

[10] Z. Sun, A Robust Stabilizing Law for Switched Linear Systems, *Int. J. Control,* vol. 77, 2004, pp 389-398.

[11] Z. Sun, S.S. Ge and T.H. Lee, Controllability and Reachability Criteria for Switched Linear Systems, *Automatica,* vol.38, 2002, pp 775-786.

[12] D. Cheng, Stabilization of Planar Switched Systems, *Syst. Contr. Lett.,* vol. 51, 2004, pp 79-88.

[13] S. Pettersson, Synthesis of Switched Linear Systems, *Proceedings of the 42nd IEEE Conference on Decision and Control,* Maui, Hawaii USA, 2003, pp 5283-5288.

[14] J.P. Hespanha, Root-Mean-Square Gains of Switched Linear Systems, *IEEE Transactions on Automatic Control,* vol. 48, 2003, pp 2040-2045.

[15] G. Zhai, B. Hu, K. Yasuda and A.N. Michel, Disturbance Attenuation Properties of Time-Controlled Switched Systems, *Journal of the Franklin Institute,* vol. 338, 2001, pp 765-779.

[16] M.A. Wicks, P. Peleties, R.A. DeCarlo. Construction of Piecewise Lyapunov Functions for Stabilizing Switched Systems, *Proceedings of the 33rd Conference on Decision and Control,* Lake Buena Vista, December 1994, pp 3492-3497.

[17] E. Feron, Quadratic Stabilizability of Switched System via State and Output Feedback", *MIT Technical Report CICS-P-468,* 1996.

[18] M.S. Branicky, Multiple Lyapunov Functions and Other Analysis Tools for Switched and Hybrid Systems. *IEEE Transactions on Automatic Control,* vol.43, 1998, pp 475-482.

[19] Z.G. Li, C.Y. Wen and Y.C. Soh, Stabilization of a Class of Switched Systems via Designing Switching Laws, *IEEE Transactions on Automatic Control,* vol.46, 2001, pp 665-670.

[20] P.P. Khargonekar, I.R. Petersen and K. Zhou, Robust Stabilization of Uncertain Linear Systems: Quadratic Stabilizability and H_∞ Control Theory, *IEEE Transactions on Automatic Control,* vol.35, 2001, pp 356-361.

[21] G. Gahinet, A. Nemirovski, A.J. Laub, M. Chilali, *LMI Control Toolbox for Use with Matlab,* The Mathworks Inc (1995).