Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Coupled Vibrations of a Varying Length Flexible Cable Transporter System with Arbitrary Axial Velocity

Yuhong Zhang and Sunil Agrawal

***Abstract*— Modeling for coupled vibrations of a varying length flexible cable transporter system with arbitrary axial velocity is presented using the extended Hamilton's principle. Both transverse and longitudinal vibrations are considered simultaneously in the derivation. The inertia and damping of the pulleys and actuators are included in the model. The derived governing equations provide guidelines for future research on cable systems. Numerical solution of the derived nonlinear coupled partial differential equations (PDEs) is obtained using Galerkin's method. Simulation results show that increase in damping of the system can reduce the longitudinal vibration dramatically, but has little effect on the transverse vibration.**

I. INTRODUCTION

Axially moving materials such as magnetic tapes, belts, drive chains, and band saws, tend to vibrate in the presence of external disturbances, degrading their performances. Researchers have studied modeling and vibration control of such axially moving systems in the last decades. These systems have been modeled as traveling tensioned rods or cables. Linear vibrations of axially moving materials have been studied extensively [1]. While the linear theory provides the natural frequencies, mode shapes and critical speeds, it presents several limitations. As the transporter speed increases, the tension variation causes oscillations which adversely affects the response. The linear theory becomes inapplicable near the critical speed [2].

The nonlinear vibrations of axially moving materials have received attention over the years. Kirchhoff was one of the first to derive a nonlinear model for the transverse vibration of a vibrating string [3]. He proposed a passive controller which was proportional to the velocity at the boundary to stabilize the string. Extensions of the model were obtained by other researchers as well ([4], [5], [6]).

Passive control usually does not provide the desired performance. Recently, active boundary control is studied by many researchers. Rahn derived a nonlinear model for a string system, and proposed a nonlinear, model-based asymptotic controller [7]. The proposed controller needed to measure the string's slope and its time derivative, its velocity at the actuated boundary, and the tension in the string. Baicu et al. proposed a three-dimensional model for a string system and presented both passive and active control laws [8]. Hagedorn studied the non-linear free vibration of an elastic cable under the action of gravity [9]. The above studies assume that the string length is a constant during the process of modeling and controller design. The energy and stability of a class of strings with varying lengths was studied in ([10], [11]).

The current research efforts have focused on transverse vibration or longitudinal vibration individually. The study of coupled vibrations of cable systems is a relatively less studied problem in literature. Tabarrok studied coupled vibrations of an axially moving beam, and proposed closed-form and semi-analytic solution for given axial speed [12]. Riedel studied coupled forced response of an axially moving strip, where the axial speed was assumed to be a constant [13]. Most previous studies only modeled the cable itself, while ignoring other discrete components, such as mass and inertias of motors or pulleys in the system.

In this paper, we study coupled vibrations of a cable transporter system with arbitrary varying axial velocity. The transverse and longitudinal vibrations are studied simultaneously. The mass, damping, and inertias of the actuator and pulleys are included in the model. The derived governing equations are shown to be coupled nonlinear PDEs. Galerkin's method is applied to obtain the numerical solution of the governing equations. Simulation results indicate that increase in the damping reduces the longitudinal vibration dramatically, but has little effect on the transverse vibration. The results also show that the magnitude of the transverse vibration is significantly larger than that of the longitudinal vibration. The organization of this paper is as follows: Section 2 derives the model of the cable transporter system. Section 3 describes the procedure for the numerical solution of the governing equations. Simulation results are presented in Section 4, followed by conclusions of the work.

II. MODELING OF THE SYSTEM

In order to get a good understanding of the response of the cable transporter system subject to control inputs and external disturbances, an accurate mathematical model is needed. Model-based feedback controllers design also require a dynamic model of the system.

A. Hamilton's Principle

Hamilton's principle states that the motion of a dynamic system between two given states at the times t_1 and t_2,

Mr. Yuhong Zhang is with the Department of Mechanical Engineering, University of Delaware, Newark, DE 19716, USA, zhangy@me.udel.edu.
Dr. Sunil Agrawal (Corresponding Author) is with the Department of Mechanical Engineering, University of Delaware, Newark, DE 19716, USA, agrawal@me.udel.edu.

TABLE I

THE VARIABLES USED IN THE DERIVATION

Variables	Description
$T_1(t)$	Torque applied by the right motor.
$T_2(t)$	Torque applied by the left motor.
r_1	Radius of the right pulley.
r_2	Radius of the left pulley.
I_1	Inertia of the right pulley.
I_2	Inertia of the left pulley.
C_1	Damping coefficient of the right pulley.
C_2	Damping coefficient of the left pulley.
E	Young's modulus.
A	Cross sectional area of the cables.
M	Mass of the slider.
ρ	Mass per unit length of the cable.
$\xi(t)$	Actuator and sensor location.
$l_m(t)$	Cable length external to the left pulley.
$v(t)$	Axial velocity of the system, $v(t) = \dot{l}_m(t)$.
L	Length between centers of the two pulleys.
$l_p(t)$	Cable length external to the right pulley.
$P(x,t)$	Tension in the cable.
C	Damping coefficient of the slider.
$u(t)$	A point wise control force in transverse direction.
$\theta_1(t)$	Angular displacement of the right pulley.
$\theta_2(t)$	Angular displacement of the left pulley.
$y(x,t)$	Transverse vibration of the cable.
$w(x,t)$	Longitudinal vibration of the cable.

minimizes the integral

$$\int_{t_1}^{t_2} (T - U + W)dt, \tag{1}$$

where T is the total kinetic energy, U is the total potential energy, and W is the total work done by external forces. Calculus of variations is then used to derive the model of a mechanical system. The variables used in the model development are shown in Table I. The schematic of the system is shown in Fig. 1. The origin is at the contact point of the left pulley, $u(t)$ is the control input in the transverse direction located at $x = \xi(t)$, for suppression of the transverse vibration. The slider position is defined by $l_m(t)$ and $v(t) = \dot{l}_m(t)$ is the axial velocity.

The velocity of a particle on the cable is composed of two parts due to the coupled vibrations. The velocity in the x direction is $v(t) + w_t + v(t)w_x$ and in the y direction is $y_t + vy_x$. Hence, the velocity of the slider at $x = l_m$, is $v(t) + w_t(l_m,t) + vw_x(l_m,t)$. The subscript '$_t$' denotes partial derivative with respect to time, and subscript '$_x$' denotes partial derivative with respect to space. In the following derivation, we omit the arguments such as time t and space x of the variables for the purpose of brevity, i.e., we express $y(x,t)$ simply as y, $w_t(x,t)$ as w_t, etc. We make the following assumptions in the derivation:

- The density of the cable ρ, the Young's modulus E, the cross sectional area of the cable A are constants.
- The damping coefficients C, C_1, C_2, and the inertias I_1, I_2 are constants.
- There is no bending and rotation in the cables.
- There are no vibrations in cables wound in pulleys.

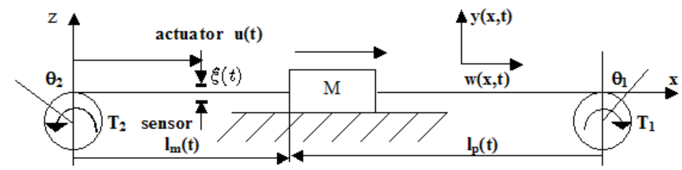

Fig. 1. A schematic of the cable transporter system.

- The motion is restricted to xz plane.
- The cable weight is neglected, i.e., no sag in cables.
- The shear strain is neglected.

The kinematic constraints of the system are as follows:

$$l_m(t) + l_p(t) = L, \tag{2}$$
$$r_1\theta_1 + l_p(t) = k1, \tag{3}$$
$$r_2\theta_2 + l_m(t) = k2, \tag{4}$$
$$r_1\theta_1 + r_2\theta_2 = 0, \tag{5}$$

where k_1 and k_2 are constants. On time differentiation of Eqs. (2)-(4), we get

$$\dot{l}_m(t) = -\dot{l}_p(t), \tag{6}$$
$$\dot{\theta_1} = -\frac{1}{r_1}\dot{l}_p(t) = \frac{1}{r_1}\dot{l}_m(t), \tag{7}$$
$$\dot{\theta}_2 = -\frac{1}{r_2}\dot{l}_m(t). \tag{8}$$

The total kinetic energy $T(t)$ of the system is composed of three parts: energy from the two pulleys, energy from the slider, and the energy from the cable vibrations.

$$\begin{aligned} T(t) = &\frac{1}{2}(I_1 + m_{s1}r_1^2)\dot{\theta}_1^2 + \frac{1}{2}(I_2 + m_{s2}r_2^2)\dot{\theta}_2^2 \\ &+ \frac{1}{2}M(v + w_t|_{l_m} + vw_x|_{l_m})^2 \\ &+ \frac{1}{2}\rho\int_0^{\xi_-}[(y_t + vy_x)^2 + (v + w_t + vw_x)^2]dx \\ &+ \frac{1}{2}\rho\int_{\xi_+}^{l_m}[(y_t + vy_x)^2 + (v + w_t + vw_x)^2]dx \\ &+ \frac{1}{2}\rho\int_{l_m}^{L}[(y_t + vy_x)^2 + (v + w_t + vw_x)^2]dx, \end{aligned} \tag{9}$$

where $m_{s1} = \rho r_1\theta_1$ and $m_{s2} = \rho r_2\theta_2$ are masses of the newly-wound cables on the pulley, $w_t|_{l_m} = w_t(l_m,t)$, and $w_x|_{l_m} = w_x(l_m,t)$. The normal strain in the x direction, according to the reference [14], is

$$\epsilon = w_x + \frac{1}{2}w_x^2 + \frac{1}{2}y_x^2. \tag{10}$$

The total potential energy $U(t)$ of the system due to the normal strain is

$$U(t)=\frac{1}{2}\int_{0}^{\xi_-}EA\epsilon^2dx+\frac{1}{2}\int_{\xi_+}^{l_m}EA\epsilon^2dx+\frac{1}{2}\int_{l_m}^{L}EA\epsilon^2dx. \quad (11)$$

The virtual work done by external forces including work done by actuator input $u(t)$, work done by torques $T_1(t)$ and $T_2(t)$, and work done by friction forces at the slider and pulleys is

$$\delta W=u(t)\delta y|_\xi+T_1\delta\theta_1+T_2\delta\theta_2-C_1\dot{\theta}_1\delta\theta_1-C_2\dot{\theta}_2\delta\theta_2 \\ -C(v+w_t|_{l_m}+vw_x|_{l_m})(\delta l_m+\delta w|_{l_m}+w_x|_{l_m}\delta l_m), \quad (12)$$

where $w|_{l_m}=w(l_m,t)$. On substitution of Eqs. (9), (11), and (12) in the extended Hamilton's principle, using part integration and calculus of variations, we obtain the following governing equations,

$$\rho\frac{\partial}{\partial t}(v+w_t+vw_x)+\rho v\frac{\partial}{\partial x}(v+w_t+vw_x) \\ =EA\frac{\partial}{\partial x}[(w_x+\frac{1}{2}w_x^2+\frac{1}{2}y_x^2)(1+w_x)], \quad (13)$$

$$\rho\frac{\partial}{\partial t}(y_t+vy_x)+\rho v\frac{\partial}{\partial x}(y_t+vy_x) \\ =EA\frac{\partial}{\partial x}[(w_x+\frac{1}{2}w_x^2+\frac{1}{2}y_x^2)y_x], \quad (14)$$

$$(\frac{T_1}{r_1}-\frac{T_2}{r_2})-(\frac{I_1}{r_1^2}+\frac{I_2}{r_2^2})\ddot{l}_m-(\frac{C_1}{r_1^2}+\frac{C_2}{r_2^2})\dot{l}_m \\ =M\frac{\partial}{\partial t}[v+w_t|_{l_m}+vw_x|_{l_m})(1+w_x|_{l_m})] \\ +C(v+w_t|_{l_m}+vw_x|_{l_m})(1+w_x|_{l_m}) \\ +\int_{0}^{\xi_-}\frac{\partial}{\partial t}\rho[(v+w_t+vw_x)(1+w_x)]dx \\ +\int_{\xi_+}^{l_m}\frac{\partial}{\partial t}\rho[(v+w_t+vw_x)(1+w_x)]dx \\ +\int_{l_m}^{L}\frac{\partial}{\partial t}\rho[(v+w_t+vw_x)(1+w_x)]dx \\ +\int_{0}^{\xi_-}\frac{\partial}{\partial t}\rho[y_x(y_t+vy_x)]dx+\int_{\xi_+}^{l_m}\frac{\partial}{\partial t}\rho[y_x(y_t+vy_x)]dx \\ +\int_{l_m}^{L}\frac{\partial}{\partial t}\rho[y_x(y_t+vy_x)]dx. \quad (15)$$

The resulting internal condition is,

$$u(t)|_\xi=\dot{\xi}\rho(y_t+vy_x)|_{\xi_-}-\rho v(y_t+vy_x)|_{\xi_-} \\ -\dot{\xi}\rho(y_t+vy_x)|_{\xi_+}+\rho v(y_t+vy_x)|_{\xi_+} \\ -EA(w_x+\frac{1}{2}w_x^2+\frac{1}{2}y_x^2)y_x|_{\xi_-} \\ +EA(w_x+\frac{1}{2}w_x^2+\frac{1}{2}y_x^2)y_x|_{\xi_+}, \quad (16)$$

where we have used continuity conditions $\delta y|_\xi=\delta y|_{\xi_+}=\delta y|_{\xi_-}$, $\delta l_m|_\xi=\delta l_m|_{\xi_+}=\delta l_m|_{\xi_-}$, and $\delta w|_\xi=\delta w|_{\xi_+}=\delta w|_{\xi_-}$. The boundary conditions are

$$w(0,t)=w(L,t)=0, \quad (17)$$

$$y(0,t)=y(l_m,t)=y(L,t)=0. \quad (18)$$

The transverse vibration in Eq. (14) can be expanded as follows:

$$\rho(y_{tt}+2vy_{xt}+v^2y_{xx}+\dot{v}y_x)=EA(y_xw_{xx} \\ +\frac{3}{2}y_x^2y_{xx}+\frac{1}{2}y_{xx}w_x^2+y_xw_xw_{xx}+y_{xx}w_x), \quad (19)$$

where $y_{tt}=\frac{\partial^2 y}{\partial t^2}$, $y_{xt}=\frac{\partial^2 y}{\partial x\partial t}$, $y_{xx}=\frac{\partial^2 y}{\partial x^2}$, and $w_{xx}=\frac{\partial^2 w}{\partial x^2}$. The result in Eq. (19) is compatible with the result in [11]. If the velocity v in Eq. (19) is assumed to be constant, Eq. (19) is also in agreement with the result in [13] with the exclusion of the external force terms.

After expressing the time and space derivatives in Eq. (13), the motion equation governing the longitudinal vibration is

$$\rho(w_{tt}+2vw_{xt}+v^2w_{xx}+\dot{v}w_x+\dot{v})=EA(w_{xx} \\ +3w_xw_{xx}+\frac{3}{2}w_x^2w_{xx}+\frac{1}{2}w_{xx}y_x^2+w_xy_xy_{xx}+y_xy_{xx}), \quad (20)$$

where $w_{tt}=\frac{\partial^2 w}{\partial t^2}$ and $w_{xt}=\frac{\partial^2 w}{\partial x\partial t}$. If the axial velocity $v(t)$ in Eq. (20) is constant, it reduces to the result in [13].

III. Discretization - Galerkin's Method

The derived motion equations are coupled nonlinear PDEs and it is impossible to obtain an exact analytical solution. Galerkin's method is applied to truncate the infinite-dimensional PDEs into a set of nonlinear finite-dimensional ordinary differential equations (ODEs) with time dependent coefficients. We normalize the x coordinate as follows:

$$\eta=\frac{x}{L}, \quad 0\leq\eta\leq 1. \quad (21)$$

The space derivative of y(x,t) can be expressed as

$$\frac{\partial y}{\partial x}=\frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial x}=\frac{1}{L}y_\eta, \quad (22)$$

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{L^2}y_{\eta\eta}, \quad (23)$$

where y_η and $y_{\eta\eta}$ are the short notations of the space derivatives of $\frac{\partial y}{\partial\eta}$ and $\frac{\partial^2 y}{\partial\eta^2}$, respectively. The transverse vibration $y(x,t)$ is discretized in the following form in terms of mode functions $\Psi_j(\eta)$:

$$y(x,t)=\sum_{j=1}^{n}q_j(t)\Psi_j(\eta), \quad (24)$$

where $q_j(t)$ is the unknown general coordinate to be determined and n is the number of mode functions. Similarly for the longitudinal vibration, we have

$$w(x,t)=\sum_{i=1}^{n}p_i(t)\Psi_i(\eta), \quad (25)$$

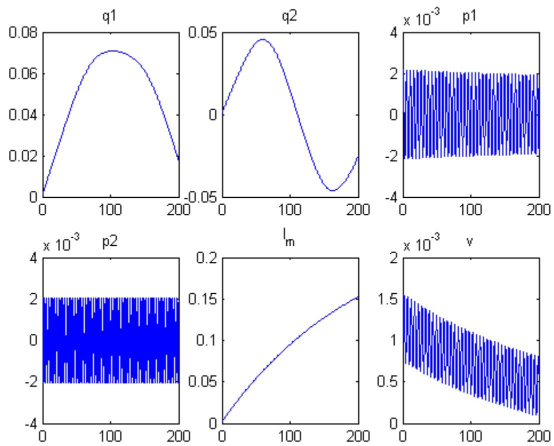

Fig. 2. States trajectories of the system.

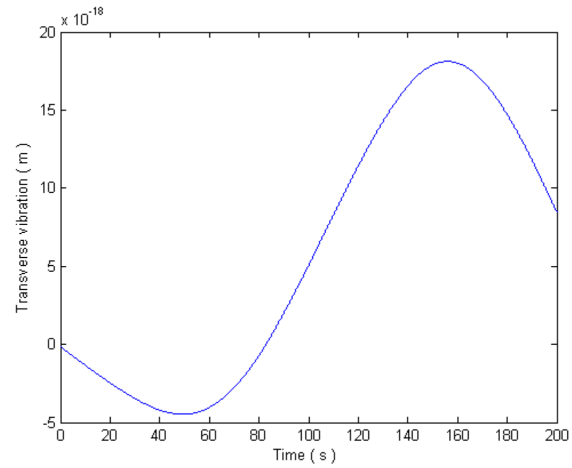

Fig. 3. Transverse vibration at one end.

where $p_i(t)$ is the unknowns to be determined. On substitution of Eqs. (22)-(25) in Eqs. (15), (19), and (20), we end up with a set of finite-dimensional discretized ODEs with $\mathbf{q}(t) = [q_1(t), q_2(t), ...q_n(t), p_1(t), p_2(t), ...p_n(t), l_m(t)]^T$ as unknown.

IV. Simulation Results and Analysis

The orthogonal normal mode functions are chosen to satisfy the boundary conditions

$$\Psi_i(\eta) = sin(i\pi\eta), \quad 0\leq\eta\leq1, \quad i = 1, 2, ...n \tag{26}$$

where n is the number of the mode functions. Given the initial conditions on $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t) = [\dot{q}_1(t), \dot{q}_2(t), ...\dot{q}_n(t), \dot{p}_1(t), \dot{p}_2(t), ...\dot{p}_n(t), v(t)]^T$, a numerical solution is easily obtained using MATLAB.

Fig. 2 shows the state trajectories $\mathbf{q}$ versus time. We see that the amplitudes of states q_1 and q_2, which correspond to the transverse vibration, are much bigger than those of states p_1 and p_2, which correspond to the longitudinal vibration. The transverse and longitudinal vibrations at one end $x = L$, which corresponds to $\eta = 1$ are shown in Figs. 3 and 4, respectively. We observe that both vibrations at the end are zero, which is reasonable since we made no vibration assumption at the end.

Figs. 5 and 6 show the transverse and longitudinal vibration at $\eta = 0.5$. It can be seen that the transverse vibration is much bigger than the longitudinal vibration. This result is in agreement with the observations from experiment. This explains why longitudinal vibration is usually ignored in literature.

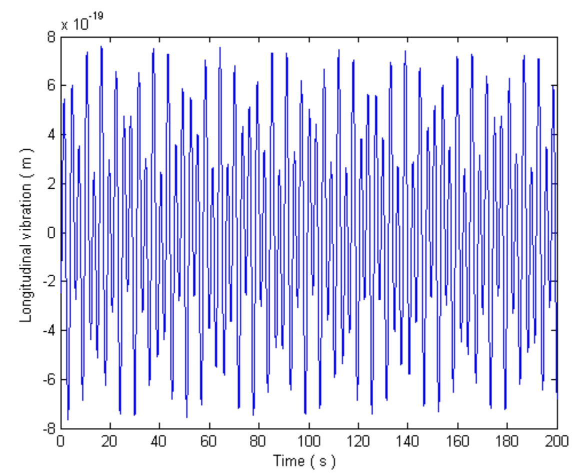

Fig. 4. Longitudinal vibration at one end.

In order to see the effects of damping on the vibrations, we increase the damping coefficients by one hundred times, the results of transverse and longitudinal vibrations are shown in Figs. 7 and 8, respectively. Comparing with the results shown in Figs. 5 and 6, we see that the increase in the damping reduces the longitudinal vibration dramatically, but has little effect on the transverse vibration. This justifies the addition of the control input $u(t)$ to suppress the transverse vibration.

The arbitrary axial velocity affects both the transverse vibration and longitudinal vibration. Figs. 9 and 10 show the transverse and longitudinal vibration when the magnitude of

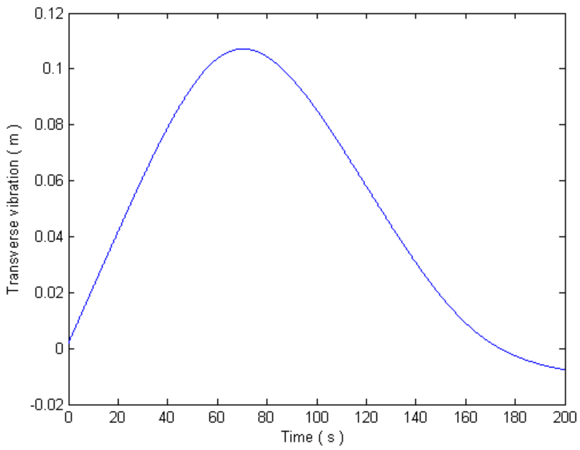

Fig. 5. Transverse vibration at $\eta = 0.5$.

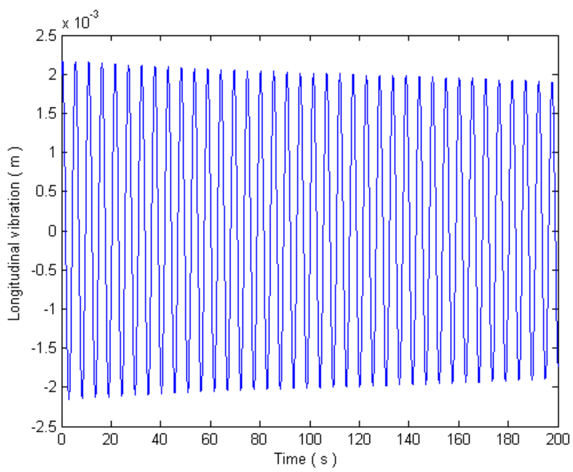

Fig. 6. Longitudinal vibration at $\eta = 0.5$.

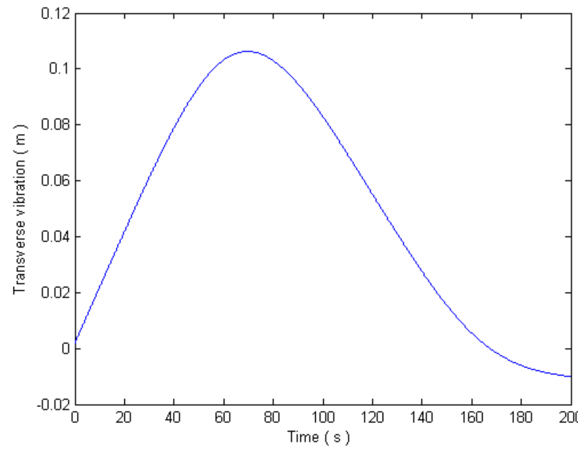

Fig. 7. Transverse vibration at $\eta = 0.5$ with large damping.

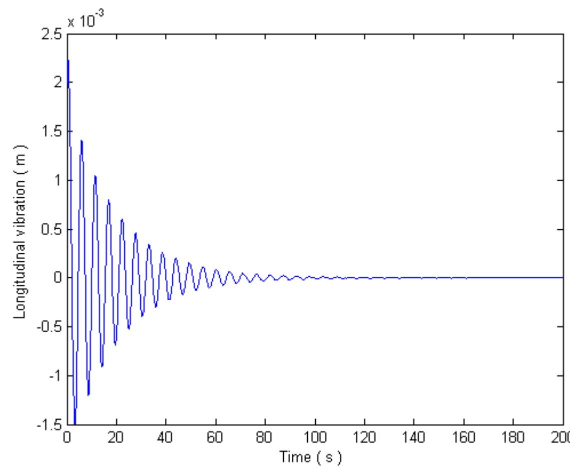

Fig. 8. Longitudinal vibration at $\eta = 0.5$ with large damping.

the initial axial velocity is increased one hundred times. In general, increase in the axial velocity will lead to increases of both vibrations. When the axial velocity reaches above a critical velocity, divergence will happen, or even chaotic motions.

V. CONCLUSIONS

In this paper, we derive the governing equations for coupled vibrations of a varying length flexible cable transporter system with arbitrary axial velocity using Hamilton's principle. The inertias of the pulleys and motor, and dampings of the system are included in the model. The derived equations provide guidelines for future study on cable systems.

It is impossible to find an exact analytical solution for the derived nonlinear governing equations. Galerkin's method is applied to truncate the infinite-dimensional PDEs into finite-dimensional nonlinear ODEs with time dependent coefficients. The numerical results demonstrate that increase in the damping can reduce the longitudinal vibration dramatically, but has little effect on the transverse vibration. The results also show that the magnitude of the transverse vibration is significantly larger than that of the longitudinal vibration, which explains why the longitudinal vibration is often ignored in literature. Future work will include: derivation of controllers $T_1(t)$, $T_2(t)$, and $u(t)$, stability

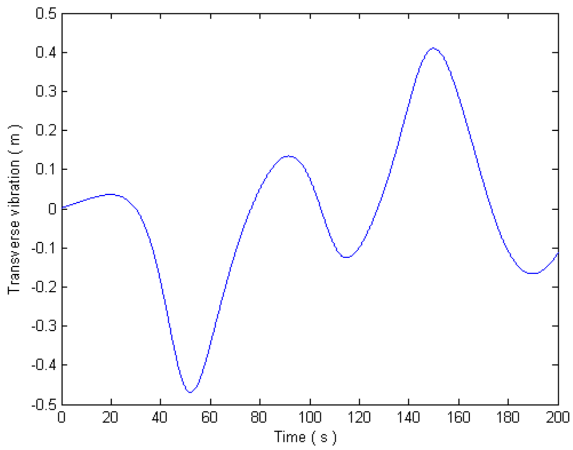

Fig. 9. Transverse vibration at $\eta = 0.5$ with large initial velocity.

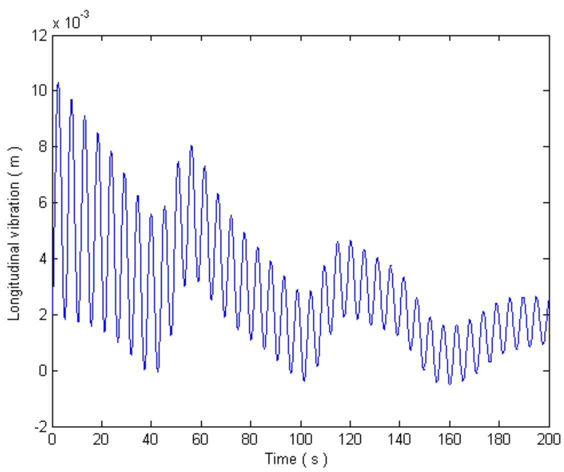

Fig. 10. Longitudinal vibration at $\eta = 0.5$ with large initial velocity.

analysis, and experiment implementation.

VI. ACKNOWLEDGMENTS

The authors gratefully acknowledge the support of NSF grant number IIS-0117733.

REFERENCES

[1] J. A. Wickert, C. D. Mote Jr., "Current research on the vibration and stability of axially moving materials," *The Shock and Vibration Digest*, vol. 20, no. 5, pp. 3-13, 1988.

[2] C. D. Mote Jr., "On the Nonlinear Oscillation of an Axially Moving String," *Journal of Applied Mechanics, Transaction of ASME*, vol. 33, pp. 463-464, 1966.

[3] G. Kirchhoff., "Vorlesungen $\ddot{u}$ber Mathematische Physik," Leipzig: Druck and Verlag von B. G. eubner, 1877.

[4] G. F. Carrier., "On the non-linear vibration problem of the elastic string," *Quarterly of Applied Mathematics*, 3, pp. 157-165, 1945.

[5] D. W. Oplinger., "Frequency response of a nonlinear stretched string," *Journal of the Acoustical Society of America*, 32, pp. 1529-1538, 1960.

[6] R. Narasimha., "Non-linear vibration of an elastic string," *Journal of Sound and Vibration*, 8, pp. 134-146, 1968.

[7] F. Zhang, D. M. Dawson, S. P. Nagarkatti, and C. D. Rahn., "Boundary Control for a General Class of Nonlinear String Actuator Systems," *ASME Transactions of Journal of Sound and Vibration*, vol. 229, no. 1, pp. 113-132, 2000.

[8] C. F. Baicu, C. D. Rahn, and B. D. Nibali., "Active Boundary Control of Elastic Cables: Theory and Experiment," *ASME Transactions of Journal of Sound and Vibration*, vol. 198, no. 1, pp. 17-26, 1996.

[9] P. Hagedorn, and B. Sch$\ddot{A}$fer., "On Non-Linear Free Vibrations of an Elastic Cable," *International Journal of Nonlinear Mechanics*, vol. 15, pp. 333-340, 1980.

[10] W. D, Zhu, and J. Ni., "Energetics and Stability of Translating Media with an Arbitrarily Varying Length," *ASME Transactions of Journal of Vibration and Acoustics*, vol. 122, pp. 295-304, 2000.

[11] W. D, Zhu, J. Ni, and J. Huang., "Active Control of Translating Media with Arbitrarily Varying Length," *ASME Transactions of Journal of Vibration and Acoustics*, vol. 123, no. 1, pp. 347-358, 2001.

[12] B. Tabarrok, C. M. Leech and Y, I. Kim., "On the Dynamics of an Axially Moving Beam," *Journal of The Franklin Institute*, vol. 297, no. 3, pp. 201-220, 1974.

[13] C. H. Riedel, and C. A. Tan., "Coupled, forced response of an axially moving strip with internal resonance," *International Journal of Non-Linear Mechanics*, vol. 37, pp. 101-116, 2000.

[14] V. V. Novozhilov., "Theory of Elasticity," *Pergamon Press*, pp. 18-19, 1961.