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Gradient-like Behavior Analysis and Synthesis of Uncertain Pendulum-like Systems

Ying Yang and Lin Huang

Abstract— **This paper focuses on uncertain pendulum-like systems subject to norm-bounded parameter uncertainty in the forward path and a vector-valued periodic nonlinearity in the feedback path, and addresses the robust gradient-like behavior analysis and synthesis problems for such systems. Sufficient conditions for robust gradient-like behavior are derived in terms of linear matrix inequalities (LMIs) and a technique for the estimation of the uncertainty bound is proposed by solving a generalized eigenvalue minimization problem. The problem of robust controller synthesis is concerned with designing a feedback controller such that the resulting closed-loop system is gradient-like for all admissible uncertainties. It is shown that a solution to the gradient-like control problem for the uncertain pendulum-like system can be obtained by solving a gradient-like control problem for an uncertainty free system. An example is presented to demonstrate the applicability and validity of the proposed approach.**

I. Introduction

In recent years, frequency-domain methods have been applied successfully for investigation of stability of stationary sets, see [1], [2], [3] and the references therein. Dynamic systems with multiple equilibria deserve investigation for theoretical development as well as practical applications. It is an essential feature of many nonlinear control systems to have multiple equilibria. Many important classes of electric and electronic systems, such as Chua's circuits [4] and systems of phase synchronization (phase-locked loops) [5] can be described by a class of dynamic systems with finite or infinite equilibria set. In reference [1], a class of pendulum-like feedback nonlinear systems with multiple equilibria was considered and frequency-domain inequalities conditions guaranteeing some global properties of solutions such as Lagrange stability, dichotomy, Bakaev stability and gradient-like behavior have been proposed. While in [6], the concept of Lagrange stability defined in [1] was extended to the case of controller synthesis and conditions of Lagrange stabilizability for pendulum-like systems have been derived based on the H_∞ sub-optimal control theory. Since pendulum-like systems always have an unbounded set of equilibrium points, they cannot be asymptotically stable. A natural analog of global asymptotical stability for such a system is the gradient-like behavior, i.e., the convergence property of all trajectories. This property corresponds to the locking-in phenomenon in nonlinear oscillations and the locking-in problem is also a fundamental problem in the control theory of oscillators [7]. In this paper, the frequency- domain inequalities condition of gradient-like behavior given by Leonov in [1] is converted into an LMI-based criterion, which enables us to take account of system uncertainties and derive feedback controllers to ensure the gradient-like behavior for the uncertain pendulum-like systems. The uncertain system under consideration will be described by a state-space model which contains parameter uncertainties in both the state and input matrices. Based on the Kalman-Yakubovich-Popov lemma connecting the frequency-domain inequality and linear matrix inequality (LMIs), sufficient conditions of robust gradient-like behavior for uncertain pendulum-like systems are given in terms of LMIs. Meanwhile the robust synthesis problem is addressed by designing a static state feedback controller and a dynamic output feedback controller such that the resulting closed-loop system is gradient-like for all admissible uncertainties respectively. It will be shown that the robust gradient-like control problem can be converted into a gradient-like control problem for an uncertainty free pendulum-like system. With this LMI approach, the largest allowable magnitude of the admissible uncertainty can also be explicitly computed by solving a generalized eigenvalue minimization problem which is essentially a convex optimization problem and numerically efficient.

In this paper, we use the following notations: $\mathbb{R}^{n\times n}$ is the set of $n\times n$ real matrices. For a matrix A, A^T denotes its transpose, A^* its complex conjugate transpose. The matrix inequality $A > B (A \geq B)$ means that A and B are square Hermitian matrices and $A-B$ is positive (semi-)definite. He is Hermit operator with $\mathrm{He}A = A + A^T$.

II. Preliminaries

Let us consider the ordinary differential equation

$$\dot{x} = f(t,x) \quad (1)$$

where $f:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous and locally Lipschitz continuous in the second argument. Suppose that every solution $x(t,t_0,x_0)$ of (1) with $t_0\geq 0$ and $x_0\in\mathbb{R}^n$ may be continued to $[t_0,+\infty)$. Let $\Gamma := \{\sum_{j=1}^m k_j d_j \mid k_j\in\mathbb{Z}, 1\leq j\leq m\}$, where $d_j\in\mathbb{R}^n$ are supposed to be linearly independent $(m\leq n)$.

Definition 2.1: We say that (1) is pendulum-like with respect to Γ if for any solution $x(t,t_0,x_0)$ of (1) we have

$$x(t,t_0,x_0+d) = x(t,t_0,x_0)+d$$

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Y. Yang is with Centre for Systems and Control, Department of Mechanics and Engineering Science, Peking University, Beijing, 100871, P.R. China yy@mech.pku.edu.cn

L. Huang is with Centre for Systems and Control, Department of Mechanics and Engineering Science, Peking University, Beijing, 100871, P.R. China hl35hj75@pku.edu.cn

for all $t \geq t_0$ and all $d \in \Gamma$.

Definition 2.2: System (1) is said to be gradient-like if every solution tends to a certain equilibrium point as t tends to $+\infty$.

In this paper we consider the pendulum-like systems of the form

$$\begin{aligned} \dot{x} &= Ax + B\varphi(z) \\ \dot{z} &= Cx + D\varphi(z) \end{aligned} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$. We suppose that (A,B) is controllable, (A,C) is observable and $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ is a vector valued function having the components $\varphi_i(z) = \varphi_i(z_i)$ with $z = (z_1, z_2, \cdots, z_m)^T$. We assume that every component $\varphi_i : \mathbb{R} \to \mathbb{R}$ is Δ_i periodic, satisfies a local Lipschitz condition and possesses a finite number of zeroes on $[0, \Delta_i)$, and for each component φ_i there exists z_{0i} such that $\varphi_i(z_{0i}) \neq 0$ and $\dot{\varphi}_i(z_{0i}) \neq 0$. Let us introduce the vector $d_i = (0, \cdots, 0, \Delta_i, 0, \cdots, 0)$ where Δ_i is the i-th component of d_i, then system (2) is pendulum-like with respect to $\Gamma = \{\sum_{j=1}^m k_j d_j, k_j \in \mathbb{Z}\}$. Assume that

$$\nu_i = \left| \int_0^{\Delta_i} \varphi_i(z)\,\mathrm{d}z \right| \Big/ \int_0^{\Delta_i} |\varphi_i(z)|\,\mathrm{d}z,\ i \in \underline{m}$$

and denotes $\nu = \mathrm{diag}\{\nu_1, \nu_2, \cdots, \nu_m\}$. The transfer function of the linear part of (2) from the input φ to the output $-\dot{z}$ is given by

$$K(s) = C(A - sI)^{-1}B - D$$

Lemma 2.1 ([1]): Suppose $K(s)$ is stable and there exist diagonal matrices $\kappa = \mathrm{diag}\{\kappa_1, \kappa_2, \cdots, \kappa_m\}$, $\delta = \mathrm{diag}\{\delta_1, \delta_2, \cdots, \delta_m\}$ and $\varepsilon = \mathrm{diag}\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m\}$ with $\delta > 0$ and $\varepsilon > 0$ satisfying the following conditions:

1° $\frac{1}{2}\mathrm{He}\left(\kappa K(i\omega)\right) - K(i\omega)\varepsilon K^*(i\omega) - \delta \geq 0$;
2° $4\varepsilon\delta > (\kappa\nu)^2$.

then system (2) is gradient-like.

Definition 2.3: System (2) is said to be gradient-like in the sense of Lemma 2.1 if the gradient-like property of (2) can be guaranteed via Lemma 2.1.

Lemma 2.2 ([8]): Given $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n}, M = M^T \in \mathbb{R}^{(n+m)\times(n+m)}$, with $\det(j\omega I - A) \neq 0$ for $\omega \in \mathbb{R}$ and (A,B) controllable, the following two statements are equivalent:

1° $\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \leq 0,\ \forall \omega \in \mathbb{R} \cup \{\infty\}$;
2° there exists a matrix $P = P^T \in \mathbb{R}^{n\times n}$ such that

$$M + \begin{bmatrix} \mathrm{He}(PA) & PB \\ B^T P & 0 \end{bmatrix} \leq 0$$

The corresponding equivalence for strict inequalities holds even if (A,B) is not controllable.

Lemma 2.3: Let $T_1 = T_1^T, T_2, T_3$ be real matrices of appropriate size, then the following statements are equivalent:

1° $T_1 + \mathrm{He}(T_2 \Delta T_3) < 0, \forall \Delta : \Delta^T \Delta \leq \lambda^2 I$;
2° There exists a positive number $\eta > 0$ such that

$$T_1 + \eta\lambda^2 T_2 T_2^T + \frac{1}{\eta} T_3^T T_3 < 0$$

3° There exists a positive number $\eta > 0$ such that

$$\begin{bmatrix} T_1 + \eta\lambda^2 T_2 T_2^T & T_3^T \\ T_3 & -\eta I \end{bmatrix} < 0$$

4° There exists a positive number $\eta > 0$ such that

$$\begin{bmatrix} T_1 + \eta\lambda^2 T_3^T T_3 & T_2 \\ T_2^T & -\eta I \end{bmatrix} < 0$$

Proof: The proof of the equivalence between 1° and 2° can be found in [9]. The equivalence between 2° and 3° or 4° follows immediately from Schur complement. ■

III. Robust analysis

In this section, we derive the robust analysis results for uncertain pendulum-like system to achieve gradient-like behavior. First, we give a theorem which establishes the connection between the frequency-domain conditions of gradient-like behavior given in Lemma 2.1 and an LMI-based criterion.

Theorem 3.1: Suppose A is Hurwitz, then system (2) is gradient-like in the sense of Lemma 2.1 if and only if there exist $P = P^T \geq 0$ and diagonal matrices κ, $\delta > 0$ and $\varepsilon > 0$ such that

$$\begin{bmatrix} C^T\varepsilon C + \mathrm{He}(PA) & C^T(\varepsilon D + \frac{1}{2}\kappa) + PB \\ (D^T\varepsilon + \frac{1}{2}\kappa)C + B^T P & \delta + D^T\varepsilon D + \frac{1}{2}\mathrm{He}(\kappa D) \end{bmatrix} \leq 0 \tag{3a}$$

$$\begin{bmatrix} 2\varepsilon & \kappa\nu \\ \nu\kappa & 2\delta \end{bmatrix} > 0 \tag{3b}$$

Proof: Let

$$M = \begin{bmatrix} C^T\varepsilon C & C^T(\varepsilon D + \frac{1}{2}\kappa) \\ (D^T\varepsilon + \frac{1}{2}\kappa)C & \delta + D^T\varepsilon D + \frac{1}{2}\mathrm{He}(\kappa D) \end{bmatrix}$$

and using Lemma 2.2, we can prove the equivalence of (3a) and condition 1° of Lemma 2.1. Note that the upper left corner of M is positive semidefinite, it follows from (3a) and Hurwitz stability of A that $P \geq 0$. (3b) is directly derived from condition 2° of Lemma 2.1. ■

Remark 3.1: The significance of this theorem is that, by using Lemma 2.2 we convert the conditions of Lemma 2.1 into an equivalent LMI requirement. From this LMI condition, it is possible to extend the results to take account of the parameter uncertainty in the linear part of the system and derive feedback control law which renders the closed-up system gradient-like by using the efficient numerical linear matrix inequalities methods.

As an immediate consequence, we have a more convenient criterion as stated in the following.

Corollary 3.1: System (2) is gradient-like in the sense of Lemma 2.1 if and only if there exist $P = P^T > 0$ and diagonal matrices κ, $\delta > 0$ and $\varepsilon > 0$ such that

$$\begin{bmatrix} C^T\varepsilon C + \mathrm{He}(PA) & C^T(\varepsilon D + \frac{1}{2}\kappa) + PB \\ (D^T\varepsilon + \frac{1}{2}\kappa)C + B^T P & \delta + D^T\varepsilon D + \frac{1}{2}\mathrm{He}(\kappa D) \end{bmatrix} < 0 \tag{4a}$$

$$\begin{bmatrix} 2\varepsilon & \kappa\nu \\ \nu\kappa & 2\delta \end{bmatrix} > 0 \tag{4b}$$

Let us now consider the class of uncertain pendulum-like systems described by state-space models of the form

$$\begin{aligned} \dot{x} &= (A+\Delta A)x + B\varphi(z) \\ \dot{z} &= Cx + D\varphi(z) \end{aligned} \tag{5}$$

where ΔA stands for the parameter uncertainties which are norm-bounded and of the form

$$\Delta A = HFE \tag{6}$$

and $H \in \mathbb{R}^{n\times i}$, $E \in \mathbb{R}^{i\times m}$ are known constant matrices and $F \in \mathbb{R}^{i\times i}$ is an unknown matrix function satisfying

$$F^T F \le \lambda^2 I \tag{7}$$

with $\lambda > 0$ a given constant. From the definition of ν in Lemma 2.1 we know $\nu \le \sigma I$, where $\sigma < 1$. In the following sections, we assume that the system is controllable and observable for all admissible uncertainties. Then we have the following result:

Theorem 3.2: There exist diagonal matrices $\varepsilon > 0, \delta > 0$ and κ such that (4) holds for the uncertain system (5) satisfying (6) if and only if there exists a scaling parameter $\eta > 0$ such that (4) holds with

$$\hat{\varepsilon} = \begin{bmatrix} \varepsilon & 0 \\ 0 & aI \end{bmatrix}, \quad \hat{\kappa} = \begin{bmatrix} \kappa & 0 \\ 0 & cI \end{bmatrix}, \quad \hat{\delta} = \begin{bmatrix} \delta & 0 \\ 0 & dI \end{bmatrix} \tag{8}$$

for the system

$$\begin{aligned} \dot{x} &= \hat{A}x + \hat{B}\tilde{\varphi}(z) \\ \dot{z} &= \hat{C}x + \hat{D}\tilde{\varphi}(z) \end{aligned} \tag{9}$$

where

$$\hat{A} = A + \frac{\lambda}{\sqrt{a+1}}HE, \quad \hat{B} = \begin{bmatrix} B & \frac{\lambda}{\sqrt{(a+1)\eta}}H \end{bmatrix}$$
$$\hat{C} = \begin{bmatrix} C \\ \sqrt{\eta}E \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} D & 0 \\ 0 & bI \end{bmatrix}$$

and a, b, c, d satisfying

$$\begin{aligned} a &> 0 \\ c^2 &> \frac{4(1+a)}{1-\sigma^2} \\ b &= -\frac{c+2}{2a} \\ d &= \frac{b(2-c)-2}{2} \end{aligned} \tag{10}$$

Proof: Inequality (4) holds for system (9) if there exists a positive definite solution $P = P^T > 0$ to the linear matrix inequality

$$\begin{bmatrix} \hat{C}^T\hat{\varepsilon}\hat{C} + \hat{A}^TP + P\hat{A} & \hat{C}^T(\hat{\varepsilon}\hat{D} + \frac{1}{2}\hat{\kappa}) + P\hat{B} \\ (\hat{D}^T\hat{\varepsilon} + \frac{1}{2}\hat{\kappa})\hat{C} + \hat{B}^TP & \hat{\delta} + \hat{D}^T\hat{\varepsilon}\hat{D} + \frac{1}{2}\mathrm{He}(\hat{\kappa}\hat{D}) \end{bmatrix} < 0 \tag{11}$$

with

$$\begin{bmatrix} 2\hat{\varepsilon} & \hat{\kappa}\tilde{\nu} \\ \tilde{\nu}\hat{\kappa} & 2\hat{\delta} \end{bmatrix} > 0 \tag{12}$$

Substituting $\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{\varepsilon}, \hat{\delta}, \hat{\kappa}$ into (11) leads to

$$\begin{bmatrix} C^T\varepsilon C & C^T(\varepsilon D + \frac{1}{2}\kappa) & (ab+\frac{c}{2})\sqrt{\eta}E^T \\ (D^T\varepsilon + \frac{1}{2}\kappa)C & \delta + D^T\varepsilon D + \frac{1}{2}\mathrm{He}(\kappa D) & 0 \\ (ab+\frac{c}{2})\sqrt{\eta}E & 0 & (d+ab^2+bc)I \end{bmatrix} +$$
$$\begin{bmatrix} a\eta E^TE + \mathrm{He}P\left(A + \frac{\lambda}{\sqrt{a+1}}HE\right) & PB & \frac{\lambda}{\sqrt{(a+1)\eta}}PH \\ B^TP & 0 & 0 \\ \frac{\lambda}{\sqrt{(a+1)\eta}}H^TP & 0 & 0 \end{bmatrix} < 0 \tag{13}$$

From (10) we have $ab + \frac{c}{2} = -1, d + ab^2 + bc = -1$, then (13) becomes

$$\begin{bmatrix} C^T\varepsilon C & C^T(\varepsilon D + \frac{1}{2}\kappa) & -\sqrt{\eta}E^T \\ (D^T\varepsilon + \frac{1}{2}\kappa)C & \delta + D^T\varepsilon D + \frac{1}{2}\mathrm{He}(\kappa D) & 0 \\ -\sqrt{\eta}E & 0 & -I \end{bmatrix} +$$
$$\begin{bmatrix} a\eta E^TE + \mathrm{He}P\left(A + \frac{\lambda}{\sqrt{a+1}}HE\right) & PB & \frac{\lambda}{\sqrt{(a+1)\eta}}PH \\ B^TP & 0 & 0 \\ \frac{\lambda}{\sqrt{(a+1)\eta}}H^TP & 0 & 0 \end{bmatrix} < 0$$

Using Schur Complement, the above inequality is equivalent to

$$\begin{bmatrix} C^T\varepsilon C & C^T(\varepsilon D + \frac{1}{2}\kappa) \\ (D^T\varepsilon + \frac{1}{2}\kappa)C & \delta + D^T\varepsilon D + \frac{1}{2}\mathrm{He}(\kappa D) \end{bmatrix} +$$
$$\begin{bmatrix} \mathrm{He}(PA) + \frac{\lambda^2}{(a+1)\eta}PHH^TP + (a+1)\eta E^TE & PB \\ B^TP & 0 \end{bmatrix} < 0 \tag{14}$$

By Lemma 2.3, (14) holds if and only if for any F satisfying (7)

$$\begin{bmatrix} C^T\varepsilon C + \mathrm{He}(A+HFE)^TP & C^T(\varepsilon D + \frac{1}{2}\kappa) + PB \\ (D^T\varepsilon + \frac{1}{2}\kappa)C + B^TP & \delta + D^T\varepsilon D + \frac{1}{2}\mathrm{He}(\kappa D) \end{bmatrix} < 0$$

By noting that (10) implies $4ad > \sigma^2c^2$, we can verify by straightforward manipulations that (12) is equivalent to (4b). Thus completes the proof. ■

Corollary 3.2: The uncertain system (5) is gradient-like in the sense of Lemma 2.1 if and if there exists a scaling parameter $\eta > 0$ such that the condition in Corollary 3.1 is satisfied for uncertainty free system (9) with the diagonal matrices of the form (8).

Remark 3.2: The above corollary show that the gradient-like behavior of the uncertain pendulum-like system (5) with the parameter uncertainty form (6) can be discussed by that of an uncertainty free pendulum-like system. This result will play a crucial role in solving the robust synthesis problem in this paper.

Next we consider the uncertain pendulum-like system described by

$$\begin{cases} \dot{x} = (A+\Delta A)x + (B+\Delta B)\varphi(z) \\ \dot{z} = Cx + D\varphi(z) \end{cases} \tag{15}$$

where ΔA and ΔB have the form of

$$\begin{bmatrix} \Delta A & \Delta B \end{bmatrix} = HF\begin{bmatrix} E_1 & E_2 \end{bmatrix} \tag{16}$$

and $H \in \mathbb{R}^{n\times i}$, $E_1 \in \mathbb{R}^{j\times n}$, $E_2 \in \mathbb{R}^{j\times m}$ are known constant matrices and $F \in \mathbb{R}^{i\times j}$ is an unknown matrix function satisfying $F^T F \le \lambda^2 I$ with $\lambda > 0$ a given constant. Then we have the following result:

Theorem 3.3: Suppose there exist $P = P^T > 0$, diagonal matrices $\kappa, \delta > 0, \varepsilon > 0$ and a positive number $\eta > 0$ such that the following linear matrix inequalities hold

$$\begin{bmatrix} C^T\varepsilon C & C^T(\varepsilon D+\frac{1}{2}\kappa) & 0 & \eta\lambda E_1^T \\ (D^T\varepsilon+\frac{1}{2}\kappa)C & \delta+D^T\varepsilon D+\frac{1}{2}\mathrm{He}(\kappa D) & 0 & \eta\lambda E_2^T \\ 0 & 0 & -\eta I & 0 \\ \eta\lambda E_1 & \eta\lambda E_2 & 0 & -\eta I \end{bmatrix} + \begin{bmatrix} \mathrm{He}(PA) & PB & PH & 0 \\ B^T P & 0 & 0 & 0 \\ H^T P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} < 0 \tag{17a}$$

$$\begin{bmatrix} 2\varepsilon & \kappa\nu \\ \nu\kappa & 2\delta \end{bmatrix} > 0 \tag{17b}$$

then system (15) is gradient-like.

Proof: By Corollary 3.1, the uncertain system (15) is gradient-like if there exist $P > 0$ and diagonal matrices $\kappa, \varepsilon > 0, \delta > 0$ such that

$$\begin{bmatrix} C^T\varepsilon C & C^T(\varepsilon D+\frac{1}{2}\kappa) \\ (D^T\varepsilon+\frac{1}{2}\kappa)C & \delta+D^T\varepsilon D+\frac{1}{2}\mathrm{He}(\kappa D) \end{bmatrix} + \begin{bmatrix} \mathrm{He}\,P(A+HFE_1) & P(B+HFE_2) \\ (B+HFE_2)^T P & 0 \end{bmatrix} < 0 \tag{18}$$

and (17b) hold. Denote

$$M = \begin{bmatrix} C^T\varepsilon C+\mathrm{He}(PA) & PB+C^T(\varepsilon D+\frac{1}{2}\kappa) \\ B^T P+(D^T\varepsilon+\frac{1}{2}\kappa)C & \delta+D^T\varepsilon D+\frac{1}{2}\mathrm{He}(\kappa D) \end{bmatrix}$$

then (18) can be written in the form of

$$M + \mathrm{He}\left(\begin{bmatrix} PH \\ 0 \end{bmatrix} F \begin{bmatrix} E_1 & E_2 \end{bmatrix}\right) < 0$$

According to Lemma 2.3, the above inequality holds if and only if there exists a positive number $\eta > 0$ such that

$$M + \eta\lambda^2 \begin{bmatrix} E_1^T \\ E_2^T \end{bmatrix} \begin{bmatrix} E_1 & E_2 \end{bmatrix} + \frac{1}{\eta} \begin{bmatrix} PH \\ 0 \end{bmatrix} \begin{bmatrix} H^T P & 0 \end{bmatrix} < 0$$

it can be easily proved that the above inequality is equivalent to (17a). Thus the uncertain system (15) is gradient-like. ■

Remark 3.3: The above result show that assessing gradient-like behavior of the uncertain pendulum-like system (15) satisfying (16) can be carried out by solving two LMIs which is essentially a convex optimization problem and numerically efficient.

From the above results, we can also derive the following gradient-like conditions based on the determination of the largest allowable magnitude of the admissible uncertainty which will not destabilize the system. The significance of this result is that it provides a basis to evaluate the quality of the design and presents an efficient way to access the robustness of a feedback system in engineering practice.

Corollary 3.3: The uncertain pendulum-like system (15) with respect to (16) for $F^T F \le -\zeta I$ is gradient-like, where ζ is the global minimum of the following generalized eigenvalue minimization problem with respect to $P = P^T > 0$ and diagonal matrices $\varepsilon > 0$, $\delta > 0$, κ and a positive number $\eta > 0$:

$$\begin{aligned} &\min \zeta \\ &s.t. \begin{bmatrix} T_1 - \zeta\eta T_3^T T_3 & T_2 \\ T_2^T & -\eta I \end{bmatrix} < 0 \\ &\begin{bmatrix} 2\varepsilon & \kappa\nu \\ \nu\kappa & 2\delta \end{bmatrix} > 0 \\ &T_1 = \begin{bmatrix} C^T\varepsilon C+\mathrm{He}(PA) & PB+C^T(\varepsilon D+\frac{1}{2}\kappa) \\ B^T P+(D^T\varepsilon+\frac{1}{2}\kappa)C & \delta+D^T\varepsilon D+\frac{1}{2}\mathrm{He}(\kappa D) \end{bmatrix} \\ &T_2 = \begin{bmatrix} PH \\ 0 \end{bmatrix}, T_3 = \begin{bmatrix} E_1 & E_2 \end{bmatrix} \end{aligned} \tag{19}$$

IV. Robust synthesis

In this section, we consider the robust synthesis problem for pendulum-like systems with parameter uncertainty in the linear part. It is concerned with designing a feedback controller such that the resulting closed-loop system is gradient-like for all admissible uncertainties. Let us first consider the following uncertain pendulum-like system

$$\begin{aligned} \dot{x} &= (A+\Delta A)x + B_1\xi + (B_2+\Delta B)u \\ \dot{z} &= Cx + D_{11}\xi + D_{12}u \\ y &= x \\ \xi &= \varphi(z) \end{aligned} \tag{20}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the control input, $\xi \in \mathbb{R}^m$ is the nonlinear input, $\dot{z} \in \mathbb{R}^m$ is the controlled output and $\varphi(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is a vector-valued nonlinear mapping. ΔA and ΔB represent the parameter uncertainty which belongs to certain bounded compact set. First we will show that if there exists a dynamic output feedback controller such that system (20) is gradient-like in the sense of Lemma 2.1, there also exists a static state feedback controller to realize the same purpose.

Theorem 4.1: For uncertain pendulum-like system (20), if there exists a dynamic output feedback controller $K(s)$ such that the resulting closed-loop system is gradient-like in the sense of Lemma 2.1 for all admissible uncertainties, there must exists a static state feedback controller that achieves the same result as well. Furthermore, if the dynamic output feedback controller is

$$\begin{aligned} \dot{x}_k &= A_k x_k + B_k x \\ u &= C_k x_k + D_k x \end{aligned} \tag{21}$$

and P is the positive definite solution to (4) corresponding to the closed-loop system of (20) with (21) , denote

$$Q = P^{-1} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}$$

then the state feedback controller is

$$K = D_k + C_k Q_{12}^T Q_{11}^{-1}$$

Proof: Denote $\hat{A} = A + \Delta A$ and $\hat{B} = B + \Delta B$. The closed-loop system of (20) with (21) is

$$\begin{aligned} \dot{x}_c &= \widetilde{A}x_c + \widetilde{B}\xi \\ \dot{z} &= \widetilde{C}x_c + D_{11}\xi \end{aligned} \tag{22}$$

where $x_c = [x^T \quad x_k^T]^T$ and

$$\widetilde{A} = \begin{bmatrix} \hat{A} + \hat{B}D_k & \hat{B}C_k \\ B_k & A_k \end{bmatrix}, \widetilde{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$
$$\widetilde{C} = \begin{bmatrix} C + D_{12}D_k & D_{12}C_k \end{bmatrix}$$

According to Corollary 3.1, system (22) is gradient-like if exist a positive definite matrix $P > 0$ and diagonal matrices $\kappa, \delta > 0, \varepsilon > 0$ such that

$$\begin{bmatrix} \widetilde{C}^T \varepsilon \widetilde{C} + \mathrm{He}(P\widetilde{A}) & P\widetilde{B} + \widetilde{C}^T(\varepsilon D_{11} + \frac{1}{2}\kappa) \\ \widetilde{B}^T P + (D_{11}^T \varepsilon + \frac{1}{2}\kappa)\widetilde{C} & \delta + D_{11}^T \varepsilon D_{11} + \frac{1}{2}\mathrm{He}(\kappa D_{11}) \end{bmatrix} < 0 \tag{23a}$$

$$\begin{bmatrix} 2\varepsilon & \kappa\nu \\ \nu\kappa & 2\delta \end{bmatrix} > 0 \tag{23b}$$

Multiplying $\mathrm{diag}(P^{-1}, I)$ on the left and right of (23a) and substituting $Q, \widetilde{A}, \widetilde{B}, \widetilde{C}$ we have

$$\begin{bmatrix} \overline{C}^T \varepsilon \overline{C} + \mathrm{He}[(\hat{A} + \hat{B}D_k)Q_{11} + \hat{B}C_k Q_{12}] & B_1 + \overline{C}^T(D_{11}^T \varepsilon + \frac{1}{2}\kappa)^T \\ B_1^T + (D_{11}^T \varepsilon + \frac{1}{2}\kappa)\overline{C} & \delta + D_{11}^T \varepsilon D_{11} + \frac{1}{2}\mathrm{He}(\kappa D_{11}) \end{bmatrix} < 0$$

where

$$\overline{C} = (C + D_{12}D_k)Q_{11} + D_{12}C_k Q_{12}^T$$

Multiplying $\mathrm{diag}(Q_{11}^{-1}, I)$ on the left and right of the above inequality and denoting $Y = Q_{11}^{-1}$ we have

$$\begin{bmatrix} (C + D_{12}K)^T \varepsilon (C + D_{12}K) & (C + D_{12}K)^T(\varepsilon D_{11} + \frac{1}{2}\kappa) \\ (D_{11}^T \varepsilon + \frac{1}{2}\kappa)(C + D_{12}K) & \delta + D_{11}^T \varepsilon D_{11} + \frac{1}{2}\mathrm{He}(\kappa D_{11}) \end{bmatrix} + \begin{bmatrix} (\hat{A} + \hat{B}K)^T Y + Y(\hat{A} + \hat{B}K) & YB_1 \\ B_1^T Y & 0 \end{bmatrix} < 0$$

Note that the above inequality with (23b) guarantees the gradient-like behavior of the closed-loop system corresponding to the system (20) with the state feedback $u = Kx$

$$\begin{aligned} \dot{x} &= (\hat{A} + \hat{B}K)x + B_1\xi \\ \dot{z} &= (C + D_{12}K)x + D_{11}\xi \end{aligned}$$

Thus completes the theorem. ■

In view of Theorem 4.1, we develop the following conditions for the existence of a static state feedback K such that system (20) satisfying (16) is gradient-like.

Theorem 4.2: Consider the uncertain system (20) satisfying (16), then there exists a state feedback controller $u = Kx$ such that the resulting closed-loop system is gradient-like in the sense of Lemma 2.1 for all admissible uncertainties if and only if for some $\eta > 0$ this controller achieves the same property for the scaled system

$$\begin{aligned} \dot{x} &= \widetilde{A}x + \widetilde{B}_1\xi + \widetilde{B_2}u \\ \dot{z} &= \widetilde{C}x + \widetilde{D}_{11}\xi + \widetilde{D}_{12}u \\ y &= x \\ \xi &= \varphi(z) \end{aligned} \tag{24}$$

where

$$\begin{aligned} &\widetilde{A} = A + \frac{\lambda}{\sqrt{a+1}}HE_1, \ \widetilde{B}_1 = \begin{bmatrix} B_1 & \frac{\lambda}{\sqrt{(a+1)\eta}}H \end{bmatrix}, \\ &\widetilde{B}_2 = B_2 + \frac{\lambda}{\sqrt{(a+1)\eta}}HE_2, \ \widetilde{C} = \begin{bmatrix} C \\ \sqrt{\eta}E_1 \end{bmatrix}, \\ &\widetilde{D}_{11} = \begin{bmatrix} D_{11} & 0 \\ 0 & bI \end{bmatrix}, \ \widetilde{D}_{12} = \begin{bmatrix} D_{12} \\ \sqrt{\eta}E_2 \end{bmatrix} \end{aligned} \tag{25}$$

with the diagonal matrices such that (4) holds for the closed-loop system having the forms of

$$\hat{\varepsilon} = \begin{bmatrix} \varepsilon & 0 \\ 0 & aI \end{bmatrix}, \quad \hat{\kappa} = \begin{bmatrix} \kappa & 0 \\ 0 & cI \end{bmatrix}, \quad \hat{\delta} = \begin{bmatrix} \delta & 0 \\ 0 & dI \end{bmatrix}$$

where a, b, c, d satisfying (10).

In the following, we will consider the output feedback controller synthesizing to achieve gradient-like behavior for pendulum-like systems with parameter uncertainty. Let us consider the following uncertain system

$$\begin{aligned} \dot{x} &= (A + \Delta A)x + B_1\xi + (B_2 + \Delta B)u \\ \dot{z} &= C_1 x + D_{11}\xi + D_{12}u \\ y &= C_2 x + D_{21}\xi \end{aligned} \tag{26}$$

where $\Delta A, \Delta B$ satisfying (16).

Theorem 4.3: There exists a linear dynamic output feedback controller $K(s)$ such that the system (26) is gradient-like in the sense of Lemma 2.1 for all admissible uncertainties if and only if for some $\eta > 0$ this controller achieve the same property for the following scaled system

$$\begin{aligned} \dot{x} &= \widetilde{A}x + \widetilde{B}_1\xi + \widetilde{B}_2u \\ \dot{z} &= \widetilde{C}_1 x + \widetilde{D}_{11}\xi + \widetilde{D}_{12}u \\ y &= C_2 x + [D_{21} \quad 0]\xi \end{aligned} \tag{27}$$

where matrices $\widetilde{A}, \widetilde{B}_1, \widetilde{B}_2, \widetilde{C}_1, \widetilde{D}_{11}, \widetilde{D}_{12}$ are given in (25) and the diagonal matrices such that (4) holds for the closed-loop system have the form of

$$\hat{\varepsilon} = \begin{bmatrix} \varepsilon & 0 \\ 0 & aI \end{bmatrix}, \quad \hat{\kappa} = \begin{bmatrix} \kappa & 0 \\ 0 & cI \end{bmatrix}, \quad \hat{\delta} = \begin{bmatrix} \delta & 0 \\ 0 & dI \end{bmatrix}$$

where a, b, c, d satisfying (10).

V. Numerical Example

Consider an uncertain pendulum-like system with

$$A = \begin{bmatrix} -63 & -20 \\ 32 & 0 \end{bmatrix}, B = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 2 & -2 \end{bmatrix}, D = -0.5$$

The uncertainties are given as

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with $F^T F \leq I$ and the nonlinear feedback $\xi = \sin z$. Solving the linear matrix inequalities in Theorem 3.3 by using LMI Toolbox [10], we get

$$P = \begin{bmatrix} 944.4 & 531.4 \\ 531.4 & 1946.6 \end{bmatrix}, \varepsilon = 1165.8, \ \delta = 101.7725,$$
$$\kappa = 7100.3, \ \eta = 301.5821$$

Thus the system is gradient-like for all admissible uncertainty. Solve the generalized eigenvalue problem corresponding to (19), we get the largest allowable uncertainty bound $|\zeta| = 1.5666$. From Corollary 3.3, this result guarantees that the uncertain pendulum-like system will be robustly gradient-like for $\forall F, F^T F \leq 1.5666I$. This estimation can be verified by Figure 1 where the numerical experiment results of the system with twenty randomly generated initial value $\mathbf{x}_0, \phi_0$ and $F(\|F\| = 1.2516)$ are given. The result presented in Figure 1 where all of x_i converge to 0 and ϕ converges to $2k\pi(k = -1, 0, 1)$ shows that the systems perturbed by those F are all gradient-like. This observation coincides with Theorem 3.3 and Corollary 3.3 and confirms the robust gradient-like behavior of the system.

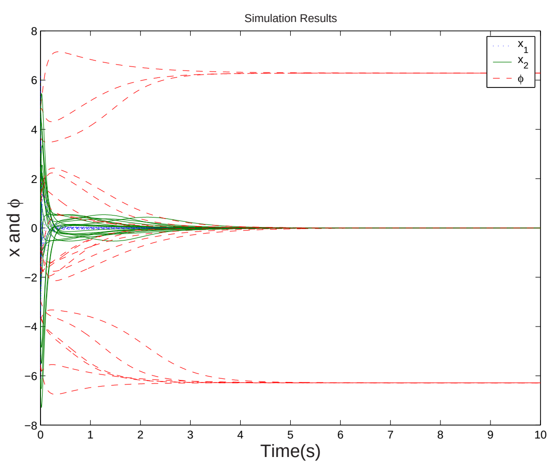

Fig. 1. Simulation for $\|F\| = 1.2516$

VI. Conclusion

In this paper, the LMI conditions of robust gradient-like behavior and the controller existence conditions guaranteeing the gradient-like behavior for a class of uncertain nonlinear systems with multiple equilibria have been first summarized. Using the Kalman-Yakubovich-Popov lemma in terms of linear matrix inequality as the analytical framework, early work performed by Leonov for nominal pendulum-like systems is extended to take account of system uncertainties and derive feedback control law which renders the closed-up system gradient-like. Under this LMI-based framework, other global properties of systems with multiple equilibria can be investigated as well as synthesizing corresponding feedback controller to ensure those global properties. These will be the subjects of further study.

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