Unknown input observers for switched linear discrete time systems

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***Abstract*— This paper is concerned with unknown input observers in the case of switched linear discrete time systems. Sufficient conditions of global convergence of such kind of observers along with a systematic procedure to design the gains of the observers is proposed. A discussion about the existence of such observers is provided.**

I. INTRODUCTION

In recent years, the study of switched systems has received a growing attention in control theory and practice. By switched systems we mean a class of hybrid dynamical systems consisting of a family of continuous (or discrete) time subsystems and a rule that governs the switching between them [1]. Most of the contributions in this field deal with stability or controllability analysis and some control design problems [2], [3], [4]. On the other hand, Unknown Input Observers (UIO) have been widely studied and commonly used in engineering applications. One of the most well-known practical interest of such kind of observers is the fault detection and isolation problem. Nevertheless, UIO have been largely investigated for linear systems [5][6][7][8] or bilinear systems [9]. To our knowledge, there is no result related to the design of UIO for switched systems. This motivates the present work.

In this paper, an unknown input observer is proposed for state reconstruction of switched linear discrete time systems. Conditions of global convergence of the observer are derived. It is shown that the conditions differ from the linear case in a substantial way and are based upon a LMI approach.

Notation

Throughout the paper, for a symmetric matrix X, $X > 0$ indicates that X is positive definite and the symbol $(\bullet)^T$ denotes each of its symmetric block. $\|.\|$ stands for the Euclidean norm. $X^\dagger$ corresponds to the Moore-Penrose generalized inverse of X given by $X^\dagger = (X^T X)^{-1} X^T$.

II. PROBLEM STATEMENT

We consider switched linear systems with the following dynamics :

$$\left\{ \begin{array}{lcl} x_{k+1} & = & A_\alpha x_k + E_\alpha + B_\alpha u_k \\ y_k & = & C_\alpha x_k \end{array} \right. \quad (1)$$

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where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^r$ is the control input and $y_k \in \mathbb{R}^m$, $m < n$, is the output vector. $\{(A_i, B_i, C_i, E_i) : i \in \mathcal{E}\}$ are a family of matrices of appropriate dimensions parameterized by an index set $\mathcal{E} = \{1, 2, ..., N\}$ and $\alpha : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathcal{E}$ is a switching signal ($i = \alpha(x_k, k)$). The switching sequence may also be generated by any strategy or supervisor. We assume that the switching signal is unknown a priori but real time available.

The problem considered in this paper can be sketched as follows : design an unknown input observer such that the reconstructed state $\hat{x}_k$ asymptotically coincides with x_k given any u_k and any initial state $\hat{x}_0$:

$$\lim_{k\to\infty} \|x_k - \hat{x}_k\| = 0 \quad \forall \hat{x}_0 \quad \text{and} \quad \forall u_k \quad (2)$$

A. The standard linear case

Consider the autonomous discrete-time linear system :

$$\left\{ \begin{array}{lcl} x_{k+1} & = & Ax_k + Bu_k \\ y_k & = & Cx_k \end{array} \right. \quad (3)$$

The observer guaranteeing the global convergence of the state reconstruction error $\epsilon_k \stackrel{\triangle}{=} x_k - \hat{x}_k$, is known to get the form :

$$\left\{ \begin{array}{l} \hat{x}_{k+1} = \hat{z}_{k+1} + Qy_{k+1} \\ \hat{z}_{k+1} = N\hat{z}_k + Ly_k \end{array} \right. \quad (4)$$

with the matrices verifying :

$$\left\{ \begin{array}{l} P = \mathbf{1}_n - QC \\ N = PA - KC \\ L = K + NQ \end{array} \right. \quad (5)$$

It is called an Unknown Input Observer since u_k is not available. Details concerning unknown input observers for linear discrete-time systems are provided for instance in [6].

It can be shown by direct substitutions that the dynamics of the state reconstruction error obeys the following equation :

$$\epsilon_{k+1} = (PA - KC)\epsilon_k + PBu_k \quad (6)$$

where P and K are two matrices which must ensure the convergence of ϵ_k towards zero.
A necessary and sufficient condition for the existence of the observer (4) is given by the following theorem :

Theorem 1. *[6] For the system (3), the observer (4) exists if and only if :*

ia) $rank(CB) = rank(B) = r$
iia) *the pair* (PA, C) *is detectable*

On one hand, the condition *ia)* ensures the existence of a gain Q causing the term PB to vanish, that is a gain Q being solution of :

$$B = QCB \tag{7}$$

The general solution of (7) is of the form :

$$Q = B(CB)^{\dagger} + Y(\mathbf{1}_m - (CB)(CB)^{\dagger}) \tag{8}$$

with Y an arbitrary matrix.
On the other hand, the additional condition *iia)* ensures the existence of a gain K which causes the matrix $PA - KC$ to be Hurwitz and so the error ϵ_k to converge towards zero.

In order to extend such an observer to switched linear systems, it is convenient to rewrite (4) into a single recursion :

$$\hat{x}_{k+1} = (PA - KC)\hat{x}_k + Ky_k + Qy_{k+1} \tag{9}$$

B. The switched linear case

For system (1), a natural extension of the unknown linear input observer structure is proposed in this paper and gets the following dynamics.

$$\hat{x}_{k+1} = (P_\alpha A_\alpha - K_\alpha C_\alpha)\hat{x}_k + K_\alpha y_k + Q_\alpha y_{k+1} + P_\alpha E_\alpha \tag{10}$$

with $P_\alpha = \mathbf{1}_n - Q_\alpha C_\alpha$.
Subtracting (10) from (1) yields :

$$\epsilon_{k+1} = (P_\alpha A_\alpha - K_\alpha C_\alpha)\epsilon_k + P_\alpha B_\alpha u_k \tag{11}$$

with $\epsilon_k = x_k - \hat{x}_k$ the state reconstruction error. In the next section, conditions of global convergence of this error towards the origin are derived. It will be shown that these conditions differ from the linear case in a substantial way.

III. MAIN RESULT

First, in order to achieve an input independence property, the terms $P_i B_i$, $i \in \mathcal{E}$, in (11) have to vanish. Thus, following reasonings of the linear case, it can be stated the following proposition.

Proposition 1. *For (1)-(10), the state reconstruction error equation (11) is input independent whenever* $rank(C_i B_i) = rank(B_i) = r$.

In fact, this proposition ensures that for all $i \in \mathcal{E}$, equations $P_i B_i = 0$ can be solved. Indeed, for $i \in \mathcal{E}$, $P_i B_i = 0$ entails that Q_i is subject to :

$$B_i = Q_i C_i B_i \tag{12}$$

Proposition 1 ensures the existence of the matrices Q_i, solutions of (12). The general expression of Q_i is :

$$Q_i = B_i(C_i B_i)^{\dagger} + Y_i(\mathbf{1}_m - (C_i B_i)(C_i B_i)^{\dagger}) \tag{13}$$

with Y_i an arbitrary matrix, $i \in \mathcal{E}$. A thorough discussion about the choice of Y_i will be subsequently carried out.

When Q_i, $i \in \mathcal{E}$, satisfies (13), (11) turns into :

$$\epsilon_{k+1} = (P_\alpha A_\alpha - K_\alpha C_\alpha)\epsilon_k \tag{14}$$

Compared with the linear case, a major distinction lies in the computation of the matrices P_i and K_i, $i \in \mathcal{E}$, for achieving the global convergence. It is worth emphasizing that ensuring each linear dynamics to be stable does not necessarily guarantee the stability of the switched linear dynamics. As a consequence, the condition *iia)* of Theorem 1 does no longer hold. That's why a new condition has to be derived. Moreover, Y_i involved in P_i plays the role of a parameterization. In some special cases discussed in the next section, an arbitrary choice of Y_i may not be suitable (See [5] in the linear case). To overcome the problem of an arbitrary choice of Y_i, the computation of a suitable Y_i must be included in the design procedure. From this perspective, the dynamical matrix involved in the state reconstruction error (14) is rewritten as follows :

$$\epsilon_{k+1} = (\tilde{A}_\alpha - \tilde{K}_\alpha \tilde{C}_\alpha)\epsilon_k \tag{15}$$

with $\tilde{A}_i = A_i - B_i(C_i B_i)^{\dagger} C_i A_i$, $\tilde{K}_i = [K_i \ Y_i]$ and $\tilde{C}_i = [C_i^T \ (C_i \tilde{A}_i)^T]^T$, $i \in \mathcal{E}$.
Let note that the rank condition of Proposition 1 ensures also $(C_i B_i)^{\dagger}$ to exist. Conditions of global stability of (15) are stated in the following Theorem.

Theorem 2. *The unknown input observer (10) ensuring that the error ϵ converges globally towards the origin exists and can be designed whenever the following conditions are satisfied :*

- *ib)* $rank(C_i B_i) = rank(B_i) = r$
- *iib) there exist matrices G_i, F_i and symmetric matrices S_i such that the LMI's (16) are feasible*

$$\begin{bmatrix} G_i + G_i^T - S_i & (\bullet)^T \\ \tilde{A}_i^T G_i - \tilde{C}_i^T F_i & S_j \end{bmatrix} > 0 \tag{16}$$

$\forall (i,j) \in \mathcal{E} \times \mathcal{E}$.

Moreover, the resulting gains are directly given by $\tilde{K}_i = (G_i^{-1})^T F_i^T$.

Proof 1. *ib) ensures the input independence property according to Proposition 1. The proof of iib) follows similar reasoning as in [10].*

The observer design involves the computation of the matrices K_i and Y_i which can be extracted from the partitioning $\tilde{K}_i = [K_i \ Y_i]$. The matrices Q_i are then computed from (13).

IV. OBSERVABILITY AND DETECTABILITY ISSUES

The following proposition gives a necessary condition of existence of the proposed unknown input observer.

Proposition 2. *A necessary condition for the existence of the proposed unknown input observer is that the pairs* $(\tilde{A}_i, \tilde{C}_i)$ *are detectable.*

The feasibility of (16) includes the necessary condition of Proposition 2. Indeed, according to Theorem 2, satisfying (16) is equivalent to guarantee stability of the error dynamics whatever the switching rule can be. This incudes the case where the switching rule leads to a linear behavior. Hence, $\tilde{A}_i - \tilde{K}_i\tilde{C}_i$ has to be Hurwitz, that is the pairs $(\tilde{A}_i, \tilde{C}_i)$ must be detectable.

The matrix Y_i plays an important role for the necessary conditions. For some special setting of Y_i, the pair (P_iA_i, C_i) might not be detectable, preventing (14) from being stable. Based on the new formulation (15), Y_i belongs to the unknown matrices through $\tilde{K}_i$. And yet, since feasibility of (16) includes the necessary condition of detectability of Proposition 2 as mentioned before, the solution of (16) enforces Y_i to belong to an admissible set of solutions. As a result, the problem of a suitable choice of Y_i does no longer hold.

Moreover, additional remarks can be made according to the respective dimensions of the input and output r and m:

a) $m < r$
In this case, the condition of Proposition 1 which is a necessary condition is not fulfilled. The observer (10) cannot exist.

b) $m = r$
In this case, $(C_iB_i)^\dagger = (C_iB_i)^{-1}$ and so $C_i\tilde{A}_i = C_iA_i - C_iB_i(C_iB_i^{-1})C_iA_i = 0$. The observability matrices Q_i^O reduce to :

$$Q_i^O = \begin{bmatrix} C_i \\ \mathbf{0}_{2m(n-1)\times n} \end{bmatrix}$$

Hence, $rank(Q_i^O) = rank(C_i) = m < n$ which implies that all the pairs $(\tilde{A}_i, \tilde{C}_i)$ are unobservable. This means that there exist $n - m$ eigenvalues which are kept unchanged whatever be $\tilde{K}_i$. On one hand, if at least one of those fixed eigenvalues get a modulus greater than one, the pairs $(\tilde{A}_i, \tilde{C}_i)$ are not detectable and in view of Proposition 2, the observer (10) cannot exist. On the other hand, if all the fixed eigenvalues get a modulus less than one, condition (16) is likely to be feasible.
Furthermore, since $C_i\tilde{A}_i = 0$, one has :

$$\tilde{A}_i - \tilde{K}_i\tilde{C}_i = \tilde{A}_i - K_iC_i - Y_iC_i\tilde{A}_i = \tilde{A}_i - K_iC_i \quad (17)$$

Consequently, the solution of (16) does no longer depend on Y_i which can be set to zero. From a practical point of view, owing to numerical problems, $C_i\tilde{A}_i$ might not strictly be zero, causing (16) to be bad conditioned. Taking into consideration (17), (16) can be equivalently reformulated :

$$\begin{bmatrix} G_i + G_i^T - S_i & (\bullet)^T \\ \tilde{A}_i^T G_i - {C_i}^T F_i & S_j \end{bmatrix} > 0 \quad (18)$$

and the resulting gains are directly given by $K_i = (G_i^{-1})^T F_i^T$, $i \in \mathcal{E}$. The gains Q_i reduce to $Q_i = B_i(C_iB_i)^{-1}$ according to (13).

c) $m > r$
In this case, the observability matrices Q_i^O may be of maximal rank n. Consequently, all the eigenvalues could be arbitrarily fixed. Besides, if the condition *iib)* of Theorem 2 is feasible then the matrices P_i and K_i of the observer (10) are designed from both the solutions of (16) and the solutions of (13).

V. ILLUSTRATIVE EXAMPLE

We want to design an unknown input observer for a system given by (1) with $x_k = [x_k^1 \ x_k^2 \ x_k^3]^T$:

- $A_i = \begin{bmatrix} 0 & 0.89 & -2 \\ h_i & 0.89 & 0 \\ -0.1 & 0 & 0.1 \end{bmatrix}$, $h_1 = -1.28$, $h_2 = 1.95$
- $E_1 = [0 \ 0 \ 0]^T$ and $E_2 = [0 \ -6(a+\lambda) \ 0]^T$
- the input matrix is chosen to be constant such that $B_1 = B_2 = [1 \ -2 \ 1]^T$

The switching rule is defined by:

- $\alpha = 1$ if $x_k^1 < 6$
- $\alpha = 2$ if $x_k^1 \geq 6$

A. First setup

The considered setup is related to the output matrix $C_1 = C_2 = [1 \ 1 \ -1]$ ($m = 1$).

Let us apply Theorem 2. The first condition *ib)* is satisfied since $rank(C_iB_i) = rank(B_i) = 1$, $i = 1, 2$. This setup corresponds to the case $m = r = 1$. Thus, according to *b)* of Section IV, $rank(Q_i^O) = rank(C_i) = 1$ and only one eigenvalue of the state reconstruction error matrix $\tilde{A}_i - \tilde{K}_i\tilde{C}_i$ can be fixed arbitrarily. The two remaining eigenvalues are $\{-3.0427, 0.6127\}$ for $i = 1$ and $\{-0.06837, -0.1313\}$ for $i = 2$. Since one of the eigenvalues has a modulus greater than one (for $i = 1$), the pair $(\tilde{A}_1, \tilde{C}_1)$ is not detectable and according to the necessary condition of Proposition 2, the observer (10) cannot exist.

B. Second setup

The considered setup is related to the output matrix $C_1 = C_2 = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$ ($m = 2$).

Let us apply Theorem 2. The first condition *ib)* is satisfied since $rank(C_iB_i) = rank(B_i) = 1$, $i = 1, 2$. This setup corresponds to the case $m > r$. Thus, according to *c)* of Section IV and since $rank(Q_i^O) = 3$, the pairs $(\tilde{A}_i, \tilde{C}_i)$ are observable and the necessary condition of Proposition 2 is satisfied. By solving (16), it turns out that *iib)* is feasible and yields :

$$K_1 = \begin{bmatrix} 0 & 0 \\ 0.3166 & -0.4893 \\ 0.3166 & -0.4893 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 \\ 0.8295 & 0.0236 \\ 0.8295 & 0.0236 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 0.2008 & 0.6647 \\ -0.5069 & 1.4647 \\ -0.8646 & 1.0257 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.9074 & -1.1019 \\ -1.2495 & 3.3212 \\ -1.2358 & 1.9537 \end{bmatrix}$$

From (13), one obtains :

$$Q_1 = Q_2 = \begin{bmatrix} 0.3333 & 0.3333 \\ 0.2060 & -0.3176 \\ -0.4606 & 0.0157 \end{bmatrix}$$

Simulation results are performed with u_k given in Figure 1. The observer behavior is depicted on Figure 2.

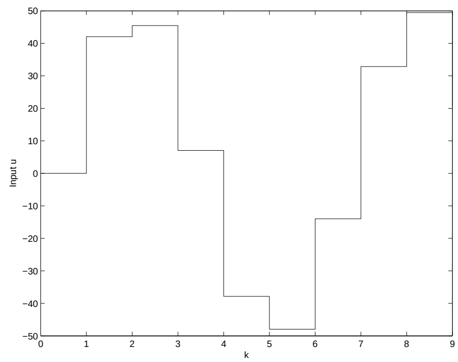

Fig. 1. The input u_k

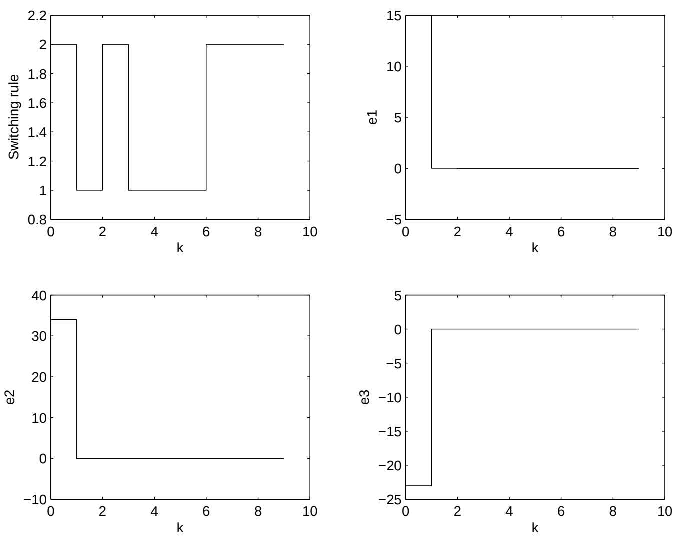

Fig. 2. The switching rule α and the observer error components

VI. CONCLUSION

The work presented in this paper deals with unknown input observers for switched linear discrete time systems. The existence of such observers, the conditions of global convergence along with a systematic procedure to compute the gains have been derived. The computation of the observer gains is performed by solving a tractable set of Linear Matrix Inequalities. Th extension of this work to nonlinear polytopic systems have been performed in [11] where the application to chaos synchronization of discrete-time systems for communications purposes is addressed. The information to be masked is embedded in the chaotic dynamics of the *transmitter* and acts as an external input. It cannot be transmitted to the *receiver* for security preservation purposes. Hence, the *receiver* system must be designed such that the information can be unmasked, given the only available output data consisting of a function of the state vector.

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