Converse Lyapunov theorems and robust asymptotic stability for hybrid systems

Chaohong Cai, Andrew R. Teel, Rafal Goebel

Abstract— **We state results on the existence of smooth Lyapunov functions for hybrid systems whose solutions satisfy a class-$\mathcal{KLL}$ estimate with respect to two measures. The class-$\mathcal{KLL}$ estimate, a natural extension of class-$\mathcal{KL}$ estimate, is in terms of the elapsed time and the number of jumps that have occurred. The main result is that a smooth Lyapunov function exists if and only if the class-$\mathcal{KLL}$ estimate is robust. In turn, sufficient conditions for robustness are given. Special cases include systems with compact attractors. Most of the results parallel, and unify, what has been developed previously for differential inclusions and difference inclusions.**

I. Introduction

Hybrid systems are ones whose trajectories can flow in continuous time and also jump at discrete instants. The system variables can be dynamical processes (states) and/or logical processes (modes). The flow can be governed by differential equations or differential inclusions, and the jumps can be described by difference equations or difference inclusions. Examples of hybrid systems are ubiquitous in science and engineering [28], [15].

Stability theory for systems depends on the definition of their solutions. To develop stability theory for hybrid systems, researchers have proposed several different solution concepts on (hybrid) time domain/space and provided related results on solutions (see [26], [20], [17], [28], [3], [1], [22], [5], [8]). As for the Lyapunov characterization of asymptotic stability for general or specific hybrid systems, many sufficient conditions have been proposed in literature (see [2], [29], [20], [6], [15], [18], [24]), and some necessary conditions (converse Lyapunov theorems) have been established as well (see [29], [19]). However, few general statements are available on the robustness of asymptotic stability for hybrid systems (for example, see [23], [7], [8]), and to the best of the authors' knowledge, no results on the existence of *smooth* Lyapunov functions have appeared. Also, converse Lyapunov theorems for stability with respect to (w.r.t.) two measures have not been addressed in the literature. In this paper, we provide such converse Lyapunov results inspired by the previous ones for differential and difference inclusions in [11], [16], [4], [27], [10], [9].

We list the following definitions and notation:

Research partially supported by the Army Research Office under Grant no. DAAD19-03-1-0144, the National Science Foundation under Grant no. CCR-0311084 and Grant no. ECS-0324679, and by the Air Force Office of Scientific Research under Grant no. F49620-03-1-0203.

Authors are with Center for Control Engineering & Computation, Department of Electrical & Computer Engineering, University of California, Santa Barbara. Email: cai, teel, rafal@ece.ucsb.edu.

- $\mathbb{R}_{\geq 0} = [0, +\infty)$, $\mathbb{N}_{\geq 0} = \{0, 1, 2, ...\}$, $\mathbb{N}^n_{\geq 0} = \mathbb{N}_{\geq 0} \times \mathbb{N}_{\geq 0} \times \cdots \times \mathbb{N}_{\geq 0}$ (n copies).
- Given a vector $v \in \mathbb{R}^n$, v' denotes the transpose of v.
- $\mathcal{B}$ is the open unit ball in Euclidean space.
- Given $\mathcal{A}, \mathcal{G} \subset \mathbb{R}^n$, $\mathcal{A} + \mathcal{G} := \{a + g : a \in \mathcal{A}, g \in \mathcal{G}\}$.
- Given a set $\mathcal{A}$, $\overline{\mathcal{A}}$ stands for the closure of $\mathcal{A}$, $\overline{\text{co}}\mathcal{A}$ stands for the closed convex hull of $\mathcal{A}$.
- Given $\mathcal{A} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_\mathcal{A} := \inf_{y \in \mathcal{A}} |x - y|$.
- Given an open set $\mathcal{X}$ containing a closed set $\mathcal{A}$, a function $\omega : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is a **proper indicator for** $\mathcal{A}$ **on** $\mathcal{X}$ if ω is continuous, $\omega(x) = 0$ if and only if $x \in \mathcal{A}$, and ω blows up when its argument approaches the boundary of $\mathcal{X}$ or goes unbounded.
- Given $\mathcal{X}_1 \subset \mathcal{X}$, $\mathcal{X}_1$ is **relatively open** (respectively, **relatively closed**) in $\mathcal{X}$ if there exists an open (respectively, closed) set $\mathcal{X}_2$ such that $\mathcal{X}_1 = \mathcal{X}_2 \cap \mathcal{X}$.
- The **domain** of a set-valued mapping $M : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is the set $\operatorname{dom} M := \{x \in \mathcal{O} : M(x) \neq \emptyset\}$.
- A set-valued mapping $M : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is outer semicontinuous at $x \in \mathcal{O}$ if for all sequences $x_i \to x$ and $y_i \in M(x_i)$, if $\lim_{i\to\infty} y_i = y$ for some y, then $y \in M(x)$. The mapping M is **outer semicontinuous** (OSC) if it is outer semicontinuous at each $x \in \mathcal{O}$.
- A set-valued mapping $M : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is **locally bounded** if for any compact $K \subset \mathcal{O}$ there exists $m > 0$ such that $M(K) := \bigcup_{x \in K} M(x) \subset m\mathcal{B}$; if M is OSC and locally bounded, then $M(K)$ is compact. For locally bounded set-valued mappings with closed values, OSC agrees with upper semicontinuity.
- A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to **class-$\mathcal{K}$** ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing. It is said to belong to **class-$\mathcal{K}_\infty$** if, in addition, it is unbounded.
- A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to **class-$\mathcal{KL}$** ($\beta \in \mathcal{KL}$) if it satisfies: $\forall t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s\to 0^+} \beta(s, t) = 0$, and $\forall s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s, t) = 0$.
- A function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to **class-$\mathcal{KLL}$** ($\gamma \in \mathcal{KLL}$) if, for each $r \geq 0$, $\gamma(\cdot, \cdot, r) \in \mathcal{KL}$ and $\gamma(\cdot, r, \cdot) \in \mathcal{KL}$.

II. Hybrid Inclusions

Consider hybrid inclusions on an open set $\mathcal{O} \subset \mathbb{R}^n$,

$$\mathcal{H} := \begin{cases} \dot{x} \in F(x) & \text{for} \quad x \in C, \\ x^+ \in G(x) & \text{for} \quad x \in D, \end{cases} \tag{1}$$

where $F : \mathcal{O} \rightrightarrows \mathbb{R}^n$ describes the flow, which can occur in the set $C \subset \mathcal{O}$, and $G : \mathcal{O} \rightrightarrows \mathcal{O}$ describes the jumps, which can occur from the set $D \subset \mathcal{O}$. The state x may include both continuous and discrete variables, the latter often consisting of logical modes like "on" and "off" which can be associated with integer values. In Section IV-B we will partially reconcile such an association with the use of an open state space in $\mathbb{R}^n$. The treatment of (1) will be based on the tools developed in [7], [8]. Similar hybrid inclusions have been addressed in [1], [13], [5].

The solutions to the hybrid inclusion (1) are defined on hybrid time domains, as used in [7], [8] and [5]. We call a subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}_{\geq 0}$ a **compact hybrid time domain** if $E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_J$. We say E is a **hybrid time domain** if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots J\})$ is a compact hybrid time domain.

A **hybrid arc** is a function ϕ defined on a hybrid time domain, and such that $\phi(\cdot, j)$ is locally absolutely continuous for each j. A hybrid arc $\phi : \operatorname{dom}\phi \mapsto \mathcal{O}$ is a **solution** to $\mathcal{H}$ if (i) for all $j \in \mathbb{N}_{\geq 0}$ and almost all $t \in \mathbb{R}_{\geq 0}$ such that $(t, j) \in \operatorname{dom}\phi$,

$$\phi(t, j) \in C, \quad \dot{\phi}(t, j) \in F(\phi(t, j)),$$

and (ii) for all $(t, j) \in \operatorname{dom}\phi$ such that $(t, j+1) \in \operatorname{dom}\phi$,

$$\phi(t, j) \in D, \quad \phi(t, j+1) \in G(\phi(t, j)).$$

A solution to $\mathcal{H}$ is **maximal** if it cannot be extended, and **complete** if its domain is unbounded. Complete solutions are maximal. Henceforth, we will make explicit the dependence of the solutions on the initial condition, which will be denoted by x. By slight abuse of notation, we will use $\phi(t, j, x)$ to denote a solution to $\mathcal{H}$ starting at x and evaluated at $(t, j) \in \operatorname{dom}\phi$. We denote by $\mathcal{S}(x)$ the set of all maximal solutions to $\mathcal{H}$ starting from x. $\mathcal{H}$ is **forward complete** on $\mathcal{X} \subset \mathcal{O}$ if, for all $x \in \mathcal{X}$, each $\phi \in \mathcal{S}(x)$ is complete and $\phi(t, j, x) \in \mathcal{X}$ for each $(t, j) \in \operatorname{dom}\phi$. Given $\mathcal{S}(\cdot)$ and any function $\omega : \mathcal{X} \to \mathbb{R}_{\geq 0}$, define $\mathbf{A}(\cdot, \cdot)$ as

$$\mathbf{A}(\mathcal{S}, \omega) := \left\{ \xi \in \mathcal{X} : \sup_{\substack{\phi \in \mathcal{S}(\xi) \\ (t,j) \in \operatorname{dom}\phi}} \omega(\phi(t, j, \xi)) = 0 \right\}.$$

In order to guarantee that the solutions to $\mathcal{H}$ will have properties suitable for establishing the existence of smooth Lyapunov functions, we will impose the following basic hybrid conditions throughout the paper:

***Standing Assumption** 1 (Hybrid basic conditions):* The data of the system $\mathcal{H}$ satisfy

- $C \subset \mathcal{O}$ and $D \subset \mathcal{O}$ are relatively closed sets in $\mathcal{O}$;
- $F : \mathcal{O} \rightrightarrows \mathbb{R}^n$ is OSC and locally bounded, and $F(x)$ is nonempty and convex for each $x \in C$;
- $G : \mathcal{O} \rightrightarrows \mathcal{O}$ is OSC and locally bounded, and $G(x)$ is nonempty for each $x \in D$.

These basic conditions are the weakest ones possible for our purposes. In particular, if any one of the conditions is not imposed then examples can be found that don't satisfy our converse Lyapunov theorems, i.e., don't admit a smooth Lyapunov function. For more details, see Section VI-D.

General statements about the existence of solutions under the hybrid basic conditions can be found in [1], [5] and [8].

III. $\mathcal{KLL}$-STABILITY, ASYMPTOTIC STABILITY AND THEIR STRONG SUFFICIENT LYAPUNOV CONDITIONS

A. $\mathcal{KLL}$-stability

Ultimately we will be interested in a property of solutions called $\mathcal{KLL}$-stability w.r.t. two measures:

***Definition** 1:* Let $\mathcal{X} \subset \mathcal{O}$ be open and $\omega_i : \mathcal{X} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous. $\mathcal{H}$ is **$\mathcal{KLL}$-stable w.r.t.** (ω_1, ω_2) **on** $\mathcal{X}$ if $\mathcal{H}$ is forward complete on $\mathcal{X}$ and there exists $\gamma \in \mathcal{KLL}$ such that for each $x \in \mathcal{X}$, all solutions $\phi \in \mathcal{S}(x)$ satisfy

$$\omega_1(\phi(t, j, x)) \leq \gamma(\omega_2(x), t, j), \ \forall (t, j) \in \operatorname{dom}\phi.$$

$\mathcal{KLL}$-stability w.r.t. two measures covers many stability concepts that have appeared in the literature. It is the generalization to hybrid systems of two measure stability, which was initially developed in [21] and studied extensively in [12]. One interesting example is "output stability", covered by taking $\omega_2(\cdot) = |\cdot|$ and $\omega_1(\cdot) = |h(\cdot)|$ where $h : \mathcal{X} \to \mathcal{X}$ is an output map; see [25]. The special case where $\omega_1 = \omega_2 =: \omega$ is called $\mathcal{KLL}$-stability with respect to a single measure. In the next subsection, we relate $\mathcal{KLL}$-stability w.r.t. a single measure to asymptotic stability of a compact set. **Uniform global asymptotic stability** of a closed set $\mathcal{A}$ (possibly compact) corresponds to $\mathcal{KLL}$-stability w.r.t. $\omega_1(x) = \omega_2(x) = |x|_{\mathcal{A}}$ on $\mathcal{X} = \mathbb{R}^n$. $\mathcal{KLL}$-stability w.r.t. two measures can be phrased in a slightly different way:

***Proposition** 1:* Let $\mathcal{X} \subset \mathcal{O}$ be open and $\omega_i : \mathcal{X} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous. We have the equivalent statements:

1) $\mathcal{H}$ is $\mathcal{KLL}$-stable w.r.t. (ω_1, ω_2) on $\mathcal{X}$.
2) All of the following hold:
   - $\mathcal{H}$ is forward complete on $\mathcal{X}$.
   - (Uniform stability and global boundedness): $\exists \alpha \in \mathcal{K}_\infty$ s.t. $\forall x \in \mathcal{X}$, all $\phi \in \mathcal{S}(x)$ satisfy
   $$\omega_1(\phi(t, j, x)) \leq \alpha(\omega_2(x)), \ \forall (t, j) \in \operatorname{dom}\phi.$$
   - (Uniform global attractivity): $\forall r > 0$, $\forall \varepsilon > 0$, $\exists \mathcal{T}(r, \varepsilon) > 0$ s.t. $\forall x \in \mathcal{X}$, all $\phi \in \mathcal{S}(x)$ satisfy
   $$\omega_2(x) \leq r, \ (t, j) \in \operatorname{dom}\phi, \ t + j \geq \mathcal{T} \implies \omega_1(\phi(t, j, x)) \leq \varepsilon.$$

B. Asymptotically stable compact sets

Let $\mathcal{A}$ be a compact subset of $\mathcal{O}$. The set $\mathcal{A}$ is **stable** if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x \in (\mathcal{A} + \delta\overline{\mathcal{B}}) \cap (C \cup D)$, each solution $\phi \in \mathcal{S}(x)$ is complete and satisfies $|\phi(t, j, x)|_{\mathcal{A}} \leq \varepsilon$ for all $(t, j) \in \operatorname{dom}\phi$; it is **attractive** if there exists $\mu > 0$ such that for each $x \in (\mathcal{A} + \mu\overline{\mathcal{B}}) \cap (C \cup D)$, each solution $\phi \in \mathcal{S}(x)$ is complete and satisfies $\lim_{(t,j) \in \operatorname{dom}\phi, \ t+j \to \infty} |\phi(t, j, x)|_{\mathcal{A}} = 0$; and it is **asymptotically stable** if it is both stable and attractive. The set of points from which all maximal solutions are complete and converge to $\mathcal{A}$ is called the **basin of attraction** for $\mathcal{A}$ and is denoted $\mathcal{X}_{\mathcal{A}}$. The set $\mathcal{A}$ is **globally asymptotically stable** if $\mathcal{A}$ is asymptotically stable and $\mathcal{X}_{\mathcal{A}} = \mathbb{R}^n$.

In [8, Proposition 6.1] it is established that the basin of attraction is open relative to $C \cup D$ when local existence of solutions is guaranteed. In this paper we focus mainly on the situation where the basin of attraction is open. It is clear that if ω is a proper indicator for $\mathcal{A}$ on an open set $\mathcal{X} \subset \mathcal{O}$ and the hybrid system $\mathcal{H}$ is $\mathcal{KLL}$-stable w.r.t. (ω, ω), then $\mathcal{A}$ is asymptotically stable with the basin of attraction containing $\mathcal{X}$. Theorem 6.2 in [8], specialized to the case where the basin of attraction is open, establishes that asymptotic stability implies $\mathcal{KLL}$-stability with respect to any proper indicator on the basin of attraction. It parallels known results for differential inclusions and difference inclusions (see [27, Proposition 3] and [10, Proposition 2]).

C. Strong sufficient Lyapunov conditions

1) $\mathcal{KLL}$-stability w.r.t. a single measure:

***Definition** 2:* Let $\mathcal{X} \subset \mathcal{O}$ be open and $\omega : \mathcal{X} \to \mathbb{R}_{\geq 0}$ be continuous. A function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is said to be a **smooth Lyapunov function w.r.t.** ω **on** $\mathcal{X}$ **for** $\mathcal{H}$ if V is smooth and there exist class-$\mathcal{K}_\infty$ functions α_1, α_2 such that

$$\alpha_1(\omega(x)) \ \leq \ V(x) \ \leq \ \alpha_2(\omega(x)) \qquad \forall x \in \mathcal{X} \ , \tag{2}$$

$$\max_{f \in F(x)} \langle \nabla V(x), f \rangle \ \leq \ -V(x) \qquad \forall x \in \mathcal{X} \cap C \ , \tag{3}$$

$$\max_{g \in G(x)} V(g) \ \leq \ e^{-1} V(x) \qquad \forall x \in \mathcal{X} \cap D \ . \tag{4}$$

We first establish that when $\mathcal{X}$ is forward complete the existence of a smooth Lyapunov function is a sufficient condition for asymptotic stability, albeit a restrictive one. For weaker sufficient conditions for asymptotic stability see [6], [18], [24]. The two last papers contain generalizations of LaSalle's invariance principle to hybrid systems.

***Proposition** 2:* Let $\omega : \mathcal{X} \to \mathbb{R}_{\geq 0}$ be continuous on $\mathcal{X}$. If $\mathcal{H}$ is forward complete on $\mathcal{X}$ and there exists a smooth Lyapunov function for $\mathcal{KLL}$-stability w.r.t. ω on $\mathcal{X}$ for $\mathcal{H}$, then $\mathcal{H}$ is $\mathcal{KLL}$-stable w.r.t. ω on $\mathcal{X}$.

When $\mathcal{X} \subset (C \cup D)$ and ω is a proper indicator function for a compact set, the forward completeness is guaranteed by the existence of a smooth Lyapunov function. Thus, the following corollary of Proposition 2 ensues.

***Corollary** 1:* Suppose there exists a smooth Lyapunov function w.r.t. ω on $\mathcal{X}$ for $\mathcal{H}$ where ω is a proper indicator for the compact set $\mathcal{A}$ on the open set $\mathcal{X}$. If $\mathcal{X} \subset (C \cup D)$ then, for $\mathcal{H}$, the set $\mathcal{A}$ is locally asymptotically stable with a basin of attraction containing $\mathcal{X}$.

2) $\mathcal{KLL}$-stability w.r.t. two measures:

***Definition** 3:* Let $\mathcal{X} \subset \mathcal{O}$ be open and $\omega_i : \mathcal{X} \to \mathbb{R}_{\geq 0}, i = 1, 2$, be continuous. A function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is a **smooth Lyapunov function for $\mathcal{KLL}$-stability w.r.t.** (ω_1, ω_2) **on** $\mathcal{X}$ **for** $\mathcal{H}$ if V is smooth and there exist class-$\mathcal{K}_\infty$ functions α_1, α_2 such that

$$\alpha_1(\omega_1(x)) \ \leq \ V(x) \ \leq \ \alpha_2(\omega_2(x)) \ \ \forall x \in \mathcal{X} \ , \tag{5}$$

and (3)-(4) hold, and

$$V(x) = 0 \iff x \in \mathbf{A}(\mathcal{S}, \omega_1). \tag{6}$$

***Remark** 1:* When $\omega_1 = \omega_2 = \omega$, the bounds (5) and the decrease conditions (3)-(4) automatically imply (6).

***Proposition** 3:* Let $\omega_i : \mathcal{X} \to \mathbb{R}_{\geq 0}, i = 1, 2$, be continuous on $\mathcal{X}$. If $\mathcal{H}$ is forward complete on $\mathcal{X}$ and there exists a smooth Lyapunov function for $\mathcal{KLL}$-stability w.r.t. (ω_1, ω_2) on $\mathcal{X}$ for $\mathcal{H}$, then $\mathcal{H}$ is $\mathcal{KLL}$-stable w.r.t. (ω_1, ω_2) on $\mathcal{X}$.

***Remark** 2:* Proposition 3 generalizes Proposition 2 and holds even without imposing (6) in Definition 3. Condition (6) will be used later to show that a smooth Lyapunov function, as we have defined it, is sufficient for robust $\mathcal{KLL}$-stability as defined later in Section VI. Proposition 3 holds when smoothness is relaxed to continuous differentiability.

IV. Converse Lyapunov Theorems for Asymptotically Stable Compact Sets

In this section, we establish that a smooth Lyapunov function exists for an asymptotically stable compact set when the basin of attraction is open. The results that can be stated are more subtle when the basin of attraction is not open, like for the bouncing ball [28], the rocking block [18], and control systems with logic variables.

A. Open basin of attraction

Our first converse Lyapunov theorem is the following:

***Theorem** 1:* If the compact set $\mathcal{A}$ is locally asymptotically stable with an open basin of attraction $\mathcal{X}_{\mathcal{A}}$ and ω is a proper indicator for $\mathcal{A}$ on $\mathcal{X}_{\mathcal{A}}$, then there exists a smooth Lyapunov function w.r.t. ω on $\mathcal{X}_{\mathcal{A}}$ for $\mathcal{H}$.

Theorem 1 will follow from our general converse Lyapunov theorem w.r.t. two measures and robustness results on $\mathcal{KLL}$-stability in Subsection VI-C. If $C = \mathcal{O}$ and $D = \emptyset$ (respectively, $D = \mathcal{O}$ and $C = \emptyset$), then Theorem 1 covers converse Lyapunov theorems for locally asymptotic stability in differential (respectively, difference) inclusions; see [4, Theorem 1.2], [27, Corollary 3], and [10, Corollary 1].

B. Non-open basin of attraction

To construct smooth Lyapunov functions for the situation where the basin of attraction is not open, we extend the definition of the system data so that the basin of attraction becomes open, without changing $\mathcal{S}(x)$ for x in the original basin of attraction. While it is not clear how to do this generally, it can be done for a wide class of systems. Rather than attempting an exhaustive classification, we will only discuss a class of systems that covers switching nonlinear systems and systems with discrete logic variables.

Using (v_1, v_2) for $[v_1' \ v_2']'$, consider the hybrid system

$$\mathcal{H}^0 := \left\{ \begin{array}{rcl} \dot{\xi} & \in F_q(\xi) & \text{for} \quad \xi \in C_q \ , \\ (\xi^+, q^+) & \in G_q(\xi) & \text{for} \quad \xi \in D_q \ , \end{array} \right.$$

with state space $\mathcal{O}^0 := \left\{ (\xi, q) : \xi \in \mathcal{O}_q, \ q \in \mathbb{N}^{n_q}_{\geq 0} \right\}$ under the following assumption :

***Assumption** 1:* For each $q \in \mathbb{N}^{n_q}_{\geq 0}$,

- $\mathcal{O}_q \subset \mathbb{R}^{n_\xi}$ is open, $C_q \subset \mathcal{O}_q$ and $D_q \subset \mathcal{O}_q$ are relatively closed in $\mathcal{O}_q$, and $C_q \cup D_q = O_q$;
- $F_q : \mathcal{O}_q \rightrightarrows \mathbb{R}^{n_\xi}$ is OSC and locally bounded, and $F_q(\xi)$ is nonempty and convex for each $\xi \in C_q$;
- $G_q : \mathcal{O}_q \rightrightarrows \mathcal{O}_q \times \mathbb{N}^{n_q}_{\geq 0}$ is OSC and locally bounded, and $G_q(\xi)$ is nonempty for each $\xi \in D_q$.

A special case that is frequently considered is when F_q and G_q are single-valued continuous maps for each q, and perhaps the first n_ξ components of G_q equal ξ so that only the variable q makes jumps.

We now let $\varepsilon \in [0, 1/2)$ and define the extended state space $\mathcal{O}^\varepsilon := \left\{(\xi, q) : \xi \in \mathcal{O}_{\Pi(q)},\ q \in \mathbb{N}^{n_q}_{\geq 0} + \varepsilon\mathcal{B}\right\}$ where $\Pi(q)$ denotes the unique (since $\varepsilon < 1/2$) closest point in $\mathbb{N}^{n_q}_{\geq 0}$ to q. Note that $\mathcal{O}^\varepsilon$ is open for each $\varepsilon \in (0, 1/2)$. For $\varepsilon = 0$, the definition matches the definition of the original state space. We also define

$$\begin{aligned} C^\varepsilon &:= \left\{(\xi, q) \in \mathcal{O}^\varepsilon : \xi \in C_{\Pi(q)}\ , q \in \mathbb{N}^{n_q}_{\geq 0} + \varepsilon\mathcal{B}\right\}, \\ D^\varepsilon &:= \left\{(\xi, q) \in \mathcal{O}^\varepsilon : \xi \in D_{\Pi(q)}\ , q \in \mathbb{N}^{n_q}_{\geq 0} + \varepsilon\mathcal{B}\right\}, \end{aligned}$$

and, for each $x \in \mathcal{O}^\varepsilon$, we define

$$\begin{aligned} F^\varepsilon(x) &:= \{(f_1, f_2) : f_1 \in F_{\Pi(q)}(\xi), \\ &\qquad f_2 = -q + \Pi(q), (\xi, q) = x\}, \\ G^\varepsilon(x) &:= \{g : g \in G_{\Pi(q)}(\xi)\ ,\ (\xi, q) = x\}, \end{aligned}$$

$$\mathcal{H}^\varepsilon := \begin{cases} \dot{x} \in F^\varepsilon(x) & \text{for} \quad x \in C^\varepsilon\ , \\ x^+ \in G^\varepsilon(x) & \text{for} \quad x \in D^\varepsilon\ . \end{cases}$$

The following observations motivate the next theorem:

- The component q satisfies $\dot{q} = -q + \Pi(q)$, the solutions of which from $q_\circ$ converge to $\Pi(q_\circ)$.
- The behavior of $\mathcal{H}^0$ on $\mathcal{O}^0$ is captured by using the state space $\mathcal{O}^\varepsilon$ for some $\varepsilon \in (0, 1/2)$ and the data $(F^\varepsilon, G^\varepsilon, C^0, D^0)$. Because of Assumption 1, the data satisfy the hybrid basic conditions. Therefore, our earlier discussions about local asymptotic stability apply to $\mathcal{H}^0$. In particular, if there is a compact set that is locally asymptotically stable for $\mathcal{H}^0$, its basin of attraction is open relative to $C^0 \cup D^0$.
- For each $\varepsilon \in (0, 1/2)$, the data $(F^\varepsilon, G^\varepsilon, C^\varepsilon, D^\varepsilon)$ for $\mathcal{H}^\varepsilon$ with state space $\mathcal{O}^\varepsilon$ satisfy the hybrid basic conditions. Moreover, since $C_q \cup D_q = \mathcal{O}_q$ for each $q \in \mathbb{N}_{\geq 0}$ it follows that $C^\varepsilon \cup D^\varepsilon = \mathcal{O}^\varepsilon$. Thus, since $\mathcal{O}^\varepsilon$ is open for each $\varepsilon \in (0, 1/2)$, if there is a compact set that is locally asymptotically stable for $\mathcal{H}^\varepsilon$, then its basin of attraction is open.
- If, for $\mathcal{H}^0$, the compact set $\mathcal{A}^0$ is asymptotically stable with the basin of attraction $\mathcal{X}^0_{\mathcal{A}} = \left\{(\xi, q) : \xi \in \mathcal{X}_q,\ q \in \mathbb{N}^{n_q}_{\geq 0}\right\}$ then, for $\mathcal{H}^\varepsilon$, $\mathcal{A}^0$ is also asymptotically stable with the basin of attraction $\mathcal{X}^\varepsilon_{\mathcal{A}} := \left\{(\xi, q) : \xi \in \mathcal{X}_{\Pi(q)}\ , q \in \mathbb{N}^{n_q}_{\geq 0} + \varepsilon\mathcal{B}\right\}$. This is because, for each $x \in \mathcal{O}^\varepsilon$, the projection of a solution of $\mathcal{H}^\varepsilon$ to $\mathcal{O}^0$ is exactly the corresponding solution of $\mathcal{H}^0$ from the initial condition projected to $\mathcal{O}^0$.

Theorem 2: Let Assumption 1 hold. For $\mathcal{H}^0$, suppose the compact set $\mathcal{A}^0 \subset \mathcal{O}^0$ is asymptotically stable with the basin of attraction $\mathcal{X}^0_{\mathcal{A}}$. Then:

1) for each $q \in \mathbb{N}_{\geq 0}$, there exist a compact (possibly empty) set $\mathcal{A}_q$ and an open (possibly empty) set $\mathcal{X}_q$ such that $\mathcal{A}_q \subset \mathcal{X}_q$, $\mathcal{A}^0 = \left\{(\xi, q) : \xi \in \mathcal{A}_q,\ q \in \mathbb{N}^{n_q}_{\geq 0}\right\}$ and $\mathcal{X}^0_{\mathcal{A}} = \left\{(\xi, q) : \xi \in \mathcal{X}_q,\ q \in \mathbb{N}^{n_q}_{\geq 0}\right\}$;
2) for each $q \in \mathbb{N}^{n_q}_{\geq 0}$, let $\omega_q : \mathcal{X}_q \to \mathbb{R}_{\geq 0}$ be a proper indicator function for $\mathcal{A}_q$ on $\mathcal{X}_q$. Then there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and, for each $q \in \mathbb{N}^{n_q}_{\geq 0}$, a smooth function $V_q : \mathcal{X}_q \to \mathbb{R}_{\geq 0}$ such that

$$\begin{aligned} \alpha_1(\omega_q(\xi)) \ &\leq\ V_q(\xi) \leq \alpha_2(\omega_q(\xi)) \quad \forall \xi \in \mathcal{X}_q, \\ \max_{f \in F_q(\xi)} \langle \nabla V_q(\xi), f\rangle &\leq -V_q(\xi) \qquad \forall \xi \in \mathcal{X}_q \cap C_q, \\ \max_{(g_1, g_2) \in G_q(\xi)} V_{g_2}(g_1) &\leq e^{-1} V_q(\xi) \quad \forall \xi \in \mathcal{X}_q \cap D_q. \end{aligned}$$

V. Converse Lyapunov Theorems for $\mathcal{KLL}$-stability w.r.t. a Single Measure

Like in the continuous-time and discrete-time cases, it is unknown whether the existence of a smooth Lyapunov function is necessary, in general, for $\mathcal{KLL}$-stability w.r.t. a single measure. We will establish that the existence of a smooth Lyapunov function is necessary for $\mathcal{KLL}$-stability w.r.t. a single measure when an additional assumption holds. This assumption parallels what is used in continuous-time systems, respectively discrete-time systems, when not assuming that F is locally Lipschitz, respectively G is continuous. It can be seen as the hybrid counterpart of the one on the backward completability by ω-normalization for differential inclusions (see Proposition 2 in [27]) or, of the one for difference inclusions (see Proposition 1 in [10]).

Assumption 2: The open set $\mathcal{X} \subset \mathcal{O}$, the continuous function $\omega : \mathcal{X} \to \mathbb{R}_{\geq 0}$, and the hybrid system $\mathcal{H}$ satisfy

(i) for each $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, there exists a locally bounded function $\psi : \mathcal{X} \to \mathbb{R}_{\geq 0}$ such that, for each $x \in \mathcal{X}$, $\phi \in \mathcal{S}(x)$ and $(t, j) \in \operatorname{dom}\phi$ with $t + j \leq 1$

$$\begin{aligned} &\underline{\alpha}(\omega(x)) \leq \omega(\phi(t, j, x)) \leq \overline{\alpha}(\omega(x)) \\ &\quad \Longrightarrow\ |\phi(t, j, x) - x| \leq \psi(\phi(t, j, x)); \end{aligned}$$

(ii) for any (finite) point z on the boundary of $\mathcal{X}$,

$$z_i \in \mathcal{X},\ z_i \to z \implies \max\left\{\frac{1}{\omega(z_i)}, \omega(z_i)\right\} \to \infty.$$

Under Assumption 2, we have the following result.

Theorem 3: Let Assumption 2 hold and $\mathcal{H}$ be $\mathcal{KLL}$-stable w.r.t. ω on $\mathcal{X}$. Then there exists a smooth Lyapunov function w.r.t. ω on $\mathcal{X}$ for $\mathcal{H}$.

The downside of Assumption 2 is that item (i) is expressed in terms of solutions. The following sufficient condition for Assumption 2 is expressed in terms of the data (F, G, C, D), the set $\mathcal{X}$, and the function ω on $\mathcal{X}$.

Assumption 3: The open set $\mathcal{X} \subset \mathcal{O}$, the continuous function $\omega : \mathcal{X} \to \mathbb{R}_{\geq 0}$, and the hybrid system $\mathcal{H}$ are such that there exists a decomposition of the state $x = [x_1' \ x_2']' \in \mathcal{X}$ satisfying the following:

- there exist $c \geq 0$ and $\theta \in \mathcal{K}_\infty$ such that

$$|x_1| \leq \theta(\omega(x)) + \theta(|x_2|) + c, \quad \forall x \in \mathcal{X};$$

- there exist $l \geq 0$ and $b \geq 0$ and a locally bounded function $\varphi : \mathcal{X} \to \mathbb{R}_{\geq 0}$ such that, for each $x \in \mathcal{X}$,

$$g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in G(x),\ x \in D \ \Rightarrow \ |g_2 - x_2| \leq \varphi(g),$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in F(x),\ x \in C \ \Rightarrow \ |v_2| \leq l|x_2| + b;$$

- item (ii) in Assumption 2 holds.

***Proposition** 4:* Assumption 3 implies Assumption 2.

***Remark** 3:* Assumption 3 can also be applied to hybrid systems with constant parameters and time-varying hybrid systems. Space limitation prevents doing this here.

VI. Robust $\mathcal{KLL}$-stability and Converse Lyapunov Theorems

All of the converse Lyapunov theorems presented so far are enabled by the fact, established in Subsection VI-C, that a smooth Lyapunov function exists if and only if the $\mathcal{KLL}$-stability is robust, and by showing that the situations considered so far guarantee robustness. In this section, we define robust $\mathcal{KLL}$-stability.

A. Robust $\mathcal{KLL}$-stability w.r.t. a single measure

Let the function $\sigma : \mathcal{X} \to \mathbb{R}_{\geq 0}$ be continuous on $\mathcal{X}$ such that $\{x\} + \sigma(x)\overline{\mathcal{B}} \subset \mathcal{X}$ for all $x \in \mathcal{X}$. Given a hybrid system $\mathcal{H}$, define its **σ-perturbed hybrid system** $\mathcal{H}_\sigma$ by

$$\begin{aligned} F_\sigma(x) &:= \overline{\text{co}} F((x + \sigma(x)\overline{\mathcal{B}}) \cap C) + \sigma(x)\overline{\mathcal{B}},\ \forall x \in \mathcal{X}, \\ G_\sigma(x) &:= \{v \in \mathcal{X} : v \in \{g\} + \sigma(g)\overline{\mathcal{B}}, \\ &\qquad g \in G((x + \sigma(x)\overline{\mathcal{B}}) \cap D)\},\ \forall x \in \mathcal{X}, \\ C_\sigma &:= \{x \in \mathcal{X} : (x + \sigma(x)\overline{\mathcal{B}}) \cap C \neq \emptyset\}, \\ D_\sigma &:= \{x \in \mathcal{X} : (x + \sigma(x)\overline{\mathcal{B}}) \cap D \neq \emptyset\}, \end{aligned}$$

$$\mathcal{H}_\sigma := \begin{cases} \dot{x} \in F_\sigma(x) & \text{for} \quad x \in C_\sigma, \\ x^+ \in G_\sigma(x) & \text{for} \quad x \in D_\sigma. \end{cases}$$

We denote by $\mathcal{S}_\sigma(\cdot)$ the set of maximal solutions to $\mathcal{H}_\sigma$.

***Remark** 4:* The perturbation of F agrees with the perturbation form of differential inclusions used in [4] and [27]. The perturbation of G agrees with the one in [10] for difference inclusions. The perturbations of C and D have the form of set perturbations; see also [8, Section V].

***Definition** 4:* Let $\omega : \mathcal{X} \to \mathbb{R}_{\geq 0}$ be continuous. $\mathcal{H}$ is **robustly $\mathcal{KLL}$-stable w.r.t.** ω **on** $\mathcal{X}$ if there exists a continuous function $\sigma : \mathcal{X} \to \mathbb{R}_{\geq 0}$ such that

- $\{x\} + \sigma(x)\overline{\mathcal{B}} \subset \mathcal{X}$ for all $x \in \mathcal{X}$,
- $\mathcal{H}_\sigma$ is $\mathcal{KLL}$-stable w.r.t. ω on $\mathcal{X}$,
- $\sigma(x) > 0$ for all $x \in \mathcal{X} \setminus \mathcal{A}_\omega$, where $\mathcal{A}_\omega := \{\xi \in \mathcal{X} : \omega(\xi) = 0\}$.

***Remark** 5:* We point out robust $\mathcal{KLL}$-stability in Definition 4 automatically implies that $\mathcal{A}_\omega = \mathbf{A}(\mathcal{S}, \omega) = \mathbf{A}(\mathcal{S}_\sigma, \omega)$. This fact will be generalized in the definition of robust $\mathcal{KLL}$-stability w.r.t. two measures below.

***Theorem** 4:* Let $\omega : \mathcal{X} \to \mathbb{R}_{\geq 0}$ be continuous. If $\mathcal{H}$ is forward complete on $\mathcal{X}$ and there exists a smooth Lyapunov function for $\mathcal{KLL}$-stability w.r.t. ω on $\mathcal{X}$ for $\mathcal{H}$, then $\mathcal{H}$ is robustly $\mathcal{KLL}$-stable w.r.t. ω on $\mathcal{X}$.

***Theorem** 5:* Let $\omega : \mathcal{X} \to \mathbb{R}_{\geq 0}$ be continuous. Under Assumption 3 (and thus Assumption 2), if $\mathcal{H}$ is $\mathcal{KLL}$-stable w.r.t. ω on $\mathcal{X}$, then it is robustly $\mathcal{KLL}$-stable w.r.t. ω on $\mathcal{X}$.

***Corollary** 2:* Suppose, for $\mathcal{H}$, the compact set $\mathcal{A}$ is locally asymptotically stable with the open basin of attraction $\mathcal{X}_\mathcal{A}$. Then, for each function ω that is a proper indicator for $\mathcal{A}$ on $\mathcal{X}_\mathcal{A}$, $\mathcal{H}$ is robustly $\mathcal{KLL}$-stable w.r.t. ω on $\mathcal{X}_\mathcal{A}$.

B. Robust $\mathcal{KLL}$-stability w.r.t. two measures

***Definition** 5:* Let $\omega_i : \mathcal{X} \to \mathbb{R}_{\geq 0}, i = 1, 2$, be continuous. $\mathcal{H}$ is **robustly $\mathcal{KLL}$-stable w.r.t.** (ω_1, ω_2) **on** $\mathcal{X}$ if there exists a continuous function $\sigma : \mathcal{X} \to \mathbb{R}_{\geq 0}$ such that

- $\{x\} + \sigma(x)\overline{\mathcal{B}} \subset \mathcal{X}$ for all $x \in \mathcal{X}$;
- $\mathcal{H}_\sigma$ is $\mathcal{KLL}$-stable w.r.t. (ω_1, ω_2) on $\mathcal{X}$;
- $\sigma(x) > 0$ for all $x \in \mathcal{X} \setminus \mathbf{A}(\mathcal{S}_\sigma, \omega_1)$;
- $\mathbf{A}(\mathcal{S}_\sigma, \omega_1) = \mathbf{A}(\mathcal{S}, \omega_1)$.

Robust $\mathcal{KLL}$-stability w.r.t. (ω_1, ω_2) is the key property for the existence of a smooth Lyapunov function w.r.t. (ω_1, ω_2). Unfortunately, in contrast to the case for continuous-time and discrete-time systems, we cannot provide any sufficient conditions in terms of the system data that guarantee robust $\mathcal{KLL}$-stability. In continuous time a sufficient condition is that the set-valued map F is locally Lipschitz [27, Theorem 2]. In discrete time a sufficient condition is that the set-valued map G is continuous [10, Theorem 2]. In each case, the assumption guarantees that the set of solutions depends on initial conditions and perturbations in a Lipschitz continuous, or continuous (not just upper semicontinuous) way. For hybrid systems, assumptions on F and G are not enough to guarantee this continuous dependence. The flow set C and the jump set D can play a prominent role in determining continuity.

C. Converse Lyapunov theorems for robust $\mathcal{KLL}$-stability w.r.t. two measures

We now come to the main result of the paper. It is the core and generalization of all other converse Lyapunov theorems in this paper (cf. [27, Theorem 1] and [10, Theorem 1]).

***Theorem** 6:* Let $\omega_i : \mathcal{X} \to \mathbb{R}_{\geq 0}, i = 1, 2$, be continuous. The following statements are equivalent:

- $\mathcal{H}$ is forward complete on $\mathcal{X}$ and $\exists$ a smooth Lyapunov function for $\mathcal{KLL}$-stability w.r.t. (ω_1, ω_2) on $\mathcal{X}$ for $\mathcal{H}$.
- $\mathcal{H}$ is robustly $\mathcal{KLL}$-stable w.r.t. (ω_1, ω_2) on $\mathcal{X}$.

D. On the hybrid basic conditions and converse theorems

If any one of the hybrid basic conditions is removed, the conclusions of our converse theorems do not hold in general. We illustrate this now with examples.

Example 1 (Flow set not closed): Define $D_1 := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ and the jump set

$$D := D_1 \cup \{x \in \mathbb{R}^2 : x_1 \geq 0 \ , \ |x_1 - 1| \geq |x_2|\} \\ \cup \{x \in \mathbb{R}^2 : x_1 \leq 0 \ , |x_1 + 1| \geq |x_2|\} \ .$$

Consider the system

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1 \end{array}\right\} \quad x \in \left(\overline{\mathbb{R}^2 \backslash D}\right) \backslash D_1 =: C \ , \\ x^+ = 0 \qquad\qquad x \in D \ .$$

We can verify that the origin is globally exponentially stable, but the stability is not robust. Indeed, for each continuous, positive definite function σ, we have $D_1 \subset C_\sigma$. Thus, since D_1 is forward invariant under the nominal flow, for any σ-perturbation, a solution starting in D_1 can have domain $[0, \infty) \times \{0\}$ and remain in D_1 on its domain. ■

Example 2 (Jump set not closed): Let $C := \{x \in \mathbb{R} : x \leq 1\}$ and consider the system

$$\begin{array}{rcll} \dot{x} & = & -x & x \in C \ , \\ x^+ & = & \min\{|x|, 1\} & x \in \mathbb{R} \backslash C =: D \ . \end{array}$$

We can verify that the origin is globally exponentially stable, but the stability is not robust. Indeed, for each continuous, positive definite function σ, we have $1 \in D_\sigma$. Thus, since $x = 1$ is an equilibrium point for the nominal jump equation, a solution starting at $x = 1$ can have domain $\{0\} \times \mathbb{N}_{\geq 0}$ and $x(0, j) = 1$ for each $j \in \mathbb{N}_{\geq 0}$. ■

Examples illustrating the necessity of G being OSC have been given in [9] for discrete-time systems, *i.e.* $C = \emptyset$ and $D = \mathcal{O}$. Similar examples are well-known for continuous-time systems, *i.e.* $D = \emptyset$ and $C = \mathcal{O}$; see [14].

For continuous time systems, when F is a locally Lipschitz set-valued map, it is not important for its values to be convex in order to have robustness and the existence of smooth Lyapunov functions. This is due to classical relaxation theorems. However, for hybrid systems this phenomenon does not hold, as illustrated by the next example.

Example 3 (Nonconvex differential inclusion): Define the set $C := \{x \in \mathbb{R}^2 : x_1 = x_2\}$ and consider the system

$$\begin{array}{rcll} \dot{x} & \in & \{[1 \ \ 0]', \ [0 \ \ 1]'\} =: F(x) & x \in C \ , \\ x^+ & = & 0 & x \in \mathbb{R}^2 \ . \end{array}$$

Continuous flow is not possible. Thus, only jumping is possible and the origin is globally exponentially stable. On the other hand, the asymptotic stability is not robust. Indeed, for each continuous, positive definite function σ, we have

$$[0.5 \ \ 0.5]' \in F_\sigma(x) \qquad \forall x \in C \ .$$

Thus, it is possible to flow in C for all $t \in [0, \infty)$ and have the trajectory diverge (to ∞). ■

ACKNOWLEDGMENT

The first author is grateful to J.P. Hespanha, S.E. Tuna, and Y. Xu for helpful discussions.

REFERENCES

[1] J.-P. Aubin, J. Lygeros, M. Quincampoix, S. Sastry S, and N. Seube. Impulse differential inclusions: a viability approach to hybrid systems. *IEEE Trans. Auto. Cont.*, 47:2–20, 2002.

[2] M.S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. Auto. Cont.*, 43(4):475–482, 1998.

[3] M.K. Camlibel, W.P.M.H. Heemels, and J.M. Schumacher. Solution concepts for hybrid dynamical systems. In *15th IFAC World Congress on Automatic Control*, pages 1217–1222, 2002.

[4] F. H. Clarke, Y. S. Ledyaev, and R. J. Stern. Asymptotic stability and smooth Lyapunov functions. *J. Diff. Equations*, 149:69–114, 1998.

[5] P. Collins. A trajectory-space approach to hybrid systems. In *MTNS*. Springer, Berlin, 2004.

[6] R. DeCarlo, M. Branicky, S. Pettersson, and B. Lennartson. Perspectives and results on the stability and stabilizability of hybrid systems. In *Proc. IEEE*, volume 88, pages 1069–1082, 2000.

[7] R. Goebel, J. Hespanha, A.R. Teel, C. Cai, and R. Sanfelice. Hybrid systems: generalized solutions and robust stability. In *IFAC Symp. Nonlinear Control Systems*, Stuttgart, Germany, September 2004.

[8] R. Goebel and A. R. Teel. Results on solution sets to hybrid systems with applicatino to stability theory. In *Proc. 24th ACC*, 2005.

[9] C. M. Kellett and A.R. Teel. Smooth Lyapunov functions and robustness of stability for difference inclusions. *Sys. & Cont. Lett.*, 52:395–405, 2004.

[10] C.M. Kellett and A.R. Teel. Results on converse Lyapunov theorems for difference inclusions. In *Proc. 42nd IEEE CDC*, pages 3627–3632, 2003.

[11] J. Kurzweil. On the inversion of Ljapunov's second theorem on stability of motion. *Amer. Math. Soc. Trans., Ser. 2*, 24:19–77, 1956.

[12] V. Lakshmikantham, S. Leela, and A.A. Martynyuk. *Stability Analysis of Nonlinear Systems*. Marcel Dekker, Inc., 1989.

[13] V. Lakshmikantham, S. Leela, and A. S. Vatsala. Setvalued hybrid differential equations and stability in terms of two measures. *Inter. J. Hybrid Systems*, 2:169–188, 2002.

[14] Y.S. Ledyaev and E.D. Sontag. A Lyapunov characterization of robust stabilization. *Nonlinear Analysis*, 37:813–840, 1999.

[15] D. Liberzon. *Switching in Systems and Control*. Birkhauser, 2003.

[16] Y. Lin, E.D. Sontag, and Y. Wang. A smooth converse Lyapunov theorem for robust stability. *SIAM J. Cont. & Opt.*, 34:124–160, 1996.

[17] J. Lygeros, K.H. Johansson, S. Sastry, and M. Egerstedt. On the existence of executions of hybrid automata. In *Proc. 38th IEEE CDC*, pages 2249–2254, 1999.

[18] J. Lygeros, K.H. Johansson, S.N. Simić, J. Zhang, and S.S. Sastry. Dynamical properties of hybrid automata. *IEEE Trans. Auto. Cont.*, 48(1):2–17, 2003.

[19] J. L. Mancilla-Aguilar and R. A. Garcia. A converse Lyapunov theorem for nonlinear switched systems. *Sys. & Cont. Lett.*, 41(1):67–71, 2002.

[20] A.N. Michel and B. Hu. Towards a stability theory of general hybrid dynamical systems. *Automatica*, 35:371–384, 1999.

[21] A.A. Movchan. Stability of processes with respect to two measures. *Prikl. Mat. Mekh.*, pages 988–1001, 1960.

[22] C. Prieur. Perturbed hybrid systems, applications to control theory. In *Nonlinear & Adaptive Cont.*, pages 285–294. Springer, Berlin, 2003.

[23] C. Prieur and A. Astolfi. Robust stabilization of chained systems via hybrid control. *IEEE Trans. Auto. Cont.*, 48(10):1768–1772, 2003.

[24] R. G. Sanfelice, R. Goebel, and A. R. Teel. Results on convergence in hybrid systems via detectability and an invariance principle. In *Proc. 24th ACC*, 2005.

[25] E.D. Sontag and Y. Wang. Notions of input to output stability. *Sys. & Cont. Lett.*, 38:235–248, 1999.

[26] L. Tavernini. Differential automata and their simulators. *Nonlinear Analysis, Theory, Methods and Applications*, 11(6):665–683, 1987.

[27] A. R. Teel and L. Praly. A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive simidefinite functions. *ESAIM: Control, Optimisation and Calculus of Variations*, 5:313–367, 2000.

[28] A. Van der Schaft and H. Schumacher. *An Introduction to Hybrid Dynamical Systems*. Springer, 2000.

[29] H. Ye, A.N. Michel, and L. Hou. Stability theory for hybrid dynamical systems. *IEEE Trans. Auto. Cont.*, 43(4):461–474, 1998.