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Asymptotic Feedback Controllability of Switched Control Systems to the Origin

P. C. Perera,
Department of Mathematics,
University of Texas Pan-American,
Edinburg, TX 78539
pperera@math.panam.edu

W. P. Dayawansa
Department of Mathematics and Statistics,
Texas Tech University
Lubbock, TX 79409
daya@math.ttu.edu

Abstract— **Trajectories of controllable switched systems consisting of linear continuous-time time-invariant subsystems are arbitrarily closely approximated by those of a controllable time-invariant non-switched polynomial systems. Examples are obtained to show that the aforementioned switched control systems are not locally asymptotically stabilizable via continuous switching strategies. Finally, asymptotic feedback controllability of such switched control systems is established.**

I. Introduction

A fundamental requirement for the design of feedback control systems is the knowledge of the structural properties of the switched control system under consideration. These properties are closely related to the concepts of controllability, observability, stability and stabilizability. There have been many studies for switched systems primarily on stability analysis and design in [2], [5], [9]. In the case of controllability, studies for low-order switched control systems consisting of linear subsystems have been presented in [10]. Moreover, some necessary and sufficient conditions for controllability of switched control systems are presented in [6] and [15] under the assumption that the switching strategy is fixed a priori. In [16], necessary and sufficient condition for the controllability and reachability of switched control systems consisting of linear continuous-time time-invariant subsystems is presented.

A. The General Form of a Switched Control System

Mathematically, a switched control system can be described by a differential equation of the form

$$\dot{x}(t) = f_{\sigma(t)}(x(t))$$

where $\{f_p : p \in \mathcal{I}\}$ is a family of sufficiently regular vector fields from $\mathbb{R}^n$ to $\mathbb{R}^n$ that is parameterized by some index set $\mathcal{I}$, and $\sigma : [0, \infty) \to \mathcal{I}$ is a piecewise constant switching signal.

The linear continuous-time version has the form

$$\dot{x}(t) = A_{\sigma(t)}x(t) \tag{1}$$

where $\{A_p : p \in \mathcal{I}\}$ is a family of $n \times n$ matrices with real entries that is parameterized by some index set $\mathcal{I}$, and σ is as above. The discrete-time counterpart of (1) takes the form

$$x(k + 1) = A_{\sigma(k)}x(k)$$

where σ is a function from nonnegative integers to a finite index set $\mathcal{I}$.

In this context, our focus is on the class of switched control systems consisting of linear continuous-time time-invariant subsystems, which in addition admits a certain algebraic condition corresponding to controllability. This subclass is denoted by $\mathcal{W}$ and is explicitly described in section 1.2.

B. The Class $\mathcal{W}$ of Switched Control Systems

Consider a switched control system consisting of linear continuous-time time-invariant subsystems of the form

$$\dot{x} = A_i x + Bu \tag{2}$$

where for each $(i \in \underline{k})$, A_i is an $n \times n$ matrices with real entries and B is an $n \times m$ matrix. To avoid trivialities, it is assumed that $B = (e_1|...|e_m)$ where e_l $(l \in \underline{m})$ denotes the l^{th} element of the standard basis for $\mathbb{R}^n$. Moreover, the space $\mathcal{U}$ of admissible inputs of the switched control system with subsystems of the form given in (2) is assumed to be $\mathbb{R}^m$.

The reachability subspace $< A_i|\mathcal{B} >$ of $\dot{x} = A_i x + Bu$ is given by

$$< A_i|\mathcal{B} >= \mathcal{B} + A_i\mathcal{B} + A_i^2\mathcal{B} + \cdots + A_i^{n-1}\mathcal{B}$$

where $\mathcal{B}$ is the column space of B. Define the finite sequence of subspaces $\{\mathcal{D}_l\}_{l=0}^n$ recursively as

$$\begin{aligned} \mathcal{D}_0 &= < A_1|\mathcal{B} > + \cdots + < A_k|\mathcal{B} > \\ \mathcal{D}_l &= < A_1|\mathcal{D}_{l-1} > + \cdots + < A_k|\mathcal{D}_{l-1} > \text{ for } l \in \underline{n}. \end{aligned}$$

The necessary and sufficient condition for the controllability of a switched control system consisting of subsystems of the form given in (2) is $\mathcal{D}_n = \mathbb{R}^n$ [16]. To avoid trivial cases, it is assumed that $\mathcal{D}_0 \neq \mathbb{R}^n$.

Remark 1.1: If $\mathcal{D}_0 \neq \mathbb{R}^n$ and $\mathcal{D}_n = \mathbb{R}^n$, then $1 \leq m \leq n - 2$ and $n \geq 3$ where $m = \dim\mathcal{B}$.

Definition 1.1: The class $\mathcal{W}$ is defined as the set of switched control systems consisting of subsystems of the form given in (2) satisfying

$$\mathcal{D}_0 \neq \mathbb{R}^n \text{ and } \mathcal{D}_n = \mathbb{R}^n. \tag{3}$$

Generating controls and stabilizing controllers for linear switched systems has been shown to be a nontrivial problem [1]. This is an attempt to show that it is, in fact, possible to relate with a given controllable switched system, a controllable non-switched time-invariant polynomial system with the property that all trajectories of the latter can be approximated arbitrarily closely by trajectories of the given switched system.

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Denote the set of polynomial control systems by $\mathcal{P}$. For a given $w \in \mathcal{W}$, a related non-switched time-invariant controllable polynomial system $\phi \in \mathcal{P}$ of which trajectories can be arbitrarily closely approximated by those of w, is constructed. That is, for a given $w \in \mathcal{W}$, we aim at defining a relation $S(S \subset \mathcal{W} \times \mathcal{P})$.

Examples are constructed to demonstrate the fact that, in general, for $w \in \mathcal{W}$, controllability does not imply local stabilizability via a continuous switching strategy. Since the trajectories of ϕ can be approximated arbitrarily closely by those of w, the asymptotic feedback controllability of w to the origin is established via the related non-switched polynomial system ϕ.

II. The General Form and the Controllability of $\phi \in \mathcal{P}$

A. The General Form of $\phi \in \mathcal{P}$

In this section, we investigate the general form of $\phi \in \mathcal{P}$ for a given $w \in \mathcal{W}$. For a given $w \in \mathcal{W}$, consider the non-switched time-invariant polynomial system ϕ given by

$$\dot{x} = \left(\sum_{i=1}^{k} \alpha_i(x) A_i \right) x + Bu \tag{4}$$

where A_i $(i \in \underline{k})$ are $n \times n$ matrices, $B = (e_1|...|e_m)$ and $\alpha_i(x) : \mathbb{R}^n \to \mathbb{R}$ $(i \in \underline{k})$ are nonnegative polynomial functions satisfying $\sum_{i=1}^{k} \alpha_i(x) > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$. The functions $\alpha_i(x)$ $(i \in \underline{k})$ are called feedback functions.

B. Controllability of Related $\phi \in \mathcal{P}$ when $m = n - 2$

Depending on the values of n and m, the related non-switched polynomial system ϕ for a given $w \in \mathcal{W}$ becomes linear or nonlinear. For a given $w \in \mathcal{W}$, a sufficient condition for ϕ to be linear is established in lemma 2.1.

Lemma 2.1: Suppose $w \in \mathcal{W}$ with $m = n - 2$. Then, there exist nonnegative constants α_i $(i \in \underline{k})$ such that the related non-switched system ϕ with $\alpha_i(x) = \alpha_i$ $(i \in \underline{k})$ is controllable.

Proof: See [12]. ■

Example 2.1: Consider the switched control system with $n > 3$ and $m < n - 2$ consisting of 2 subsystems $\dot{x} = A_i x + Bu$ for $i = 1, 2$ where $B = e_1$ and $A_1 = \left[a_{ij}^{(1)}\right]_{n \times n}$ and $A_2 = \left[a_{ij}^{(2)}\right]_{n \times n}$ are given by

$$a_{ij}^{(1)} = \begin{cases} 1 & i - j = 2 \\ 0 & \text{elsewhere} \end{cases} \qquad a_{ij}^{(2)} = \begin{cases} 1 & (i,j) = (2,1) \\ 0 & \text{elsewhere} \end{cases} \tag{5}$$

It is left to the reader to verify that $\mathcal{D}_0 \neq \mathbb{R}^n$ and $\mathcal{D}_n = \mathbb{R}^n$. Thus, $w \in \mathcal{W}$.

Letting $\alpha_i(x) = \alpha_i$ for $i = 1, 2$, in system ϕ given in (4), we get

$$\dot{x} = (\alpha_1 A_1 + \alpha_2 A_2) + Bu. \tag{6}$$

Let the controllability matrix of ϕ given in (6) be C. Straightforward calculations yield that for $n > 3$, $\max_{(\alpha_1, \alpha_2) \in \mathbb{R}^2} (\text{rank}(C)) = \left[\frac{n}{2}\right] + 1 < n$. Thus, for $n > 3$, there exist $w \in \mathcal{W}$ such that there are no constants α_i $(i \in \underline{k})$ for which ϕ with $\alpha_i(x) = \alpha_i$ $(i \in \underline{k})$ is controllable.

The above example indicates the fact that for a given $w \in \mathcal{W}$ with $m < n-2$, the related ϕ is not controllable with constant feedback functions $\alpha_i(x) = \alpha_i$ $(i \in \underline{k})$ in general. This motivates us to seek some nonnegative nonconstant functions for $\alpha_i(x)$ $(i \in \underline{k})$ which make ϕ nonlinear.

C. Suitable Choices for $\alpha_i(x)$ When $m < n - 2$

It is required to choose smooth functions for $\alpha_i(x)$ $(i \in \underline{k})$ such that

(a) ϕ is globally controllable for any $w \in \mathcal{W}$ and

(b) $\sum_{i=1}^{k} \alpha_i(x) > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$.

Let $r = \begin{cases} m & \text{if } m \geq 2 \\ 2 & \text{if } m = 1 \end{cases}$. Also, let $p_l^{(i)} = 2\mu_l^{(i)}$ where $\mu_l^{(i)} \in \mathbb{N}$ for all $i \in \underline{k}$, $l \in \underline{r}$. Then, in multi-index notation, $u^{p^{(i)}} = (u_1^{p_1^{(i)}}, ..., u_r^{p_r^{(i)}})$ and $|p^{(i)}| = p_1^{(i)} + \cdots + p_1^{(i)}$ for all $1 \leq i \leq k$. Then, the above requirements can be met by letting $\alpha_i(x) = c_i + u^{p^{(i)}}$ for all $(i \in \underline{k})$ with $u = (x_1, ..., x_r)$ satisfy

$$\begin{array}{rl} \text{(i)} & ||p^{(i)} - p^{(j)}||_t \geq 4 \text{ for all } i \neq j (i, j \in \underline{k}), \\ \text{(ii)} & p_r^{(i)} \neq 0 \text{ for all } i \in \underline{k}, \\ \text{(iii)} & c_1 > 0 \text{ and } c_i = 0 \text{ for all } 1 < i \leq k. \end{array} \tag{7}$$

where $||.||_t$ is the taxi-cab metric in $\mathbb{Z}^r$.

D. Controllability of Related $\phi \in \mathcal{P}$ when $m < n - 2$

Theorem 2.1: If $w \in \mathcal{W}$ is a multi-input switched control system $(m \geq 2)$, then there exist distinct positive semi-definite polynomials $\alpha_i(x)$ $(i \in \underline{k})$ satisfying (7) such that the related non-switched polynomial system ϕ is globally controllable.

Proof: Since $w \in \mathcal{W}$ is a multi-input switched control system, it consists subsystems of the form given in (2) which satisfy the condition given in (3) with $r \geq 2$. Recall that the related non-switched system ϕ given in (4) has the form

$$\dot{x} = f(x) + \sum_{l=1}^{m} g_l u_l = \left(\sum_{i=1}^{k} \alpha_i(x) A_i \right) x + Bu.$$

By choosing $\alpha_i(x)$ $(i \in \underline{k})$ as in (7), straightforward calculations yield that

$$\text{ad}_{g_r}^{p_r^{(i)}} \text{ad}_{g_{r-1}}^{p_{r-1}^{(i)}} ... \text{ad}_{g_2}^{p_2^{(i)}} \text{ad}_{g_1}^{p_1^{(i)}} f = A_i x + \sum_{l=1}^{r} k_l x_l A_i b_l \text{ for all } i \in \underline{k}. \tag{8}$$

Also note that $k_l \in \mathbb{N}$ for all $l \in \underline{r}$. Letting

$$h_i = \text{ad}_{g_r}^{p_r^{(i)}} \text{ad}_{g_{r-1}}^{p_{r-1}^{(i)}} ... \text{ad}_{g_2}^{p_2^{(i)}} \text{ad}_{g_1}^{p_1^{(i)}} f \text{ for } i \in \underline{k},$$

it can be deduced that

$$\text{ad}_{g_l}^{p_l^{(i)}} h_i = p_1^{(i)}! p_2^{(i)}! ... p_{l-1}^{(i)}! (p_l^{(i)} + 1)! p_{l+1}^{(i)}! ... p_r^{(i)}! A_i b_l \tag{9}$$

for all $i \in \underline{k}, l \in \underline{r}$. Letting

$$\overline{h}_{il} = A_i b_l = \frac{\text{ad}_{g_l}^{p_l^{(i)}} h_i}{p_1^{(i)}! p_2^{(i)}! ... p_{l-1}^{(i)}! (p_l^{(i)} + 1)! p_{l+1}^{(i)}! ... p_r^{(i)}!} \tag{10}$$

for all $i \in \underline{k}, l \in \underline{r}$, from (8), it yields that

$$A_i x = h_i - \sum_{l=1}^{r} k_l x_l \overline{h}_{il} \text{ for all } i \in \underline{k}. \quad (11)$$

It is obvious that $\overline{h}_{il}$, $h_j - \sum_{l=1}^{r} k_l x_l \overline{h}_{jl} \in \mathcal{S}$ for all $i, j \in \underline{k}$. The basis vectors of $\mathcal{D}^n$ can hence be obtained as constant vector fields of the strong accessibility Lie algebra $\mathcal{S}$ of ϕ by computing appropriate Lie brackets using (10) and (11). Since $\mathcal{D}_n = \mathbb{R}^n$, the constant vector fields of strong accessibility Lie algebra $\mathcal{S}$ has full rank. Thus, the system ϕ with $\alpha_i(x)$ given in (7) is globally controllable [11]. Also see [4], [7], [8], [12] and [14]. ∎

Hitherto, the controllability of the class $\mathcal{W}$ of switched control systems were considered except when $m = 1$ and $n > 3$. To analyze the controllability properties of such systems, we adhere to a different strategy described as follows.

Since $m = 1$, in this case, $B = e_1$. Without loss of generality, it can be assumed that there exist $i \in \underline{k}$ and $j \in \underline{n}$ such that

$$A_i b_1 = \gamma_1 e_1 + \gamma_j e_j. \quad (12)$$

If the system does not inherit this property, by means of an appropriate coordinate transformation, (12) can be obtained. Moreover, by means of another coordinate transformation, (12) can be obtained as

$$A_1 b_1 = e_2. \quad (13)$$

Theorem 2.2: If $w \in \mathcal{W}$ is a switched control system consisting of single-input linear subsystems which evolve in $\mathbb{R}^n$ $(n > 3)$ satisfying (13), then there exist distinct positive semi-definite polynomials $\alpha_i(x)$ $(i \in \underline{k})$ satisfying (7) such that the related non-switched polynomial system ϕ is globally controllable.

Proof: The lines of this proof are the same as those of theorem 2.1 with the exception that, in this case, $r = 2$. ∎

E. Approximation of Trajectories of $\phi \in \mathcal{P}$ by Those of $w \in \mathcal{W}$

In proposition 2.1, it is established that for a given $w \in \mathcal{W}$, the trajectories of the related ϕ of the form given in (4) can be arbitrarily closely approximated by those of w.

Proposition 2.1: For any $w \in \mathcal{W}$ and $T < \infty$, the trajectories of a related non-switched polynomial system ϕ in the form of (4) can be approximated arbitrarily closely by those of w for all $t \in [0, T]$.

Moreover, the feedback functions $\alpha_i(x)$ $(i \in \underline{k})$ contain information of the switching strategy that should be employed in order to follow the trajectory of ϕ arbitrary closely by the switched control system w of our interest.

Proof: Suppose ϕ is globally controllable for a given $w \in \mathcal{W}$. Thus for given $\hat{x} \in \mathbb{R}^n \backslash \{0\}$ and $\hat{\hat{x}} \in \mathbb{R}^n$, there exist $\hat{u} \in \mathbb{R}^m$ and $T > 0$ such that $x(T, 0, \hat{x}, \hat{u}) = \hat{\hat{x}}$.

Furthermore, $K = \{x(t, 0, \hat{x}, \hat{u}) : t \in [0, T]\}$ is compact. Thus, for given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$K \subset \bigcup_{j=1}^{N} B(x_j, \epsilon) \quad (14)$$

where $\{x_j\}_{j=1}^{N}$ can be chosen as follows.

(a) $x_1 = \hat{x}$, $x_N = \hat{\hat{x}}$ and

$$x_j \in K \backslash \{0\} \text{ for all } j \in \underline{N-1}. \quad (15)$$

(b) If $x_1 = \hat{x}$, then $x_j \in B(x_{j-1}, \epsilon)$ for all $j = 2, ..., N$.

Suppose t_j $(j \in \underline{N})$ are given by $x(t_j, 0, \hat{x}, \hat{u}) = x_j$ for all $j \in \underline{N}$. By β_{ij} we denote $\alpha_i(x_j)$. Let $\gamma_j = \left(\sum_{i=1}^{k} \beta_{ij}\right)^{-1}$. (By definition of $\alpha_i(x)$ $(i \in \underline{k})$, γ_j $(j \in \underline{N-1})$ are well defined since from (15) it follows that $\sum_{i=1}^{k} \beta_{ij} > 0$ for all $j \in \underline{N-1}$.)

For the sake of notational simplicity, by denote ψ_1^t, $\hat{\psi}^t$ and ψ_2^t, we denote

$$\begin{aligned} \psi_1^t &= \phi^t_{\left(\sum_{i=1}^{k} \alpha_i(x) A_i\right) x + B\hat{u}} \\ \hat{\psi}^t &= \phi^t_{\left(\sum_{i=1}^{k} \beta_{ij} A_i x\right) + B\hat{u}} \\ \psi_2^t &= \phi^{\beta_{kj} t}_{A_k x + B\gamma_j \hat{u}} \circ ... \circ \phi^{\beta_{1j} t}_{A_1 x + B\gamma_j \hat{u}} \end{aligned}$$

Let $||.||$ be the Euclidean metric in $\mathbb{R}^n$. For given K, defined as above, there exist a pair $\hat{\epsilon} > 0$ and $\hat{N}(\hat{\epsilon})$ such that for all $\epsilon (0 < \epsilon < \hat{\epsilon})$ and $N(\epsilon) [N(\epsilon) > \hat{N}(\hat{\epsilon})]$ satisfying (14), the following are true.

$$\left|\left|\psi_1^t(x_j) - \hat{\psi}^t(z_j)\right|\right| < \frac{(j-1)\epsilon}{N} + \frac{\epsilon}{2N} \quad (16)$$

and

$$\left|\left|\hat{\psi}^t(z_j) - \psi_2^t(z_j)\right|\right| < \frac{\epsilon}{2N} \quad (17)$$

for $t \in [t_j, t_{j+1}]$ and $x \in B(x_j, \epsilon)$ where z_j $(j \in \underline{N})$ are given as $z_1 = x_1 = \hat{x}$ and

$$z_{j+1} = \psi_2^{(t_{j+1} - t_j)}(z_j) = \phi^{\beta_{kj}(t_{j+1} - t_j)}_{A_k x + B\gamma_j \hat{u}} \circ ... \circ \phi^{\beta_{1j}(t_{j+1} - t_j)}_{A_1 x + B\gamma_j \hat{u}}(z_j)$$

for $j \in \underline{N-1}$. (Note that (17) is a direct consequence of Baker-Campbell-Hausdorff formula.)

From (16) and (17), it follows that

$$\left|\left|\psi_1^t(x_j) - \psi_2^t(z_j)\right|\right| < \frac{j\epsilon}{N} \quad (18)$$

for $t \in [t_j, t_{j+1}]$ and $x \in B(x_j, \epsilon)$.

Letting $t = t_{j+1}$ in (18), it yields that $||x_{j+1} - z_{j+1}|| < \frac{j\epsilon}{N}$ for $j \in \underline{N-1}$. Letting $j = N-1$, the assertion immediately follows since $||z_N - x_N|| < \frac{(N-1)\epsilon}{N} < \epsilon$.

Letting $j = 1$, (18) yields that

$$\left|\left|\psi_1^t(x_1) - \psi_2^t(z_1)\right|\right| < \frac{\epsilon}{N}.$$

for $t \in [t_1, t_2]$ and $x \in B(x_1, \epsilon)$. Note that $t_1 = 0$. Thus, it can be easily understood the fact that steering the state from x_1 for a small time $t \in [0, t_2]$ by means of the system ϕ given in (4), is arbitrarily approximately equivalent to steering the state from x_1 using the subsystems of $w \in \mathcal{W}$ given in (2) sequentially in such a way that the i^{th} subsystem is employed for a duration of $\beta_{i1} t$ with inputs scaled down by a factor γ_1. Since $\beta_{i1} = \alpha_i(x_1)$ for $i \in \underline{k}$, it is obvious that the functions $\alpha_i(x)$ $(i \in \underline{k})$ contain information of switching strategy. ■

Remark 2.1: In proposition 2.1, the functions $\alpha_i(x)$ $(i \in \underline{k})$ could be assumed to be nonnegative of arbitrary but sufficiently smooth functions. Nevertheless, they are assumed to be nonnegative polynomials since this assumption enables us to establish the global controllability of ϕ easily in theorem 2.1.

Remark 2.2: Note that when $t \to \infty$, the arbitrary close approximation of trajectories of a given $w \in \mathcal{W}$ and those of a related $\phi \in \mathcal{P}$ is not guaranteed by proposition 2.1. If ϕ is globally controllable, it will not be an issue since the state can be steered between any two arbitrary points in finite time.

Theorem 2.3: Suppose $w \in \mathcal{W}$.

(a) If $m = n-2$, then there exists a related controllable linear system of which trajectories can be arbitrarily closely approximated by those of w.

(b) If $w \in \mathcal{W}$ with $m < n-2$, then there exists a related globally controllable non-switched polynomial system of which trajectories can be arbitrarily closely approximated by those of w. In particular, the nonlinear system can be chosen to be a homogeneous system.

Proof:

(a) This follows directly from proposition 2.1 and lemma 2.1.

(b) This is immediate from proposition 2.1, theorem 2.1 and 2.2.

■

III. Stabilizability and Asymptotic Feedback Controllability of Linear Switched Control Systems to the Origin

A. Approximation of trajectories of $w \in \mathcal{W}$ by those of stabilizable linear systems when $m = n-2$

In theorem 2.3, it was proved that for a given $w \in \mathcal{W}$ with $m = n-2$, there exists a related linear time-invariant controllable system ϕ given by

$$\dot{x} = \left(\sum_{i=1}^{k} \alpha_i A_i \right) x + Bu \tag{19}$$

of which trajectories can be approximated arbitrarily closely by those of w where α_i $(i \in \underline{k})$ are nonnegative constants.

Theorem 3.1: Let $w \in \mathcal{W}$ with $m = n-2$. Then, there exists a related stabilizable linear time-invariant system of the form given in (19) of which trajectories can be approximated arbitrarily closely by those of w.

Proof: This immediately follows from proposition 2.1 and lemma 2.1. ■

B. Non-Stabilizability of $w \in \mathcal{W}$ via Continuous Switching Strategy

A linear time-invariant controllable system is always stabilizable and the system poles can be placed arbitrarily. On the contrary, in the case of nonlinear systems, controllability does not even imply local stabilizability, in general.

In example 3.1, it is demonstrated that if $m < n-2$, there exists $w \in \mathcal{W}$ for which all related non-switched time-invariant polynomial systems are not stabilizable.

Example 3.1: Consider the switched system w whose subsystems are of the form

$$\begin{aligned} \dot{x} &= A_1 x + Bu \\ \dot{x} &= A_2 x + Bu \end{aligned}$$

where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$B = [1, 0, 0, 0]^T$.

It can be easily verified that the above system satisfies (3). Thus, $w \in \mathcal{W}$. Moreover, there exists a related non-switched globally controllable polynomial system of the form

$$\dot{x} = \alpha_1(x) A_1 x + \alpha_2(x) A_2 x + Bu \tag{20}$$

of which trajectories can arbitrarily closely approximated by those of w where $\alpha_1(x)$ and $\alpha_2(x)$ are positive semi-definite polynomial feedback functions.

The system (20) can explicitly be given as

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= \alpha_2(x) x_1 \\ \dot{x}_3 &= \alpha_1(x) x_1 \\ \dot{x}_4 &= \alpha_1(x) x_2 \end{aligned}$$

If the system has the form $\dot{x} = F(x, u)$, then for all nonnegative polynomials $\alpha_1(x)$ and $\alpha_2(x)$, $F(\mathbb{R}^4 \times \mathbb{R})$ does not contain an open neighborhood of the origin. This implies that there is no pair of positive semi-definite polynomial feedback functions $\alpha_1(x)$ and $\alpha_2(x)$ for which (20) is locally stabilizable at the origin. (See [3].)

For every $n, m \in \mathbb{N}$ with $m < n-2$, similar examples can be constructed. Thus, by virtue of example 3.1, it implies that there exists $w \in \mathcal{W}$ with $m < n-2$ for which there is no related stabilizable non-switched polynomial system of which trajectories can be arbitrarily closely approximated by those of w.

Definition 3.1: A **continuous switching signal** is a switching strategy which is employed in such a way that at the time of switching between two subsystems, the system vector field remains continuous.

Observe that any (possibly nonunique) trajectory of a switched system which is forced with a continuous switching strategy, is always smooth.

Now, it is shown that controllability does not imply existence of a continuous switching strategy for an arbitrary $w \in \mathcal{W}$ with $m < n-2$ via a .

Example 3.2: Consider the switched system w given in example 3.1. Consider the system

$$\dot{x} = \psi_1(t)A_1x + \psi_2(t)A_2x + Bu \tag{21}$$

where for any fixed $t \in [0, \infty)$, $\psi_1(t)\psi_2(t) = 0$ and $\psi_1(t) + \psi_2(t) = 1$. If $\psi_1(t) = 1$ and $\psi_2(t) = 0$, then

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= 0 \\ \dot{x}_3 &= x_1 \\ \dot{x}_4 &= x_2 \end{aligned} \tag{22}$$

By the same argument used in example 3.1, it follows that the system given in (22) is not locally asymptotically stabilizable.

If $\psi_1(t) = 0$ and $\psi_2(t) = 1$, then

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= 0 \\ \dot{x}_4 &= 0 \end{aligned} \tag{23}$$

By the same argument as above, it follows that the system given in (23) is not locally stabilizable. Thus, the system given in (21) is not locally asymptotically stabilizable via a continuous switching strategy.

C. Asymptotic Feedback Controllability of $w \in \mathcal{W}$ when $m < n - 2$

Since the condition given in (3) is not sufficient for the smooth local asymptotic stabilizability of (4) via a continuous switching strategy, one has to switch to asymptotic feedback controllability of such systems to the origin which is described as follows.

A submanifold M of $\mathbb{R}^n$ which contains the origin, is constructed in such a way that, on M, the related non-switched polynomial system ϕ is invariant and is asymptotically stabilizable to the origin. Moreover, on M, the systems ϕ and w are equivalent. Then, the arbitrary close approximation of trajectories of $w \in \mathcal{W}$ and those of ϕ is utilized to drive the state by means of w from an arbitrary $x_0 \in \mathbb{R}^n$ to a point $\overline{y}$ at the vicinity of a point y where $0 \neq y \in M$. If $\overline{y} \in M$, then the system w can be employed to steer the state from $\overline{y}$ to the origin asymptotically along M. This phenomenon is called the asymptotic feedback controllability of switched control systems to the origin.

Let $w \in \mathcal{W}$. Without loss of generality, in addition to the condition given in (3), it can be assumed that A_1 has the form

$$\begin{bmatrix} A_{11}^{(1)} & | & A_{12}^{(1)} \\ ---- & -- & ---- \\ 0 & | & A_{22}^{(1)} \end{bmatrix} \tag{24}$$

where the pair $(A_{11}^{(1)}, B_1)$ is controllable where B_1 is the submatrix of B consisting its first m_1 $(m_1 \geq m)$ rows.

Define the submanifold M of $\mathbb{R}^n$ as

$$M = \{(x_1, ..., x_n) \in \mathbb{R}^n | x_r = x_{m_1+1} = \cdots = x_n = 0\} \tag{25}$$

where $r = \begin{cases} m & \text{if } m \geq 2 \\ 2 & \text{if } m = 1 \end{cases}$

Lemma 3.1: Let $w \in \mathcal{W}$ with $m < n - 2$ satisfies the condition given in (24). Suppose that the feedback functions $\alpha_i(x)$ $(i \in \underline{k})$ of the related non-switched polynomial system ϕ in the form of (4) satisfy the condition given in (7). Also suppose M is as in (25). Then for given $x_0 \in \mathbb{R}^n$ and $R > 0$, using ϕ, the state can be steered from x_0 to $y \in M$ $(||y|| = R)$ with piecewise constant inputs in finite time. M is invariant for ϕ. Moreover, ϕ is asymptotically stabilizable on M.

Proof: See [13]. ■

Theorem 3.2: Suppose the switched control system $w \in \mathcal{W}$. Also suppose that A_1 has the form given in (24). Then, w can be used to steer the state satisfying the following. For a given $x_0 \in \mathbb{R}^n$, $R > 0$ and $\delta > 0$, there exists $T > 0$ such that

$$||x(T) - y|| < \delta$$

where $y \in M$ with $||y|| = R$ and $x(0) = x_0$. If $x(\overline{T}) \in M$, then for every ϵ there exists $\hat{T}$ such that

$$||x(t)|| < \epsilon \text{ whenever } t > \hat{T}$$

where $\hat{T}$ is given by $\hat{T} = \overline{T} + \frac{1}{\lambda}\ln\left(\frac{K}{\epsilon}\right)$. ($\lambda > 0$ and $K > R$)

Proof: Proposition 2.1 implies that for arbitrary finite times, the trajectories of w can be arbitrarily closely approximated by those of a related non-switched globally controllable polynomial system ϕ of the form given in (4). The global controllability of ϕ implies that for given $x_0 \in \mathbb{R}^n$, $R > 0$ and $y \in M$ there exist $T < \infty$ such that $x(t) = y \in M$ if $x(0) = x_0$. Then, the first assertion immediately follows from the proposition 2.1.

For the second assertion, since A_1 has the form given in (24), it can be easily verified that, on M, (4) is a controllable linear system, namely,

$$\dot{x} = c_1 A_{11}^{(1)} x + B_1 u.$$

Moreover M is invariant for this system if

$$u_r = -c_1 \sum_{j=1}^{n} a_{rj}^{(1)} x_j.$$

Since (4) is a controllable linear system on M, its poles can arbitrarily be placed such that

$$\sigma(c_1 A_{11}^{(1)} + B_1 \hat{K}) < -\lambda < 0 \qquad (\lambda > 0)$$

for some matrix $\hat{K}$ of order $m \times m_1$. Then, from linear system theory, it follows that

$$||x(t)|| < \epsilon \text{ for } t > \overline{T} + \frac{1}{\lambda}\ln\left(\frac{K}{\epsilon}\right).$$

■

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