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Stability and Robustness of a Class of Nonlinear Controllers for Robot Manipulators

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***Abstract*— The stability and robustness analysis of PD-plus-feedforward controllers comprised of a nonlinear gain PD feedback controller and a nominal manipulator dynamics feedforward term is presented for the tracking control of rigid robot manipulators. Exponential convergence and the uniform ultimate boundedness of the tracking errors are established. Simulation results are presented.**

I. INTRODUCTION

Control schemes for the trajectory tracking control of rigid robot manipulators can be broadly classified as [1]: PD-plus-feed-forward (PD+) control schemes, robust control designs and adaptive control methods. PD+ control schemes are comprised of a feedback component for closed-loop stability and a feedforward component for tracking performance [2]. The feedback component is often a linear constant gain PD controller and the feedforward component may consist of complete manipulator dynamics, partial dynamics, or it may be absent completely. For the case of the feedforward term consisting of complete manipulator dynamics, stability analysis of linear PD+ control schemes has established the global asymptotic and exponential stability for the tracking control of rigid manipulators [2], [3], [4], [5]. When the feedforward term does not contain the complete manipulator dynamics, the origin of the state-space is no longer guaranteed to be an equilibrium point of the closed-loop system and the tracking errors do not vanish as time increases. For these cases, stability and robustness analysis of the linear PD+ controllers has established exponential convergence and the *uniform ultimate boundedness* of the errors for tracking control of manipulators[2].

Although the stability of closed-loop systems using the PD+ control laws is assured, performance is governed by the choice of controller gains. The constant gain PD+ controller requires comparatively large initial actuator torques and actuator size can become a limiting factor for controller performance. To improve closed-system performance, independent joint nonlinear gain PD controllers have been introduced [6], [7], [8], [9]. These controllers have proportional and derivative gains that are nonlinear functions of the joint position and velocity errors. The benefits of using the nonlinear gain PD+ controllers have been demonstrated using simulation and experimental studies on rigid manipulators and global asymptotic stability results have been established for position control with independent joint nonlinear gain PD feedback controllers and a feedforward term containing complete manipulator dynamics [7], [8], [9].

II. MANIPULATOR DYNAMICS AND PROPERTIES

The dynamics of an n-joint rigid robot manipulator can be described by the Euler-Lagrange equations [10]

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) + \mathbf{F}_d\dot{\mathbf{q}} + \mathbf{f}_s(\dot{\mathbf{q}}) + \mathbf{u}_d = \mathbf{u} \quad (1)$$

where $\mathbf{q} \in \mathbb{R}^n$ denotes joint coordinates, $\mathbf{M}(\mathbf{q})$ is the symmetric positive definite inertia matrix, $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}$ is the vector of centripetal and Coriolis terms, $\mathbf{g}(\mathbf{q})$ is the vector of gravity terms, $\mathbf{F}_d$ is the diagonal positive semi-definite matrix of dynamic friction coefficients, $\mathbf{f}_s(\dot{\mathbf{q}})$ is the vector of static friction terms, $\mathbf{u}$ is the vector of control inputs, and $\mathbf{u}_d$ is the vector of unknown but bounded disturbance terms. The model (1) exhibits some important properties [1] which can be exploited to facilitate controller design. For subsequent developments we use the following notation: $\lambda_m(\mathbf{A})$ and $\lambda_M(\mathbf{A})$ denote the smallest and largest eigenvalues, respectively, of a symmetric matrix $\mathbf{A}$, $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$ denotes the norm of a vector $\mathbf{x}$, and $\|\mathbf{A}\| = \sqrt{\lambda_M(\mathbf{A}^T\mathbf{A})}$ denotes the induced matrix norm of any real matrix $\mathbf{A}$.

Property 1: $\dot{\mathbf{M}}(\mathbf{q}) = \mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{C}^T(\mathbf{q},\dot{\mathbf{q}})$ [10].

Property 2: There exists a positive constant k_C such that

$$\|\mathbf{C}(\mathbf{x},\mathbf{y})\mathbf{z}\| = \|\mathbf{C}(\mathbf{x},\mathbf{z})\mathbf{y}\| \leq k_C\|\mathbf{y}\|\|\mathbf{z}\| \quad \forall \mathbf{x},\mathbf{y},\mathbf{z} \quad (2)$$

The constant k_C satisfies the following bound [11]:

$$k_C \geq n^2 \left(\max_{i,j,k,\mathbf{q}} |c_{ijk}(\mathbf{q})| \right) \quad (3)$$

where $c_{ijk}(\mathbf{q})$ is the (i,j,k)-th Christoffel symbol used in the definition of matrix $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$.

Property 3: Since the inertia matrix $\mathbf{M}(\mathbf{q})$ is symmetric positive definite and bounded for any $\mathbf{q}$, there exist positive constants m_1 and m_2 such that

$$m_1\|\mathbf{x}\|^2 \leq \mathbf{x}^T\mathbf{M}(\mathbf{q})\mathbf{x} \leq m_2\|\mathbf{x}\|^2, \quad \forall \mathbf{q},\mathbf{x} \in \mathbb{R}^n \quad (4)$$

where $m_1 = \inf_{\mathbf{q}\in\mathbb{R}^n} \lambda_m(\mathbf{M}(\mathbf{q}))$, $m_2 = \sup_{\mathbf{q}\in\mathbb{R}^n} \lambda_M(\mathbf{M}(\mathbf{q}))$.

For the following control design it is assumed that the friction effects, disturbance terms, and joint angle velocities are bounded as follows:

$$0 \leq k_{fd1} = \lambda_m(\mathbf{F}_d) \quad \leq \quad \lambda_M(\mathbf{F}_d) = k_{fd2} \quad (5)$$

$$k_{fs} = \sup_{\dot{\mathbf{q}}\in\mathbb{R}^n} \|\mathbf{f}_s(\dot{\mathbf{q}})\| \qquad k_{ud} = \sup_{t\in\mathbb{R}_+} \|\mathbf{u}_d(t)\| \quad (6)$$

$$k_{qd1} = \sup_{t\in\mathbb{R}_+} \|\dot{\mathbf{q}}_d(t)\|, \qquad k_{qd2} = \sup_{t\in\mathbb{R}_+} \|\ddot{\mathbf{q}}_d(t)\|. \quad (7)$$

The Control Problem: Given the desired joint angle trajectory $\mathbf{q}_d(t)$ with finite first and second derivatives $\dot{\mathbf{q}}_d(t)$ and $\ddot{\mathbf{q}}_d(t)$, design a control law which assures that for any initial conditions and any admissible uncertainties, the position and velocity trajectories $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$ exponentially track the reference trajectories $\mathbf{q}_d(t)$ and $\dot{\mathbf{q}}_d(t)$ with some desired rate of convergence and within some desired degree of accuracy. The joint angles and rates are assumed available for feedback.

III. Nonlinear Controllers for Trajectory Tracking of Rigid Manipulators

The goal of the tracking control problem for robotic manipulators is to design a control law $\mathbf{u}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_d, \dot{\mathbf{q}}_d, \ddot{\mathbf{q}}_d)$ so that $(\mathbf{q}, \dot{\mathbf{q}})$ tracks $(\mathbf{q}_d, \dot{\mathbf{q}}_d)$ in some sense. To accomplish this, the following general control structure is considered:

$$\mathbf{u} = \mathbf{u}_{fb}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_d, \dot{\mathbf{q}}_d) + \mathbf{u}_{ff}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_d, \dot{\mathbf{q}}_d, \ddot{\mathbf{q}}_d) \qquad (8)$$

where $\mathbf{u}_{fb}$ is the feedback portion and $\mathbf{u}_{ff}$ is the feedforward term. Here we investigate the stability robustness of a class of nonlinear controllers with the feedforward term:

$$\mathbf{u}_{ff} = \mathbf{M}_n(\mathbf{q})\ddot{\mathbf{q}}_d + \mathbf{C}_n(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}_d + \mathbf{g}_n(\mathbf{q}) \qquad (9)$$

where $\mathbf{M}_n(\mathbf{q})$, $\mathbf{C}_n(\mathbf{q}, \dot{\mathbf{q}})$ and $\mathbf{g}_n(\mathbf{q})$ represent the nominal or estimated values for $\mathbf{M}(\mathbf{q})$, $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ and $\mathbf{g}(\mathbf{q})$, respectively, and are subject to the following bounds:

$$\delta_M = \sup_{\mathbf{q} \in \mathbb{R}^n} \|\mathbf{M}(\mathbf{q}) - \mathbf{M}_n(\mathbf{q})\| \qquad (10)$$

$$\delta_C = \sup_{\mathbf{q} \in \mathbb{R}^n, \|\dot{\mathbf{q}}_d\| \le k_{qd1}} \|\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}_d) - \mathbf{C}_n(\mathbf{q}, \dot{\mathbf{q}}_d)\| \qquad (11)$$

$$\delta_g = \sup_{\mathbf{q} \in \mathbb{R}^n} \|\mathbf{g}(\mathbf{q}) - \mathbf{g}_n(\mathbf{q})\| \qquad (12)$$

Even though the subsequent stability and robustness analysis requires the existence of the above bounds, only δ_C needs to be known explicitly and *a priori* to ensure stability and robustness. In the case of manipulators with a known range of uncertainty only in the inertia parameters, the bound δ_C can be determined using (2) and (3). In many applications an unknown payload is the main source of uncertainty thus leaving a single uncertain inertia parameter affecting the computation of δ_C.

The general structure of $\mathbf{u}_{fb}$ is motivated by the fact that rigid manipulators belong to a class of mechanical systems that can be stabilized by PD-type control laws. The proposed feedback term is a general affine function of the tracking errors as per:

$$\mathbf{u}_{fb} = \mathbf{K}_p(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_d, \dot{\mathbf{q}}_d)\tilde{\mathbf{q}} + \mathbf{K}_d(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_d, \dot{\mathbf{q}}_d)\dot{\tilde{\mathbf{q}}} \qquad (13)$$

where $\tilde{\mathbf{q}} = \mathbf{q}_d - \mathbf{q}$ and $\dot{\tilde{\mathbf{q}}} = \dot{\mathbf{q}}_d - \dot{\mathbf{q}}$ are the joint position and velocity errors, and $\mathbf{K}_p(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_d, \dot{\mathbf{q}}_d)$ and $\mathbf{K}_d(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_d, \dot{\mathbf{q}}_d)$ are function matrices selected to satisfy stability and performance requirements. In their simplest form, $\mathbf{K}_p$ and $\mathbf{K}_d$ are diagonal positive definite constant matrices yielding the most common independent joint linear PD control law. To improve certain performance characteristics or satisfy constraints on the control torques the class of representations for $\mathbf{u}_{fb}$ has been extended to nonlinear gain PD controllers [7], [8], [9].

Here a more general structure of the gain matrices is considered for further extending the class of representations of $\mathbf{u}_{fb}$ for trajectory tracking control of n-joint rigid robotic manipulators with nonlinear and coupled dynamics as given by equation (1). The nonlinear controllers of this class are defined according to

$$\mathbf{u}_{fb} = \mathbf{K}_p(\tilde{\mathbf{q}})\tilde{\mathbf{q}} + \mathbf{K}_d(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_d, \dot{\mathbf{q}}_d)\dot{\tilde{\mathbf{q}}} \qquad (14)$$

where the derivative gain matrix $\mathbf{K}_d(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_d, \dot{\mathbf{q}}_d)$ is assumed symmetric positive definite and bounded for all $\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_d, \dot{\mathbf{q}}_d$. In the following, we denote the symmetric positive definite derivative gain matrix as $\mathbf{K}_d(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$, but the subsequent developments equally hold for the more general case, $\mathbf{K}_d(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_d, \dot{\mathbf{q}}_d)$. To ensure global asymptotic stability of nonlinear gain PD controllers for manipulator position control, previous studies [7], [8], [9] have considered positive definite diagonal proportional gain matrix $\mathbf{K}_p(\tilde{\mathbf{q}})$ with $k_{pii}(\tilde{q}_i) > 0, i = 1 \ldots n$ as its diagonal elements. Here, the proportional gain matrix $\mathbf{K}_p(\tilde{\mathbf{q}})$ is assumed symmetric with the structure:

$$\mathbf{K}_p(\tilde{\mathbf{q}}) = \begin{bmatrix} k_{p11}(\tilde{q}_1) & k_{p12} & \cdots & k_{p1n} \\ k_{p21} & k_{p22}(\tilde{q}_2) & \cdots & k_{p2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{pn1} & k_{pn2} & \cdots & k_{pnn}(\tilde{q}_n) \end{bmatrix} \qquad (15)$$

The diagonal elements of $\mathbf{K}_p(\tilde{\mathbf{q}})$ have an upper bound and a positive lower bound *i.e.* they satisfy

$$k_{pii}^M \ge k_{pii}(\tilde{q}_i) \ge k_{pii}^m > 0, \quad \forall \tilde{q}_i \in \mathbb{R}, \qquad (16)$$

for $i = 1, \ldots, n$. Define constant symmetric positive definite matrices $\mathbf{K}_p^m$ and $\mathbf{K}_p^M$ as

$$\mathbf{K}_p^m = \mathbf{K}_p(\tilde{\mathbf{q}}) - \mathrm{diag}[\mathbf{K}_p(\tilde{\mathbf{q}})] + \mathrm{diag}(k_{p11}^m, \ldots, k_{pnn}^m) \qquad (17)$$

and

$$\mathbf{K}_p^M = \mathbf{K}_p(\tilde{\mathbf{q}}) - \mathrm{diag}[\mathbf{K}_p(\tilde{\mathbf{q}})] + \mathrm{diag}(k_{p11}^M, \ldots, k_{pnn}^M) \qquad (18)$$

Given $k_{pii}^M > k_{pii}^m > 0$, $i = 1, \cdots, n$, one can always construct $\mathbf{K}_p^m$ and $\mathbf{K}_p^M$ as defined in (17) and (18). For example, $\mathbf{K}_p^m$ and $\mathbf{K}_p^M$ can be defined as *diagonally dominant* symmetric positive definite matrices satisfying

$$k_{pii}^M > k_{pii}^m > \sum_{j=1, j \ne i}^{n} |k_{pij}|$$

with possibly non-zero off-diagonal elements k_{pij}.

The following properties of the above gain matrices are used for the stability analysis of nonlinear controllers.

Property 4: Since the proportional gain matrix $\mathbf{K}_p(\tilde{\mathbf{q}})$ and the constant matrices $\mathbf{K}_p^m$ and $\mathbf{K}_p^M$ are symmetric and positive definite, there exist positive constants k_{p1} and k_{p2} such that $\forall \tilde{\mathbf{q}}, \mathbf{x} \in \mathbb{R}^n$,

$$k_{p1}\|\mathbf{x}\|^2 \le \mathbf{x}^T \mathbf{K}_p^m \mathbf{x} \le \mathbf{x}^T \mathbf{K}_p(\tilde{\mathbf{q}}) \mathbf{x} \le \mathbf{x}^T \mathbf{K}_p^M \mathbf{x} \le k_{p2}\|\mathbf{x}\|^2$$

where $k_{p1} = \lambda_m(\mathbf{K}_p^m)$ and $k_{p2} = \lambda_M(\mathbf{K}_p^M)$.

Property 5: Since the derivative gain matrix $\mathbf{K}_d(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$ is symmetric and positive definite, there exist positive constants k_{d1} and k_{d2} such that for all $\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, \mathbf{x} \in \mathbb{R}^{\mathbf{n}}$,

$$k_{d1}\|\mathbf{x}\|^2 \le \mathbf{x}^T\mathbf{K}_d(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})\mathbf{x} \le k_{d2}\|\mathbf{x}\|^2$$

where $k_{d1} = \inf \lambda_m(\mathbf{K}_d(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}))$, $k_{d2} = \sup \lambda_M(\mathbf{K}_d(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}))$.

IV. Stability and Robustness Analysis

Here we present the results concerning the stability of the closed-loop system when the controller (14) is used for trajectory tracking with the dynamics described by (1). The closed-loop system of (1), (9) and (14) is given as

$$\mathbf{M}(\mathbf{q})\ddot{\tilde{\mathbf{q}}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\tilde{\mathbf{q}}} + \mathbf{K_p}(\tilde{\mathbf{q}})\tilde{\mathbf{q}} + [\mathbf{K_d}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) + \mathbf{F_d}]\dot{\tilde{\mathbf{q}}} = \mathbf{\Delta u} \tag{19}$$

where

$$\begin{aligned}\mathbf{\Delta u} \;=\; & [\mathbf{M}(\mathbf{q}) - \mathbf{M}_n(\mathbf{q})]\ddot{\mathbf{q}}_d + [\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{C}_n(\mathbf{q}, \dot{\mathbf{q}})]\dot{\mathbf{q}}_d \\ & + \; [\mathbf{g}(\mathbf{q}) - \mathbf{g}_n(\mathbf{q})] + \mathbf{F}_d\dot{\mathbf{q}}_d + \mathbf{f}_s(\dot{\mathbf{q}}) + \mathbf{u}_d \end{aligned} \tag{20}$$

and (19) is a *non-autonomous* differential equation since $\mathbf{q}_d$ and $\dot{\mathbf{q}}_d$ are time-varying trajectories. Before stating the stability results, we present the following lemmas.

Lemma 1: Consider a dynamical system

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_1, \ldots, \mathbf{x}_m, t) \tag{21}$$

where $\mathbf{x}_i \in \mathbb{R}^{n_i}$, for $i = 1, \ldots, m$ and $t \ge 0$. Let $\mathbf{f}_i$ be locally Lipschitz with respect to $\mathbf{x}_1, \ldots, \mathbf{x}_m$ uniformly in t on bounded intervals and continuous in t for $t \ge 0$. Suppose a scalar function $V(\mathbf{x}, t) : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}_+$ is given such that

$$V(\mathbf{x}, t) \ge c_i\|\mathbf{x}_i\|^2 \tag{22}$$

where $\mathbf{x}^T = [\mathbf{x}_1^T, \ldots, \mathbf{x}_m^T]$, $N = n_1 + \ldots + n_m$, $c_i > 0$ for $i = 1, \ldots, m$, and along the solution trajectories of (21)

$$\dot{V}(\mathbf{x}, t) \le -\sum_{i \in I_1}\left(\gamma_i - \sum_{j \in I_{2i}} \gamma_{ij}\|\mathbf{x}_j\|^{r_{ij}}\right)\|\mathbf{x}_i\|^2 + \varepsilon$$

where $\gamma_i, \gamma_{ij}, r_{ij} > 0$, $\varepsilon \ge 0$ and $I_{2i} \subseteq I_1 \subseteq \{1, \ldots, m\}$. If $\forall i \in I_1$ (with reference to (22))

$$\gamma_i > \sum_{j \in I_{2i}} \gamma_{ij}\left(\frac{V_0}{c_j}\right)^{r_{ij}/2} \tag{23}$$

where $V_0 = V(\mathbf{x}_1(0), \ldots, \mathbf{x}_m(0), 0)$, then

$$\forall \beta_i \in \left(0, \gamma_i - \sum_{j \in I_{2i}} \gamma_{ij}\left(\frac{V_0}{c_j}\right)^{r_{ij}/2}\right),$$

the following inequality holds

$$\dot{V}(\mathbf{x}, t) \le -\sum_{i \in I_1} \beta_i\|\mathbf{x}_i\|^2 + \varepsilon \tag{24}$$

for $\|\mathbf{x}\| > R$ where $R = \sqrt{\varepsilon/(\min \beta_i)}$.

Proof: The proof follows from Lemma 2.1 in [5]. ■

Definition 1 Uniform Ultimate Boundedness (u.u.b.) [12]: A solution $\mathbf{x}(t) : [t_0, \infty) \to \mathbb{R}^N$ of (21) with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ is said to be *uniformly ultimately bounded* if there exist positive constants b and c, and for every $\sigma \in (0, c)$ there is a positive constant $T(\sigma)$ such that $\|\mathbf{x}_0\| < \sigma$ implies that $\|\mathbf{x}(t)\| \le b$ for all $t \ge t_0 + T(\sigma)$. Here the constant b is referred to as the *ultimate bound*. Uniform ultimate boundedness says that the solution trajectory of the system (21) beginning at $\mathbf{x}_0$ at time t_0 will ultimately enter and remain within the closed ball $B(b)$. If $B(b)$ is a small region about the equilibrium, then u.u.b. is a practical notion of stability, which is also called *practical stability*. The next lemma contains conditions that guarantee the u.u.b. and the global exponential convergence (to a closed ball) of the solution trajectories of (21) [13].

Lemma 2: Suppose there exists a continuously differentiable scalar function $V(\mathbf{x}, t) : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties: (i) there are positive constants $\underline{c}$ and $\bar{c}$ such that $\forall \mathbf{x} \in \mathbb{R}^N$ and $t \in \mathbb{R}_+$,

$$\underline{c}\|\mathbf{x}\|^2 \le V(\mathbf{x}, t) \le \bar{c}\|\mathbf{x}\|^2,$$

(ii) there are constants $\mu > 0$ and $\varepsilon \ge 0$ such that along the solution trajectories of (21)

$$\dot{V}(\mathbf{x}, t) \le -\mu V(\mathbf{x}, t) + \varepsilon \tag{25}$$

for all $\mathbf{x}$ s.t. $\mu V(\mathbf{x}, t) > \varepsilon$ and $t \in \mathbb{R}_+$. Then the solution trajectories of (21) are uniformly ultimately bounded and globally exponentially convergent to the closed ball $B(r)$ of radius $r = \sqrt{\varepsilon/(\mu\underline{c})}$. If in addition, $\varepsilon = 0$, then the system (21) is globally exponentially stable about its origin [13].

In the following, sufficient conditions are given for the u.u.b. and exponential convergence of the solution $\bar{\mathbf{q}} = [\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}]^{\mathbf{T}}$ of the closed-loop system (19).

Theorem 1: Consider the robot model in (1) together with the nonlinear gain PD+ controller in (14). Let the derivative gain matrix $\mathbf{K}_d(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$ be symmetric positive definite for all $\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}} \in \mathbb{R}^{\mathbf{n}}$ with $k_{d1} > \delta_C$, and the proportional gain matrix $\mathbf{K}_p(\tilde{\mathbf{q}})$ be symmetric and has the structure given by (15) with diagonal elements satisfying (16). If the symmetric matrices $\mathbf{K}_p^m$ and $\mathbf{K}_p^M$ as described in (17) and (18) are positive definite, then the solution $\bar{\mathbf{q}}$ of the closed-loop system (19) is uniformly ultimately bounded and exponentially convergent to the closed ball $B(r)$ defined below.

Proof: Consider the following scalar function:

$$V(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) = \frac{1}{2}\dot{\tilde{\mathbf{q}}}^T\mathbf{M}(\mathbf{q})\dot{\tilde{\mathbf{q}}} + \int_{\mathbf{0}}^{\tilde{\mathbf{q}}} \mathbf{z}^T\mathbf{K}_p(\mathbf{z})d\mathbf{z} + \alpha\tilde{\mathbf{q}}^T\mathbf{M}(\mathbf{q})\dot{\tilde{\mathbf{q}}} \tag{26}$$

where the integral term can written as

$$\sum_{i=1}^{n}\left(\int_0^{\tilde{q}_i} z_i k_{pii}(z_i)dz_i\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} k_{pij}\tilde{q}_i\tilde{q}_j$$

and α is a sufficiently small positive constant such that

$$\min\left\{\frac{k_{p1}}{m_2}, \frac{m_1}{m_2}, \frac{2(k_{d1} + k_{fd1} - \delta_C)}{3\omega k_C + 2m_2 + \rho k^*}\right\} > \alpha > 0 \tag{27}$$

where $k^* = k_{d2} + k_C k_{qd1} + k_{fd2} + \delta_C$ and constant ρ is

$$\rho > \frac{k_{d2} + k_C k_{qd1} + k_{fd2} + \delta_C}{2k_{p1}} > 0 \qquad (28)$$

and constant $\omega > 0$ which will be defined later.

The first term of $V(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$ is a positive definite function with respect to $\dot{\tilde{\mathbf{q}}}$ because $\mathbf{M}(\mathbf{q})$ is a positive definite matrix. To show that the integral term in (26) is a positive definite function, rewrite it as

$$\int_{\mathbf{0}}^{\tilde{\mathbf{q}}} \mathbf{z}^T \mathbf{K}_p(\mathbf{z}) d\mathbf{z} = \sum_{i=1}^{n} \left(\int_0^{\tilde{q}_i} z_i k_{pii}(z_i) dz_i \right) + \frac{1}{2}\tilde{\mathbf{q}}^T \mathbf{K}_p^m \tilde{\mathbf{q}} - \frac{1}{2}\sum_{i=1}^{n} k_{pii}^m \tilde{q}_i^2 = \sum_{i=1}^{n} \left(\int_0^{\tilde{q}_i} z_i k_{pii}(z_i) dz_i - \frac{1}{2} k_{pii}^m \tilde{q}_i^2 \right) + \frac{1}{2}\tilde{\mathbf{q}}^T \mathbf{K}_p^m \tilde{\mathbf{q}} \qquad (29)$$

From (16), we have for $i = 1, \ldots, n$

$$\int_0^{\tilde{q}_i} z_i k_{pii}(z_i) dz_i \geq \int_0^{\tilde{q}_i} z_i k_{pii}^m dz_i = \frac{1}{2} k_{pii}^m \tilde{q}_i^2 \qquad (30)$$

which yields using, (29) and Property 4,

$$\int_{\mathbf{0}}^{\tilde{\mathbf{q}}} \mathbf{z}^T \mathbf{K}_p(\mathbf{z}) d\mathbf{z} \geq \frac{1}{2}\tilde{\mathbf{q}}^T \mathbf{K}_p^m \tilde{\mathbf{q}} \geq \frac{1}{2} k_{p1} \|\tilde{\mathbf{q}}\|^2 \qquad (31)$$

Therefore, the lower-bound of (26) can be given as

$$V(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) \geq \frac{1}{2} m_1 \|\dot{\tilde{\mathbf{q}}}\|^2 + \frac{1}{2} k_{p1} \|\tilde{\mathbf{q}}\|^2 + \alpha \tilde{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\tilde{\mathbf{q}}}$$

Now, the cross term in (26) can be upper-bounded as

$$|\alpha \tilde{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\tilde{\mathbf{q}}}| \leq \alpha m_2 \|\tilde{\mathbf{q}}\| \|\dot{\tilde{\mathbf{q}}}\| \leq \frac{1}{2} \alpha m_2 (\|\tilde{\mathbf{q}}\|^2 + \|\dot{\tilde{\mathbf{q}}}\|^2) \qquad (32)$$

Hence, we have $\alpha \tilde{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\tilde{\mathbf{q}}} \geq -\frac{1}{2} \alpha m_2 (\|\tilde{\mathbf{q}}\|^2 + \|\dot{\tilde{\mathbf{q}}}\|^2)$. Therefore, we can lower-bound $V(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$ as

$$V(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) \geq \frac{1}{2} m_1 \|\dot{\tilde{\mathbf{q}}}\|^2 + \frac{1}{2} k_{p1} \|\tilde{\mathbf{q}}\|^2 - \frac{1}{2} \alpha m_2 (\|\tilde{\mathbf{q}}\|^2 + \|\dot{\tilde{\mathbf{q}}}\|^2) \geq c_1 \|\tilde{\mathbf{q}}\|^2 + c_2 \|\dot{\tilde{\mathbf{q}}}\|^2 \qquad (33)$$

$$\text{where } c_1 = \frac{1}{2}(k_{p1} - \alpha m_2) \text{ and } c_2 = \frac{1}{2}(m_1 - \alpha m_2) \qquad (34)$$

Since α satisfies (27) implying $\min\{c_1, c_2\} > 0$, we have ensured that $V(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$ in (26) is globally positive definite and radially unbounded and is zero at the equilibrium point ($\tilde{\mathbf{q}} = \mathbf{0}, \dot{\tilde{\mathbf{q}}} = \mathbf{0}$). Therefore, the scalar function $V(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$ in (26) is a Lyapunov function candidate. To show that the scalar function (26) is decresent, rewrite the integral term as

$$\int_{\mathbf{0}}^{\tilde{\mathbf{q}}} \mathbf{z}^T \mathbf{K}_p(\mathbf{z}) d\mathbf{z} = \sum_{i=1}^{n} \left(\int_0^{\tilde{q}_i} z_i k_{pii}(z_i) dz_i \right) + \frac{1}{2}\tilde{\mathbf{q}}^T \mathbf{K}_p^M \tilde{\mathbf{q}} - \frac{1}{2}\sum_{i=1}^{n} k_{pii}^M \tilde{q}_i^2 = \sum_{i=1}^{n} \left(\int_0^{\tilde{q}_i} z_i k_{pii}(z_i) dz_i - \frac{1}{2} k_{pii}^M \tilde{q}_i^2 \right) + \frac{1}{2}\tilde{\mathbf{q}}^T \mathbf{K}_p^M \tilde{\mathbf{q}} \qquad (35)$$

From (16), we have for $i = 1, \ldots, n$

$$\int_0^{\tilde{q}_i} z_i k_{pii}(z_i) dz_i \leq \int_0^{\tilde{q}_i} z_i k_{pii}^M dz_i = \frac{1}{2} k_{pii}^M \tilde{q}_i^2 \qquad (36)$$

which yields using, (36) and Property 4,

$$\int_{\mathbf{0}}^{\tilde{\mathbf{q}}} \mathbf{z}^T \mathbf{K}_p(\mathbf{z}) d\mathbf{z} \leq \frac{1}{2}\tilde{\mathbf{q}}^T \mathbf{K}_p^M \tilde{\mathbf{q}} \leq \frac{1}{2} k_{p2} \|\tilde{\mathbf{q}}\|^2. \qquad (37)$$

Therefore, using (32) with Properties 3 and 4, we can place an upper-bound on $V(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$ as

$$V(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) \leq \frac{1}{2} m_2 \|\dot{\tilde{\mathbf{q}}}\|^2 + \frac{1}{2} \alpha m_2 (\|\tilde{\mathbf{q}}\|^2 + \|\dot{\tilde{\mathbf{q}}}\|^2) + \frac{1}{2} k_{p2} \|\tilde{\mathbf{q}}\|^2 \leq c_3 \|\tilde{\mathbf{q}}\|^2 + c_4 \|\dot{\tilde{\mathbf{q}}}\|^2 \qquad (38)$$

$$\text{where } c_3 = \frac{1}{2}(\alpha m_2 + k_{p2}) \text{ and } c_4 = \frac{1}{2}(\alpha + 1) m_2 \qquad (39)$$

Thus the candidate Lyapunov function $V(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$ in (26) is a globally positive definite, radially unbounded and decresent function satisfying the inequalities:

$$\underline{c}(\|\tilde{\mathbf{q}}\|^2 + \|\dot{\tilde{\mathbf{q}}}\|^2) \leq V(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) \leq \bar{c}(\|\tilde{\mathbf{q}}\|^2 + \|\dot{\tilde{\mathbf{q}}}\|^2) \qquad (40)$$

where $\underline{c} = \min\{c_1, c_2\} > 0$ and $\bar{c} = \max\{c_3, c_4\} > 0$.

From Property 1, the time derivative of $V(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$ along the solution trajectories of (19) is given by

$$\begin{aligned} \dot{V}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) &= -\dot{\tilde{\mathbf{q}}}^T [\mathbf{K}_d(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) + \mathbf{F}_d] \dot{\tilde{\mathbf{q}}} + \alpha \dot{\tilde{\mathbf{q}}}^{\mathbf{T}} \mathbf{M}(\mathbf{q}) \dot{\tilde{\mathbf{q}}} \\ &+ \alpha \tilde{\mathbf{q}}^T \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})^T \dot{\tilde{\mathbf{q}}} - \alpha \tilde{\mathbf{q}}^T \mathbf{K}_p(\tilde{\mathbf{q}}) \tilde{\mathbf{q}} \\ &- \alpha \tilde{\mathbf{q}}^T [\mathbf{K}_d(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) + \mathbf{F}_d] \dot{\tilde{\mathbf{q}}} + (\dot{\tilde{\mathbf{q}}}^T + \alpha \tilde{\mathbf{q}}^T) \mathbf{\Delta u} \end{aligned} \qquad (41)$$

Now, we establish upper bounds on the following terms. Using Property 2, we have

$$\begin{aligned} \alpha \tilde{\mathbf{q}}^T \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})^T \dot{\tilde{\mathbf{q}}} &= \alpha \tilde{\mathbf{q}}^T \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}_d - \dot{\tilde{\mathbf{q}}})^T \dot{\tilde{\mathbf{q}}} \\ &\leq \alpha k_C \|\dot{\mathbf{q}}_d - \dot{\tilde{\mathbf{q}}}\| \|\tilde{\mathbf{q}}\| \|\dot{\tilde{\mathbf{q}}}\| \\ &\leq \alpha k_C k_{qd1} \|\tilde{\mathbf{q}}\| \|\dot{\tilde{\mathbf{q}}}\| + \alpha k_C \|\tilde{\mathbf{q}}\| \|\dot{\tilde{\mathbf{q}}}\|^2 \end{aligned} \qquad (42)$$

and since $\mathbf{M}(\mathbf{q})$, $\mathbf{K}_p(\tilde{\mathbf{q}})$ and $\mathbf{K}_d(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$ are positive definite matrices, using Properties 3-5, we obtain

$$\alpha \dot{\tilde{\mathbf{q}}}^T \mathbf{M}(\mathbf{q}) \dot{\tilde{\mathbf{q}}} \leq \alpha m_2 \|\dot{\tilde{\mathbf{q}}}\|^2 \qquad (43)$$

$$-\dot{\tilde{\mathbf{q}}}^T [\mathbf{K}_d(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) + \mathbf{F}_d] \dot{\tilde{\mathbf{q}}} \leq -(k_{d1} + k_{fd1}) \|\dot{\tilde{\mathbf{q}}}\|^2 \qquad (44)$$

$$-\alpha \tilde{\mathbf{q}}^T \mathbf{K}_p(\tilde{\mathbf{q}}) \tilde{\mathbf{q}} \leq -\alpha k_{p1} \|\tilde{\mathbf{q}}\|^2 \qquad (45)$$

$$| - \alpha \tilde{\mathbf{q}}^T [\mathbf{K}_d(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) + \mathbf{F}_d] \dot{\tilde{\mathbf{q}}}| \leq \alpha (k_{d2} + k_{fd2}) \|\tilde{\mathbf{q}}\| \|\dot{\tilde{\mathbf{q}}}\| \qquad (46)$$

From Property 2 and the bounds (5)-(7), (10)-(12), we have

$$\begin{aligned} \|[\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{C}_n(\mathbf{q}, \dot{\mathbf{q}})] \dot{\mathbf{q}}_d\| &= \|[\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}_d) - \mathbf{C}_n(\mathbf{q}, \dot{\mathbf{q}}_d)] \dot{\mathbf{q}}\| \\ &\leq \delta_C \|\dot{\mathbf{q}}_d - \dot{\tilde{\mathbf{q}}}\| \end{aligned} \qquad (47)$$

$$\begin{aligned} \|\mathbf{\Delta u}\| &\leq \delta_M k_{qd2} + \delta_C (k_{qd1} + \|\dot{\tilde{\mathbf{q}}}\|) + \delta_g \\ &+ k_{fd2} k_{qd1} + k_{fs} + k_{ud} \\ &= \eta_1 + \eta_2 \|\dot{\tilde{\mathbf{q}}}\| \end{aligned} \qquad (48)$$

where $\eta_1 = \delta_M k_{qd2} + (\delta_C + k_{fd2}) k_{qd1} + \delta_g + k_{fs} + k_{ud}$ and $\eta_2 = \delta_C$, and

$$\begin{aligned} (\dot{\tilde{\mathbf{q}}}^T + \alpha \tilde{\mathbf{q}}^T) \mathbf{\Delta u} &\leq \|\dot{\tilde{\mathbf{q}}}\| \|\mathbf{\Delta u}\| + \alpha \|\tilde{\mathbf{q}}\| \|\mathbf{\Delta u}\| \\ &\leq \alpha \eta_1 \|\tilde{\mathbf{q}}\| + \eta_1 \|\dot{\tilde{\mathbf{q}}}\| \\ &+ \alpha \eta_2 \|\tilde{\mathbf{q}}\| \|\dot{\tilde{\mathbf{q}}}\| + \eta_2 \|\dot{\tilde{\mathbf{q}}}\|^2 \end{aligned} \qquad (49)$$

It now follows from the inequalities (42)-(46) and (49) that the time derivative of $V(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}})$ becomes

$$\begin{aligned}\dot{V}(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}}) \;\leq\; & -(k_{d1}+k_{fd1}-\alpha m_2-\eta_2)\|\dot{\tilde{\mathbf{q}}}\|^2-\alpha k_{p1}\|\tilde{\mathbf{q}}\|^2\\ &+\;\alpha(k_{d2}+k_{fd2}+k_C k_{qd1}+\eta_2)\|\tilde{\mathbf{q}}\|\|\dot{\tilde{\mathbf{q}}}\|\\ &+\;\eta_1\|\dot{\tilde{\mathbf{q}}}\|+\alpha\eta_1\|\tilde{\mathbf{q}}\|+\alpha k_C\|\tilde{\mathbf{q}}\|\|\dot{\tilde{\mathbf{q}}}\|^2 \end{aligned} \quad (50)$$

which can be rewritten as

$$\begin{aligned}\dot{V}(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}}) \leq & -(k_{d1}+k_{fd1}-a)\|\dot{\tilde{\mathbf{q}}}\|^2-\alpha k_{p1}\|\tilde{\mathbf{q}}\|^2\\ &+\alpha b\|\tilde{\mathbf{q}}\|\|\dot{\tilde{\mathbf{q}}}\|+\eta_1\|\dot{\tilde{\mathbf{q}}}\|+\alpha\eta_1\|\tilde{\mathbf{q}}\|+\alpha k_C\|\tilde{\mathbf{q}}\|\|\dot{\tilde{\mathbf{q}}}\|^2\\ \leq & -(k_{d1}+k_{fd1}-a)\|\dot{\tilde{\mathbf{q}}}\|^2-\alpha k_{p1}\|\tilde{\mathbf{q}}\|^2\\ &+\frac{\alpha b}{2}\left(\frac{\|\tilde{\mathbf{q}}\|^2}{\rho}+\rho\|\dot{\tilde{\mathbf{q}}}\|^2\right)\\ &+\eta_1\|\dot{\tilde{\mathbf{q}}}\|+\alpha\eta_1\|\tilde{\mathbf{q}}\|+\alpha k_C\|\tilde{\mathbf{q}}\|\|\dot{\tilde{\mathbf{q}}}\|^2\\ \leq & -\nu_1\|\dot{\tilde{\mathbf{q}}}\|^2-\alpha\nu_2\|\tilde{\mathbf{q}}\|^2\\ &+\eta_1\|\dot{\tilde{\mathbf{q}}}\|+\alpha\eta_1\|\tilde{\mathbf{q}}\|+\alpha k_C\|\tilde{\mathbf{q}}\|\|\dot{\tilde{\mathbf{q}}}\|^2\end{aligned}$$

where $a=\alpha m_2+\eta_2$, $b=k_{d2}+k_C k_{qd1}+k_{fd2}+\eta_2$, $\nu_1=k_{d1}+k_{fd1}-a-\frac{1}{2}\alpha\rho b$, $\nu_2=k_{p1}-\frac{b}{2\rho}$, and ρ is a positive constant satisfying (28). Since the conditions in (27) and (28) imply that $\nu_1,\nu_2>0$, now by completing the squares, the following inequalities hold:

$$\eta_1\|\dot{\tilde{\mathbf{q}}}\| \;\leq\; \left(\frac{\eta_1}{\sqrt{\nu_1}}\right)^2+\left(\frac{\sqrt{\nu_1}}{2}\right)^2\|\dot{\tilde{\mathbf{q}}}\|^2 \quad (51)$$

$$\eta_1\|\tilde{\mathbf{q}}\| \;\leq\; \left(\frac{\eta_1}{\sqrt{\nu_2}}\right)^2+\left(\frac{\sqrt{\nu_2}}{2}\right)^2\|\tilde{\mathbf{q}}\|^2 \quad (52)$$

and using the above inequalities, the upper bound of $\dot{V}(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}})$ can be written as

$$\begin{aligned}\dot{V}(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}}) \;\leq\; & -\frac{3}{4}\nu_1\|\dot{\tilde{\mathbf{q}}}\|^2-\frac{3}{4}\alpha\nu_2\|\tilde{\mathbf{q}}\|^2+\alpha k_C\|\tilde{\mathbf{q}}\|\|\dot{\tilde{\mathbf{q}}}\|^2\\ &+\;\frac{\eta_1^2}{\nu_1}+\frac{\alpha\eta_1^2}{\nu_2} \\ =\; & -\gamma_1\|\tilde{\mathbf{q}}\|^2-\gamma_2\|\dot{\tilde{\mathbf{q}}}\|^2+\gamma_{21}\|\tilde{\mathbf{q}}\|\|\dot{\tilde{\mathbf{q}}}\|^2+\varepsilon\end{aligned} \quad (53)$$

where $\gamma_1=3\alpha\nu_2/4,\ \gamma_2=3\nu_1/4,$ and $\gamma_{21}=\alpha k_C$ (54)

$$\varepsilon=\frac{\eta_1^2}{\nu_1}+\frac{\alpha\eta_1^2}{\nu_2} \quad (55)$$

Define ω in (27) as $\omega=(V_0/c_1)^{1/2}>0$ where $V_0=V(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}})|_{t=0}>0$ for positive definite $V(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}})$ and $c_1>0$ is given by (34). Since α satisfies (27), we have

$$\gamma_2>\gamma_{21}\omega \quad (56)$$

and hence, using Lemma 1 for $\beta_2\in(0,\gamma_2-\gamma_{21}\omega)$ with $\|\bar{\mathbf{q}}\|>\sqrt{\varepsilon/\min\{\gamma_1,\beta_2\}}$, the following inequality holds

$$\begin{aligned}\dot{V}(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}}) &\leq -\gamma_1\|\tilde{\mathbf{q}}\|^2-\beta_2\|\dot{\tilde{\mathbf{q}}}\|^2+\varepsilon\\ &\leq -\min\{\gamma_1,\beta_2\}(\|\tilde{\mathbf{q}}\|^2+\|\dot{\tilde{\mathbf{q}}}\|^2)+\varepsilon\\ &\leq -\mu V(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}})+\varepsilon\end{aligned} \quad (57)$$

where $\mu=\min\{\gamma_1,\beta_2\}/\bar{c}$. Now using Lemma 2, we can conclude that the solution $\bar{\mathbf{q}}$ of the closed-loop system (19) is uniformly ultimately bounded and exponentially convergent to the closed ball $B(r)$ where $r=\sqrt{\varepsilon/(\mu\underline{c})}$ ($\bar{c}$ and $\underline{c}$ are given by (40)). ■

Remark 1: Since the lower-bound of α is zero, (56) can be satisfied for any initial conditions V_0 by choosing arbitrarily small α. Therefore, the domain of convergence is in fact the entire state space and the solution $\bar{\mathbf{q}}$ of the closed-loop system (19) is globally exponentially convergent to the closed ball $B(r)$ though without a uniform rate.

V. Simulation Results

Results are presented to illustrate the exponential convergence and the u.u.b. of tracking errors when using the nonlinear PD+ controller for the tracking control of a two-link rigid manipulator. Results are also presented to allow comparison, in terms of tracking error convergence rates and ultimate bounds, between the performance of the nonlinear PD+ controller and the linear PD+ controller.

The planar elbow manipulator used in the simulation is assumed to be actuated by two direct drive motors: motor 1 is attached to the ground and motor 2 is attached to link 1 and has a mass of $m_{a2}=0.5kg$. The links have lengths $l_1=l_2=1m$, masses $m_1=m_2=1kg$ concentrated at their mid-points ($l_{c1}=l_{c2}=0.5m$), and gravity is taken as 9.81 m/s^2. The manipulator carries a payload of $m_l=1kg$ located at the distal end of link 2. The moments of inertia of both links are taken to be $I_1=I_2=0.0833kg.m^2$. For the purpose of illustration, the manipulator dynamics without friction and disturbance terms are used with the only uncertainty due to payload varying in the range of $[0-1]kg$.

The desired trajectories of the joints are specified as

$$\begin{aligned}q_{d1}&=0.25\pi+0.5(1-\cos(0.5\pi t))\\ q_{d2}&=0.5\pi+0.25(1-\cos(\pi t))\end{aligned} \quad (58)$$

These trajectories were tracked using two controllers whose feedforward component is given by (9) with the nominal dynamic terms $\mathbf{M}_n(\mathbf{q})$, $\mathbf{C}_n(\mathbf{q},\dot{\mathbf{q}})$, and $\mathbf{g}_n(\mathbf{q})$ are computed using a nominal payload of $m_l=0.5kg$ which is different from the actual payload of 1kg used for this simulation. The feedback components of these controllers are $\mathbf{u}_{fb-pd}=\mathbf{K}_p\tilde{\mathbf{q}}+\mathbf{K}_d\dot{\tilde{\mathbf{q}}}$ for the linear PD controller, and

$$\mathbf{u}_{fb-npd}=\mathbf{K}_p(\tilde{\mathbf{q}})\tilde{\mathbf{q}}+\mathbf{K}_d(\tilde{\mathbf{q}},\dot{\tilde{\mathbf{q}}})\dot{\tilde{\mathbf{q}}} \quad (59)$$

for the nonlinear PD controller.

As stated in the previous section, the *only* requirements for ensuring the *stability* of the above controllers are positive definite proportional and derivative gain matrices chosen such that $k_{d1}>\delta_C$. From (58), the bound on the joint velocities is given as $\|\dot{\mathbf{q}}_d\|\leq 1.1$ rad/sec. Using the above numerical values for manipulator parameters and Property 2, the positive constant k_C calculated based on $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}_d)-\mathbf{C}_n(\mathbf{q},\dot{\mathbf{q}}_d)$ is given as $k_C=2$. Therefore, we obtain the bound δ_C using Property 2 and (11) as $\delta_C=2.2$. Now, the *performance* of the feedback controllers are dictated by the type of nonlinear gain functions and the actual parameters chosen for entries of the gain matrices.

In this section, the feedback controllers are designed using a *numerical optimization technique* [14] and the Lyapunov function based conditions are used only as stability *constraints* for the optimization problem.

Using this optimization process, the following diagonal gain matrices were obtained

$$\mathbf{K}_p = \text{diag}[197.06, 41.94] \quad \text{and} \quad \mathbf{K}_d = \text{diag}[70.11, 14.6]$$

for $\mathbf{u}_{fb-pd}$, and $\mathbf{K}_p(\tilde{\mathbf{q}}) = \text{diag}\left[\frac{194.2}{0.2+|\tilde{q}_1|}, \frac{74.3}{0.2+|\tilde{q}_2|}\right]$ and $\mathbf{K}_d(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}})$ chosen as

$$\text{diag}\left[\frac{74.4}{(0.5+|\tilde{q}_1|)(1+0.07|\dot{\tilde{q}}_1|)}, \frac{7.6}{(0.1+|\tilde{q}_2|)(1+0.07|\dot{\tilde{q}}_2|)}\right]$$

for $\mathbf{u}_{fb-npd}$. For the range of values considered for the tracking and velocity errors in the simulation, namely for $|\tilde{q}_i| \leq 1.6$ rad and $|\dot{\tilde{q}}_i| \leq 10$ rad/sec, it is readily verified that the diagonal entries are positive and $k_{d1} > \delta_C$ for the selected gain matrices thus satisfying the stability requirements. The parameter values in $\mathbf{u}_{fb-npd}$ and $\mathbf{u}_{fb-pd}$ were selected to achieve minimal tracking errors subject to maximum joint torques $u_1^{\max} = 200N.m$, $u_2^{\max} = 80N.m$, and the above stated stability constraints. The particular choice of nonlinear gain functions in (59) is motivated by the task of tracking the desired trajectories (58) quickly and accurately subject to the maximum torque and stability constraints.

The results of the simulation are shown in Fig. 1. Fig. 1(a) shows the exponential convergence of tracking errors for both controllers and demonstrates that improved convergence is obtained using the nonlinear PD+ controller while satisfying actuator limits (Fig. 1(d)). Fig. 1(b) & (c) illustrate the improved ultimate bounds obtained for the tracking errors of joint 1 & 2 using nonlinear gain PD+ controller. The improvements in the closed-system performance using nonlinear gain PD controllers are attributed to their exploitation of nonlinear gain variations (that depend on $\tilde{\mathbf{q}}$ and $\dot{\tilde{\mathbf{q}}}$) as depicted by the difference in torque profiles in Fig. 1(d).

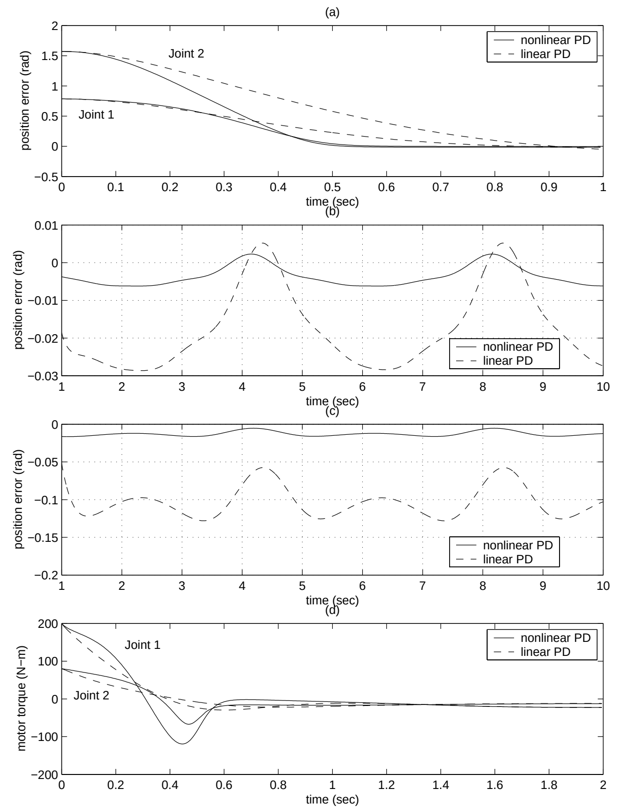

Fig. 1. Tracking performance comparison: (a) exponential convergence of position errors; (b) u.u.b. of position error for joint 1; (c) u.u.b. of position error for joint 2; (d) joint torques

VI. Conclusions

The tracking control of rigid manipulators using a general nonlinear PD+ controller with incomplete feedforward dynamics has been shown to have exponential convergence and uniform ultimate boundedness of the tracking errors and sufficient conditions have been established using a modification to the energy Lyapunov function and a lemma for addressing higher order terms in Lyapunov function derivatives. Simulation results illustrate the stability and robustness analysis and demonstrate the potential for performance improvement with this class of nonlinear PD+ controllers.

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