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Numerical Computation of Nash Strategy for Large–Scale Systems

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***Abstract*— In this paper, the linear quadratic N–players Nash games for infinite horizon large–scale systems are discussed. Nash strategies are obtained by solving the cross–coupled algebraic Riccati equations (CARE) via the numerical technique. The asymptotic expansions for the CARE are newly established. The main contribution in this paper is that the linear convergence of the proposed algorithm which is based on the fixed point algorithm is proved. In order to demonstrate the efficiency of the algorithm, numerical example is given for the practical power systems.**

I. INTRODUCTION

The linear quadratic Nash games and their applications have been studied widely in many literatures (see e.g., [1]). It is well–known that in order to obtain a Nash equilibrium strategy, the cross–coupled algebraic Riccati equations (CARE) must be solved by means of the numerical algorithm. In [2], the Newton–type algorithm for solving the CARE has been applied. However, this research has concentrated on determining feedback gain matrices for the 2–players Nash games. It should be noted that for the general N–players Nash games, it is hard to solve the N–coupled CARE (see e.g., [3] and reference therein). That is, when the N–players Nash games are considered via the Newton's method, the required workspace is needed to N times of the dimension of the full–systems. Recently, an algorithm which is called the Lyapunov iterations for solving the CARE has been introduced [4]. Although the Lyapunov iterations can be computed in the same subsystem dimension, there are no results for the convergence rate of the Lyapunov iterations. In order to improve the convergence rate of the Lyapunov iterations, the Riccati iterations for solving the CARE have been proposed [5]. However, the proof of the convergence has not been shown.

The control problems of the large–scale systems have been investigated extensively (see e.g., [6], [7]). These large–scale system situations in practice are illustrated by the multiarea power systems [7]. When the N–players Nash games are applied to such systems, the reduction of the algebraic manipulation must be needed because the large–scale systems include numerous subsystems. A popular approach to deal with the large–scale systems is the hierarchical technique (see e.g., [7]). In particular, a near–optimal controller has been proposed [7]. However, when the coupling parameter ε is not small enough, it is known from [8] that the optimality of the cost is not guaranteed. In order to avoid such drawback, the 2–players Nash games for the large–scale systems via the recursive approach have been investigated in [9]. Although the recursive approach has the advantage that the required workspace for computing the solution is the same as the dimension of each subsystem, it is very hard to apply the recursive approach to the generalized N–players Nash games because the solution of the algorithm depends on the other solutions.

This paper studies the linear quadratic N–players Nash games for the infinite horizon large–scale interconnected systems. After establishing the asymptotic structure for the CARE, a new algorithm for solving the CARE is proposed. It should be noted that our algorithm is based on the fixed point algorithm which are quite different from the recursive algorithm. Therefore, the computation of the algorithm is very simple and independent of the other solutions. As another important feature, it is shown that the new algorithm has the linear convergence property. In particular, it is worth pointing out that the convergence rate of the proposed algorithm and its exact proof are first given. Finally, in order to demonstrate the efficiency of the algorithm, numerical example is given for the practical power systems [7].

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\|\cdot\|$ denotes its Euclidean norm for a matrix. $\det M$ denotes the determinant of the matrix M. $\mathrm{vec}M$ denotes the column vector of the matrix M [11]. $\mathrm{Re}\lambda\ M$ denotes the real part of the eigenvalue of the matrix M. $\otimes$ denotes Kronecker product. δ_{ij} denotes the Kroneker delta.

II. PROBLEM FORMULATION

Consider the large–scale linear systems with N–players

$$\begin{aligned}\dot{x}_i(t) &= A_{ii}x_i(t) + B_{ii}u_i(t) + \varepsilon \sum_{j=1,\ j\neq i}^{N} A_{ij}x_j(t) \\ &\quad + \varepsilon \sum_{j=1,\ j\neq i}^{N} B_{ij}u_j(t), \\ x_i(0) &= x_i^0,\ i = 1,\ \ldots,N, \end{aligned} \tag{1}$$

where $x_i \in \mathbf{R}^{n_i},\ i = 1,\ \ldots,N$ represent i–th state vectors. $u_i \in \mathbf{R}^{m_i},\ j = 1,\ \ldots,N$ represent i–th control inputs. ε denotes a small positive weak coupling parameter which connect the other subsystems. Each player is trying to minimize its own cost performance subject to (1) by exploiting the available information to take the correct decision according to the sought strategy. The cost performance for each strategy subset is defined by

$$\begin{aligned} &J_i(u_1,\ \ldots,u_N) \\ &= \int_0^\infty \left[x^T(t)Q_{i\varepsilon}x(t) + u_i^T(t)R_{ii}u_i(t)\right] dt, \end{aligned} \tag{2}$$

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where

$$Q_{i\varepsilon} = \begin{bmatrix} \varepsilon^{1-\delta_{i1}}Q_{i1} & \varepsilon Q_{i12} & \cdots & \varepsilon Q_{i1N} \\ \varepsilon Q_{i12}^T & \varepsilon^{1-\delta_{i2}}Q_{i2} & \cdots & \varepsilon Q_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon Q_{i1N}^T & \varepsilon Q_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}}Q_{iN} \end{bmatrix} \in \mathbf{R}^{\bar{n}\times\bar{n}},$$

$$R_{ii} = R_{ii}^T > 0 \in \mathbf{R}^{m_i\times m_i},\ i = 1,\ \cdots, N,$$

$$x(t)^T := \begin{bmatrix} x_1(t)^T & \cdots & x_N(t)^T \end{bmatrix}^T \in \mathbf{R}^{\bar{n}},\ \bar{n} := \sum_{i=1}^{N} n_i.$$

The Nash equilibrium strategies $(u_1^*, \cdots, u_N^*)$ are defined as satisfying the following conditions

$$\begin{aligned} & J_i(u_1^*, \cdots, u_{i-1}^*, u_i^*, u_{i+1}^*, \cdots, u_N^*) \\ \leq\ & J_i(u_1^*, \cdots, u_{i-1}^*, u_i, u_{i+1}^*, \cdots, u_N^*), \\ i\ =\ & 1, 2, \cdots, N. \end{aligned} \tag{3}$$

It should be noted that the following assumption guarantees the existence of the admissible strategies.

Assumption 1: Each player uses the linear feedback strategies $u_i(t) = K_{i\varepsilon}x(t),\ i = 1, \cdots, N$ such that the closed–loop system is asymptotically stable for sufficiently small ε.

The optimal state strategies of the Nash games are given by

$$u_i^*(t) = -R_{ii}^{-1}B_{i\varepsilon}^T P_{i\varepsilon}x(t),\ i = 1, \cdots, N, \tag{4}$$

where $P_{i\varepsilon}$ are the positive semidefinite solutions of the following N–cross–coupled algebraic Riccati equations (CARE)

$$\begin{aligned} & \mathcal{F}_i(P_{1\varepsilon}, \cdots, P_{N\varepsilon}) \\ :=\ & P_{i\varepsilon}\left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}\right) + \left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}\right)^T P_{i\varepsilon} \\ & + P_{i\varepsilon}S_{i\varepsilon}P_{i\varepsilon} + Q_{i\varepsilon} = 0, \end{aligned} \tag{5}$$

with

$$A_\varepsilon := \begin{bmatrix} A_{11} & \varepsilon A_{12} & \cdots & \varepsilon A_{1N} \\ \varepsilon A_{21} & A_{22} & \cdots & \varepsilon A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon A_{N1} & \varepsilon A_{N2} & \cdots & A_{NN} \end{bmatrix},$$

$$B_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{1i}}B_{1i} \\ \varepsilon^{1-\delta_{2i}}B_{2i} \\ \vdots \\ \varepsilon^{1-\delta_{Ni}}B_{Ni} \end{bmatrix},\ S_{i\varepsilon} := B_{i\varepsilon}R_{ii}^{-1}B_{i\varepsilon}^T.$$

Since A_ε and $S_{i\varepsilon}$ include the term of the small weak coupling parameter ε, the solution $P_{i\varepsilon}$ of the CARE (5), if it exists, must contain terms of order ε. Taking this fact into account, the solution $P_{i\varepsilon}$ of the CARE (5) with the following structure is considered [8].

$$P_{i\varepsilon} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}}P_{i1} & \varepsilon P_{i12} & \cdots & \varepsilon P_{i1N} \\ \varepsilon P_{i12}^T & \varepsilon^{1-\delta_{i2}}P_{i2} & \cdots & \varepsilon P_{i2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{i1N}^T & \varepsilon P_{i2N}^T & \cdots & \varepsilon^{1-\delta_{iN}}P_{iN} \end{bmatrix} \in \mathbf{R}^{\bar{n}\times\bar{n}}.$$

In the following analysis, the basic assumption is needed.

Assumption 2: The triples $(A_{ii},\ B_{ii},\ \sqrt{Q_{ii}}),\ i = 1, \cdots, N$ are stabilizable and detectable.

III. ASYMPTOTIC STRUCTURE OF THE CARE

Firstly, in order to obtain the strategies, the asymptotic structure of the CARE (5) is established. Substituting the matrices A_ε, $S_{i\varepsilon}$, $Q_{i\varepsilon}$ and $P_{i\varepsilon}$ into the CARE (5), setting $\varepsilon = 0$ and partitioning the CARE (5), the following reduced–order AREs are obtained, where $\bar{P}_{ii},\ i = 1, \cdots, N$ be the limiting solutions of the CARE (5) as $\varepsilon \to +0$.

$$\bar{P}_{ii}A_{ii} + A_{ii}^T\bar{P}_{ii} - \bar{P}_{ii}S_{ii}\bar{P}_{ii} + Q_{ii} = 0, \tag{6}$$

where $S_{ii} := B_{ii}R_{ii}^{-1}B_{ii}^T$.

The limiting behavior of $P_{i\varepsilon}$ as the parameter $\varepsilon \to +0$ is described by the following theorem.

Theorem 1: Under Assumption 2, there exists a small σ^* such that for all $\varepsilon \in (0,\ \sigma^*)$ the CARE (5) admits a positive semidefinite solution $P_{i\varepsilon}$ which can be written as

$$\begin{aligned} P_{i\varepsilon} &= \bar{P}_i + O(\varepsilon) \\ &= \mathrm{block-diag}\left(\ \varepsilon^{1-\delta_{i1}}\bar{P}_{i1} \quad \cdots \quad \varepsilon^{1-\delta_{ii}}\bar{P}_{ii} \right. \\ &\quad \left. \cdots \quad \varepsilon^{1-\delta_{iN}}\bar{P}_{iN}\ \right) + O(\varepsilon). \end{aligned} \tag{7}$$

Proof: The proof can be done by using the implicit function theorem [9] to the CARE (5). To do so, it is enough to show that the corresponding Jacobian is nonsingular at $\varepsilon = 0$. The derivative of the function $\mathcal{F}_i(P_{1\varepsilon}, \cdots, P_{N\varepsilon})$ at the matrix $P_{i\varepsilon}$ is given by

$$\begin{aligned} \mathbf{J}_{ii} &:= \frac{\partial}{\partial \mathrm{vec}P_{i\varepsilon}}\mathrm{vec}\mathcal{F}_i(P_{1\varepsilon}, \cdots, P_{N\varepsilon}) \\ &= \left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}\right)^T \otimes I_{\bar{n}} \\ &\quad + I_{\bar{n}} \otimes \left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}\right)^T, \end{aligned} \tag{8a}$$

$$\begin{aligned} \mathbf{J}_{ij} &:= \frac{\partial}{\partial \mathrm{vec}P_{j\varepsilon}}\mathrm{vec}\mathcal{F}_i(P_{1\varepsilon}, \cdots, P_{N\varepsilon}) \\ &= -(S_{j\varepsilon}P_{i\varepsilon})^T \otimes I_{\bar{n}} - I_{\bar{n}} \otimes (S_{j\varepsilon}P_{i\varepsilon})^T. \end{aligned} \tag{8b}$$

Using the fact that

$$\begin{aligned} A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon} &= \mathrm{block-diag}\left(\ A_{11} - S_{11}\bar{P}_{11} \right. \\ &\quad \left. \cdots \quad A_{NN} - S_{NN}\bar{P}_{NN}\ \right) + O(\varepsilon) \end{aligned}$$

and $S_{j\varepsilon}P_{i\varepsilon} = O(\varepsilon)$, it can be shown, after some algebra, that the Jacobian of the CARE (5) in the limit as $\varepsilon \to +0$ is given by

$$\begin{aligned}\mathbf{J} &= \begin{bmatrix} \mathbf{J}_{11}|_{\varepsilon=0} & \cdots & \mathbf{J}_{1N}|_{\varepsilon=0} \\ \vdots & \ddots & \vdots \\ \mathbf{J}_{N1}|_{\varepsilon=0} & \cdots & \mathbf{J}_{NN}|_{\varepsilon=0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{J}_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{J}_0 \end{bmatrix}, \end{aligned} \tag{9}$$

where

$$\begin{aligned}\mathbf{J}_0 &= \text{block}-\text{diag}\left(\begin{array}{ccc} \mathbf{D}_{11} & \cdots & \mathbf{D}_{NN} \end{array} \right), \\ \mathbf{D}_{ii} &:= (A_{ii} - S_{ii}\bar{P}_{ii})^T \otimes I_{n_i} + I_{n_i} \otimes (A_{ii} - S_{ii}\bar{P}_{ii})^T. \end{aligned}$$

Obviously, $D_{ii} := A_{ii} - S_{ii}\bar{P}_{ii}$ is nonsingular because the ARE (6) has the positive semidefinite stabilizing solution under Assumption 2. Thus, $\det\mathbf{J} \neq 0$, i.e., $\mathbf{J}$ is nonsingular at $\varepsilon = 0$. The conclusion of Theorem 1 is obtained directly by using the implicit function theorem. On the other hand, taking into account the fact that $\bar{P}_{ii}$ is the positive semidefinite matrix, for sufficiently small parameter ε, $P_{i\varepsilon}$ is also the positive semidefinite solution. The detailed proof can be done by using the similar technique in [12]. ■

IV. ITERATIVE ALGORITHM FOR SOLVING CARE

In order to obtain the optimal strategies, the following useful algorithm is given.

Theorem 2: Consider the following iterative algorithm

$$\begin{aligned} P_{i\varepsilon}^{(k+1)} &\left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)} \right) \\ &+ \left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)} \right)^T P_{i\varepsilon}^{(k+1)} \\ &+ P_{i\varepsilon}^{(k)} S_{i\varepsilon} P_{i\varepsilon}^{(k)} + Q_{i\varepsilon} = 0, \quad k = 0,\ 1,\ \ldots, \end{aligned} \tag{10a}$$

$$P_{i\varepsilon}^{(k)} := \begin{bmatrix} \varepsilon^{1-\delta_{i1}} P_{i1}^{(k)} & \varepsilon P_{i12}^{(k)} & \cdots & \varepsilon P_{i1N}^{(k)} \\ \varepsilon P_{i12}^{(k)T} & \varepsilon^{1-\delta_{i2}} P_{i2}^{(k)} & \cdots & \varepsilon P_{i2N}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon P_{i1N}^{(k)T} & \varepsilon P_{i2N}^{(k)T} & \cdots & \varepsilon^{1-\delta_{iN}} P_{iN}^{(k)} \end{bmatrix} \tag{10b}$$

with the initial condition

$$\begin{aligned} P_{i\varepsilon}^{(0)} &= \bar{P}_i \\ &= \text{block}-\text{diag}\left(\begin{array}{ccc} \varepsilon^{1-\delta_{i1}}\bar{P}_{i1} & \cdots & \varepsilon^{1-\delta_{ii}}\bar{P}_{ii} \\ \cdots & \varepsilon^{1-\delta_{iN}}\bar{P}_{iN} \end{array} \right). \end{aligned} \tag{11}$$

Under Assumption 2, there exists a small $\bar{\sigma}$ such that for all $\varepsilon \in (0,\ \bar{\sigma})$, $\bar{\sigma} \leq \sigma^*$ the iterative algorithm (10) converges to the exact solution of $P_{i\varepsilon}$ with the rate of the linear convergence, where $P_{i\varepsilon}^{(k)}$ is positive semidefinite and $A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)}$ is stable. That is, the following conditions are satisfied.

$$\|P_{i\varepsilon}^{(k)} - P_{i\varepsilon}\| = O(\varepsilon^{k+1}), \tag{12a}$$

$$\text{Re}\lambda \left[A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(k)} \right] < 0, \quad k = 0,\ 1,\ \ldots \tag{12b}$$

Proof: The proof of this theorem can be done by using mathematical induction. When $k = 0$ for the iterative algorithms (10), taking (7) into account, it is easy to verify that the first order approximations $P_{i\varepsilon}$ corresponding to the small parameter ε are $\bar{P}_i$. Moreover, since

$$\begin{aligned} A_{i\varepsilon} - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(0)} &= \text{block}-\text{diag}\left(\begin{array}{cc} D_{11} & \\ & \cdots \quad D_{NN} \end{array} \right) + O(\varepsilon) \end{aligned}$$

is satisfied, there exists the small perturbation parameter σ_0 such that $A_{i\varepsilon} - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(0)}$ is stable because D_{ii} is stable for sufficiently small ε. When $k = h,\ h \geq 1$, it is assumed that

$$\|P_{i\varepsilon}^{(h)} - P_{i\varepsilon}\| = O(\varepsilon^{h+1}), \tag{13a}$$

$$\text{Re}\lambda \left[A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(h)} \right] < 0. \tag{13b}$$

Subtracting (5) from (10a) and setting $k = h$, the following equations are satisfied.

$$\begin{aligned} &\left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon} \right) \left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(h)} \right) \\ &+ \left(A_\varepsilon - \sum_{j=1}^N S_{j\varepsilon}P_{j\varepsilon}^{(h)} \right)^T \left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon} \right) \\ &+ \sum_{j=1,\ j\neq i}^N P_{i\varepsilon}S_{j\varepsilon} \left(P_{j\varepsilon} - P_{j\varepsilon}^{(h)} \right) \\ &+ \sum_{j=1,\ j\neq i}^N \left(P_{j\varepsilon} - P_{j\varepsilon}^{(h)} \right) S_{j\varepsilon}P_{i\varepsilon} \\ &+ \left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon} \right) S_{i\varepsilon} \left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon} \right) = 0. \end{aligned} \tag{14}$$

Using the fact that the assumption (13a) hold, it is easy to derive that

$$\begin{aligned} &\sum_{j=1,\ j\neq i}^N P_{i\varepsilon}S_{j\varepsilon} \left(P_{j\varepsilon} - P_{j\varepsilon}^{(h)} \right) = O(\varepsilon^{h+2}), \\ &\left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon} \right) S_{i\varepsilon} \left(P_{i\varepsilon}^{(h)} - P_{i\varepsilon} \right) = O(\varepsilon^{2h+2}). \end{aligned}$$

Thus, the following relation is satisfied.

$$\left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}\right)\left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}^{(h)}\right) + \left(A_\varepsilon - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}^{(h)}\right)^T \left(P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}\right) + O(\varepsilon^{h+2}) = 0. \quad (15)$$

Taking into account the fact that the stability assumption (13b) holds and using the standard properties of the algebraic Lyapunov equation (ALE) [10], it is easy to verify that

$$\|P_{i\varepsilon}^{(h+1)} - P_{i\varepsilon}\| = O(\varepsilon^{h+2}). \quad (16)$$

Furthermore, using the relation (16), it is shown that there exists the small perturbation parameter σ_{h+1} such that

$$A_{i\varepsilon} - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon}^{(h+1)} = A_{i\varepsilon} - \sum_{j=1}^{N} S_{j\varepsilon}P_{j\varepsilon} + O(\varepsilon^{h+2})$$
$$= A_{i\varepsilon} - \sum_{j=1}^{N} S_{j\varepsilon}\bar{P}_j + O(\varepsilon)$$

is stable. Consequently, choosing $\bar{\sigma} := \min\{\sigma_0, \ \dots \ , \sigma_{h+1}\}$, the relation (12b) holds for all $k \in \mathbf{N}$. This completes the proof of Theorem 2 concerned with the fixed point algorithm. ■

It should be noted that if the coupling effect between subsystems are strong, the proposed approach may not be applied.

When the algebraic Lyapunov equation (ALE) (10a) is solved, the dimension $\bar{n} := \sum_{i=1}^{N} n_i$ larger than the dimensions $n_i, \ i = 0, \ 1, \dots \ , N$ is needed. Thus, in order to reduce the dimension of the workspace, the new algorithm for solving the ALE (10a) which is based on the fixed point algorithm is established. Let us consider the following ALE (17), in a general form.

$$\mathcal{G}(\varepsilon, \ X_\varepsilon) = X_\varepsilon\Lambda_\varepsilon + \Lambda_\varepsilon^T X_\varepsilon + U_\varepsilon = 0, \quad (17)$$

In particular, the following special matrices X_ε, Λ_ε and U_ε which are related to the ALE (17) are considered because the other case $i = 2, \ \dots \ , N$ can be changed into the similar form by using the similarity transformation $\mathcal{T}_i$,
where

$$\mathcal{T}_i := \begin{bmatrix} 0 & \dots & I_{n_1} \\ \vdots & \text{block} - \text{diag}(1 \ \dots \ 1) & \vdots \\ I_{n_i} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix},$$

$$X_\varepsilon := \begin{bmatrix} X_{11} & \varepsilon X_{12} & \cdots & \varepsilon X_{1N} \\ \varepsilon X_{12}^T & \varepsilon X_{22} & \cdots & \varepsilon X_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon X_{1N}^T & \varepsilon X_{2N}^T & \cdots & \varepsilon X_{NN} \end{bmatrix},$$

$$\Lambda_\varepsilon := \begin{bmatrix} \Lambda_{11} & \varepsilon\Lambda_{12} & \cdots & \varepsilon\Lambda_{1N} \\ \varepsilon\Lambda_{21} & \Lambda_{22} & \cdots & \varepsilon\Lambda_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon\Lambda_{N1} & \varepsilon\Lambda_{N2} & \cdots & \Lambda_{NN} \end{bmatrix},$$

$$U_\varepsilon := \begin{bmatrix} U_{11} & \varepsilon U_{12} & \cdots & \varepsilon U_{1N} \\ \varepsilon U_{12}^T & \varepsilon U_{22} & \cdots & \varepsilon U_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon U_{1N}^T & \varepsilon U_{2N}^T & \cdots & \varepsilon U_{NN} \end{bmatrix}.$$

In order to guarantee the existence of the solution and the convergence of the algorithm, another assumption is needed.

Assumption 3: $\Lambda_{11}, \ \cdots, \ \Lambda_{NN}$ are stable.

Without loss of generality, it should be noted that the above assumption is satisfied automatically under the condition of Theorem 2.

The ALE (17) can be changed as follows by partitioning.

$$X_{11}\Lambda_{11} + \Lambda_{11}^T X_{11} + \varepsilon^2 \sum_{l=2}^{N}(X_{1l}\Lambda_{l1} + \Lambda_{l1}^T X_{l1}) + U_{11} = 0, \quad (18a)$$

$$X_{1j}\Lambda_{jj} + \Lambda_{11}^T X_{1j} + X_{11}\Lambda_{1j} - \varepsilon X_{1j}\Lambda_{jj} + \varepsilon\sum_{l=2}^{N}(X_{1l}\Lambda_{lj} + \Lambda_{l1}^T X_{lj}) + U_{1j} = 0, \quad (18b)$$

$$X_{ii}\Lambda_{ii} + \Lambda_{ii}^T X_{ii} + \varepsilon \sum_{l=1, \ l\neq i}^{N} (X_{il}\Lambda_{li} + \Lambda_{li}^T X_{li}) + U_{ii} = 0, \quad (18c)$$

$$X_{ij}\Lambda_{jj} + \Lambda_{ii}^T X_{ij} + \varepsilon(X_{ii}\Lambda_{ij} - X_{ij}\Lambda_{jj}) + \varepsilon \sum_{l=1, \ l\neq i}^{N} (X_{il}\Lambda_{lj} + \Lambda_{li}^T X_{lj}) + U_{ij} = 0, \quad i, \ j = 2, \ \cdots, N. \quad (18d)$$

Taking the form of (18) into account, the algorithm (19) for solving the ALE (17) is given.

$$X_{11}^{(k+1)}\Lambda_{11} + \Lambda_{11}^T X_{11}^{(k+1)} + \varepsilon^2 \sum_{l=2}^{N}(X_{1l}^{(k)}\Lambda_{l1} + \Lambda_{l1}^T X_{l1}^{(k)}) + U_{11} = 0, \quad (19a)$$

$$X_{1j}^{(k+1)}\Lambda_{jj} + \Lambda_{11}^T X_{1j}^{(k+1)} + X_{11}^{(k+1)}\Lambda_{1j} - \varepsilon X_{1j}^{(k)}\Lambda_{jj}$$

$$+\varepsilon\sum_{l=2}^{N}(X_{1l}^{(k)}\Lambda_{lj}+\Lambda_{l1}^{T}X_{lj}^{(k)})+U_{1i}=0, \tag{19b}$$

$$X_{ii}^{(k+1)}\Lambda_{ii}+\Lambda_{ii}^{T}X_{ii}^{(k+1)}+\varepsilon\sum_{l=1,\ l\neq i}^{N}(X_{il}^{(k)}\Lambda_{li}+\Lambda_{li}^{T}X_{li}^{(k)}) +U_{ii}=0, \tag{19c}$$

$$X_{ij}^{(k+1)}\Lambda_{jj}+\Lambda_{ii}^{T}X_{ij}^{(k+1)}+\varepsilon(X_{ii}^{(k)}\Lambda_{ij}-X_{ij}^{(k)}\Lambda_{jj}) +\varepsilon\sum_{l=1,\ l\neq i}^{N}(X_{il}^{(k)}\Lambda_{lj}+\Lambda_{li}^{T}X_{lj}^{(k)})+U_{ij}=0, \tag{19d}$$

where $k=0,\ 1,\ \cdots,\ X_{ii}^{(0)}=\bar{X}_{ii},\ X_{ij}^{(0)}=\bar{X}_{ij},\ i<j,\ \bar{X}_{ij}=\bar{X}_{ji}^{T}$,

$$\bar{X}_{ii}\Lambda_{ii}+\Lambda_{ii}^{T}\bar{X}_{ii}+U_{ii}=0,\ i=1,\ \dots,N,$$
$$\bar{X}_{1j}\Lambda_{jj}+\Lambda_{11}^{T}\bar{X}_{1j}+\bar{X}_{11}\Lambda_{1j}+U_{1j}=0,\ j=2,\ \dots,N,$$
$$\bar{X}_{ij}\Lambda_{jj}+\Lambda_{ii}^{T}\bar{X}_{ij}+U_{ij}=0,\ i,\ j=2,\ \cdots,N.$$

The following theorem indicates the convergence of the algorithm (19).

Theorem 3: Under Assumption 3, the fixed point algorithm (19) converges to the exact solution X_{ij} with the rate of

$$\|X_{11}^{(k)}-X_{11}\|=O(\varepsilon^{2k+2}), \tag{20a}$$
$$\|X_{ij}^{(k)}-X_{ij}\|=O(\varepsilon^{k+1}), \tag{20b}$$
$$i\leq j,\ ij\neq 11,\ k=1,\ 2,\ \dots$$

Proof: The proof of Theorem 3 can be done by using mathematical induction. When $k=0$ for the algorithms (19), it is easy to verify that the first order approximations X_{ii} and X_{ij} corresponding to the small parameter ε are $\bar{X}_{ii}$ and $\bar{X}_{ij}$, respectively. It follows from these equations that

$$\|X_{11}^{(0)}-X_{11}\|=\|\bar{X}_{11}-X_{11}\|=O(\varepsilon^{2}), \tag{21a}$$
$$\|X_{ij}^{(0)}-X_{ij}\|=\|\bar{X}_{ij}-X_{ij}\|=O(\varepsilon). \tag{21b}$$

When $k=h,\ h\geq 1$, it is assumed that

$$\|X_{11}^{(h)}-X_{11}\|=O(\varepsilon^{2h+2}), \tag{22a}$$
$$\|X_{ij}^{(h)}-X_{ij}\|=O(\varepsilon^{h+1}). \tag{22b}$$

Subtracting (18) from (19) and setting $k=h$, the following equations hold.

$$(X_{11}^{(h+1)}-X_{11})\Lambda_{11}+\Lambda_{11}^{T}(X_{11}^{(h+1)}-X_{11}) +O(\varepsilon^{2h+4})=0, \tag{23a}$$
$$(X_{ij}^{(h+1)}-X_{ij})\Lambda_{jj}+\Lambda_{ii}^{T}(X_{ij}^{(h+1)}-X_{ij}) +O(\varepsilon^{h+2})=0. \tag{23b}$$

After the cancellation takes place, since $\Lambda_{ii},\ i=1,\ 2,\dots,N$ are stable from Assumption 3, the following relations hold

$$\|X_{11}^{(h+1)}-X_{11}\|=O(\varepsilon^{2h+4}), \tag{24a}$$
$$\|X_{ij}^{(h+1)}-X_{ij}\|=O(\varepsilon^{h+2}). \tag{24b}$$

Consequently, the error equations (20) hold for all $k\in\mathbf{N}$. This completes the proof of Theorem 3. ■

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V. NUMERICAL EXAMPLE

In order to demonstrate the efficiency of the proposed algorithm, an illustrative example is given. Consider a practical power systems plant which are known as the large–scale system (1) composed of three four–dimensional subsystems [7]. The system matrices are given as follows.

$$A_{11}=\begin{bmatrix}0&1&-0.266&-0.009\\-2.75&-2.78&-1.36&-0.037\\0&0&0&1\\-4.95&0&-55.5&-0.039\end{bmatrix},$$
$$\varepsilon A_{12}=\begin{bmatrix}0.0024&0&-0.087&0.002\\-0.185&0&1.11&-0.011\\0&0&0&0\\0.222&0&8.17&0.004\end{bmatrix},$$
$$\varepsilon A_{13}=\begin{bmatrix}0.073&0&-0.25&0.003\\-0.46&0&2.8&-0.02\\0&0&0&0\\0.924&0&17.5&0.02\end{bmatrix},$$
$$\varepsilon A_{21}=\begin{bmatrix}0.021&0&0.121&0.003\\-1.1&0&-1.62&-0.015\\0&0&0&0\\-2.43&0&1.37&-0.034\end{bmatrix},$$
$$A_{22}=\begin{bmatrix}-0.21&1&-1.6&-0.005\\-1.9&-1.8&9.3&-0.12\\0&0&0&1\\-3.1&0&-56&0.032\end{bmatrix},$$
$$\varepsilon A_{23}=\begin{bmatrix}0.06&0&0.46&0.002\\-1&0&1.49&-0.04\\0&0&0&0\\0.12&0&29.8&-0.028\end{bmatrix},$$
$$\varepsilon A_{31}=\begin{bmatrix}-0.002&0&0.83&0\\-6.78&0&-10.1&0.09\\0&0&0&0\\-1.24&0&0.498&-0.017\end{bmatrix},$$
$$\varepsilon A_{32}=\begin{bmatrix}0.011&0&0.22&0\\-2.1&0&1.7&-0.123\\0&0&0&0\\-0.07&0&6.38&-0.011\end{bmatrix},$$
$$A_{33}=\begin{bmatrix}-0.197&1&-1.2&-0.003\\-54.5&-20&70.1&-2.37\\0&0&0&1\\-3.4&0&-21.0&-0.017\end{bmatrix},$$
$$B_{11}=\begin{bmatrix}0\\36.1\\0\\0\end{bmatrix},\ B_{22}=\begin{bmatrix}0\\78.9\\0\\0\end{bmatrix},\ B_{33}=\begin{bmatrix}0\\1000\\0\\0\end{bmatrix},$$
$$B_{ij}=0,\ i\neq j.$$

The small parameter is chosen as $\varepsilon=0.5065$. The weighting matrices of the cost performance are given by $R_{11}=$

Table 1.

k	$\|\mathcal{F}^{(k)}(0.5065)\|$	$\|\mathcal{F}^{(k)}(1.0e-01)\|$	$\|\mathcal{F}^{(k)}(1.0e-02)\|$	$\|\mathcal{F}^{(k)}(1.0e-03)\|$	$\|\mathcal{F}^{(k)}(1.0e-04)\|$
0	$2.6662e+01$	5.2643	$5.2643e-01$	$5.2643e-02$	$5.2643e-03$
1	8.3958	$3.4809e-01$	$3.3150e-03$	$3.3014e-05$	$3.3003e-07$
2	1.9676	$1.1748e-02$	$1.0246e-05$	$1.0105e-08$	$3.5639e-11$
3	$7.9849e-02$	$9.3374e-05$	$9.5683e-09$	$2.8657e-11$	
4	$3.6930e-03$	$3.1704e-06$	$5.8256e-11$		
5	$4.2296e-04$	$1.0000e-07$			
6	$8.1772e-05$	$1.0304e-09$			
7	$1.1519e-05$	$6.5768e-11$			
8	$8.9876e-07$				
9	$6.9238e-08$				
10	$1.7170e-08$				
11	$2.2137e-09$				
12	$1.8077e-10$				
13	$3.9460e-11$				

$R_{22} = R_{33} = 1$, $Q_1 = \text{block}-\text{diag}\begin{pmatrix} 0.5I_4 & O_{8\times 8} \end{pmatrix}$, $Q_2 = \text{block}-\text{diag}\begin{pmatrix} O_{4\times 4} & 0.5I_4 & O_{4\times 4} \end{pmatrix}$, $Q_3 = \text{block}-\text{diag}\begin{pmatrix} O_{8\times 8} & 0.5I_4 \end{pmatrix}$. It should be noted that the algorithm (10a) converges to the exact solution with accuracy of $\|\mathcal{F}^{(k)}(\varepsilon)\| < 1.0e-10$ after 13 iterations, where

$$\begin{aligned}\|\mathcal{F}^{(k)}(\varepsilon)\| \quad := \quad & \|\mathcal{F}_1(P_{1\varepsilon}^{(k)},\ P_{2\varepsilon}^{(k)},\ P_{3\varepsilon}^{(k)})\| \\ & +\|\mathcal{F}_2(P_{1\varepsilon}^{(k)},\ P_{2\varepsilon}^{(k)},\ P_{3\varepsilon}^{(k)})\| \\ & +\|\mathcal{F}_3(P_{1\varepsilon}^{(k)},\ P_{2\varepsilon}^{(k)},\ P_{3\varepsilon}^{(k)})\|.\end{aligned}$$

In order to verify the exactitude of the solution, the remainder per iteration by substituting $P_{i\varepsilon}^{(k)}$ into the CARE (5) is computed. In Table 1, the results for the error $\|\mathcal{F}^{(k)}(\varepsilon)\|$ per iterations are given. It can be seen that the algorithm (10a) has the linear convergence. Table 2 shows the result for the effect of the residual error in the second iterations (19) to the convergence of the first iterations.

Table 2.

k	$\|\mathcal{G}(1.0e-02,\ P_{1\varepsilon}^{(1)})\|$
0	$2.2640e-02$
1	$1.4673e-04$
2	$7.4634e-07$
3	$8.0243e-09$
4	$1.0138e-10$
5	$5.0296e-12$

It can be also seen that the algorithm (19) has the linear convergence.

From this example point of view, it is worth pointing out that even if number of the subsystems is more than four but not three, the required workspace for calculating the strategies is the same as the dimension of the subsystems. That is, even if the large–scale system (1) is composed of N four–dimensional subsystems, the required workspace is four.

VI. CONCLUSION

In this paper, the Nash games for the large–scale systems which are connected by the weak small coupling parameter has been studied. The main contribution of this paper is to propose the new algorithm for solving the large–scale CARE. It should be noted that the proposed design method is quite different from the existing method such as the recursive approach [9]. As a result, we have succeeded in improving the convergence rate dramatically because the proposed algorithm has linear convergence.

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