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Disturbance Rejection of Switched Systems

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***Abstract*— In this paper, we investigate the disturbance rejection of switched systems with unit-amplitude disturbance input by designing a state-dependent switching law. All the results in this paper are expressed in terms of LMIs, which can be easily tested with efficient LMI algorithm.**

I. INTRODUCTION

Switched systems are a class of hybrid systems consisting of several subsystems and a switching law that specifies which subsystem will be activated along the system trajectory at each instant of time. Switched systems deserve investigation for theoretical development as well as for practical applications. It is an essential feature of many control systems to switch among different system structures, for example, power systems and power electronics [1], transmission and stepper motors [2], constrained robotics [3], automated highway systems [4]. Switched systems also arise from the application of multiple controller, which have been widely used in adaptive control [5], where a high-level, logic-based supervisor provides switching between a family of candidate controllers so as to achieve desired performance for the closed-loop systems.

In the last decade, switched systems have received growing attention [6]-[15] and have been studied from various viewpoints. One interesting viewpoint is that the switching signal is an exogenous variables, and then the problem is to investigate whether there exists a switching signal such that switched systems have desired performance such as stability, certain disturbance attenuation and so on. [11] provided a survey of recent development in stability and design of switched systems.

In this paper, we consider the disturbance rejection of the following linear switched system:

$$\dot{x}(t) = A_{r(t)}x(t) + B_{r(t)}\omega(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\omega(t) \in \mathbb{R}^m$ is the exogenous disturbance input, $r(t) : [0, \infty) \to \mathcal{I} := \{1, 2, \cdots, L\}$ is the switching path to be designed. Furthermore, $r(t) = i$ means that the i-th subsystem (A_i, B_i) is chosen as the system realization at time t and $L > 1$ is the number of subsystems.

We first review some notions and results concerning the disturbance rejection problem of linear system without switchings:

$$\dot{x}(t) = Ax(t) + B\omega(t). \tag{2}$$

The reachable set of system (2) is defined as the set of all states reachable from the origin in finite time by unit-amplitude disturbance input, i.e., $\|\omega(t)\|_\infty \le 1$ [16]. A set $\mathcal{F}$ is said to be inescapable if (I) $0 \in \mathcal{F}$; (II) $x(0) \in \mathcal{F}$ and $\|\omega(t)\|_\infty \le 1$ implies that $x(t) \in \mathcal{F}$ for all the future time $t > 0$ [16]. Based on these basic definitions, [16] established a necessary and sufficient condition to determine whether an ellipsoid $\varepsilon = \{x|\ x^T Px \le 1, P \ge 0\}$ is an inescapable set of system (2) with $\|\omega(t)\|_\infty \le 1$. [18] studied the input/output properties of system (2) with $\int_0^\nu \omega^T(t)\omega(t)dt \le 1$, such as L_2 and RMS (root-mean square) gains, dissipativity and so on by linear matrix inequality approach. [17] investigated some problems of persistent bounded disturbance rejection for linear uncertain systems, Lur'e systems and nonlinear systems.

Although there have been many existing results on the disturbance rejection of systems without switchings, to the best of our knowledge, there're few results concerning the disturbance rejection of switched systems. In this paper, we focus on designing a switching signal such that a given ellipsoid $\varepsilon := \{x|x^T Px \le 1\}$ is an inescapable set of switched system (1) with unit-amplitude disturbance input.

The contribution of this paper is that, under the assumption that ε isn't an inescapable set of any subsystem of switched system (1) (otherwise the switching problem will be trivial by always choosing the subsystem that has ε as its inescapable set), we derive a criterion under which ε is an inescapable set of switched system (1) by designing a switching law. The criterion is necessary and sufficient for the case that $L = 2$. Furthermore, based on an important lemma, these results can be extended to switched systems with norm-bounded uncertainties. All the results in this paper are expressed in terms of LMIs, which can be easily tested with efficient LMI algorithm software [18]-[19].

This paper is organized as follows. Section 2 introduces some definitions and lemmas as the preliminaries of the paper. Some criteria to determine whether an ellipsoid is an inescapable set of a switched system are obtained in Section 3. An extension to uncertain switched systems is presented in Section 4. Two examples are given in Section 5 to motivate and exemplify our results. We give the conclusion in Section 6.

Notations: We use standard notations throughout this paper. M^T is the transpose of the matrix M. $M > 0$ ($M < 0$) means that M is positive definite (negative definite). $\mathbb{R}^n$ is the n-dimensional Euclidean space. $\mathcal{S}$ denotes the set of all the switching signals. $\mathbb{R}^{m\times n}$ is the set of all real $m \times n$

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matrices.

II. PRELIMINARIES

In this section, we present some definitions, lemmas as the starting point of our research.

In the sequel, switched system (1) is assumed to have unit-amplitude disturbance input, i.e., $\omega(t) \in \Omega$, where $\Omega := \{\omega(t) |\ \omega(t) \in \mathbb{R}^m, \|\omega(t)\|_\infty \leq 1\}$.

Definition 1. *A set $\mathcal{F}$ is said to be an inescapable set of switched system (1) if*

(I) $0 \in \mathcal{F}$;

(II) There exists a switching signal $r^*(t) \in \mathcal{S}$ such that $x(0) \in \mathcal{F}$, $\omega(t) \in \Omega$ implies that $x(t) \in \mathcal{F}$ for all $t \geq 0$.

Remark 1. *Definition 1 can be regarded as an extension of the notion of inescapable set for system (2) in [16].*

In the sufficiency proof of Theorem 2.1 in [16], it states "This means that there exists a $\omega \in BL_\infty$ and a time T such that starting from $x(0) = 0$, the state is driven to $x(T) = x_f$." Here, $x(0) = 0$ is a mistake, which contradicts $x(0) \in \varepsilon$ in the definition of inescapable set. Furthermore, to prove Theorem 2.1 in [16] correctly, we need to modify the Lemma 4.1 in [16] as follows:

Lemma 1. *Let $f : [0, \nu] \to \mathbb{R}$ be differentiable, with $f(0) \leq 1, f(\nu) > 1$, then there exists $\bar{t} \in (0, \nu)$ such that $f(\bar{t}) > 1$ and $f'(\bar{t}) > 0$.*

Proof: The proof is similar to Lemma 4.1 in [16].

Based on the above discussions and Lemma 1, we can prove Theorem 2.1 in [16]. Here, we omit the proof.

Lemma 2. *Given $T_1 = T_1^T, T_2 = T_2^T \in \mathbb{R}^{n\times n}$, the following two conditions are equivalent:*

(i) $\forall\ x \in \Pi \subseteq \mathbb{R}^n$, either $x^T T_1 x \leq 0$ or $x^T T_2 x \leq 0$.

(ii) There exist $\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 > 0$ such that $x^T(\sum_{i=1}^2 \alpha_i T_i)x \leq 0, \forall\ x \in \Pi$.

Proof: The proof is similar to Lemma 2.1 in [10].

lemma 3[20] Given matrices $\boldsymbol{G}, \boldsymbol{M}, \boldsymbol{N}$ of compatible dimensions, with $\boldsymbol{G}$ symmetric and $\boldsymbol{M} \neq 0, \boldsymbol{N} \neq 0$, then

$$\boldsymbol{G} + \boldsymbol{M}\boldsymbol{\Delta}\boldsymbol{N} + \boldsymbol{N}^T\boldsymbol{\Delta}^T\boldsymbol{M}^T \leq 0$$

holds for all $\boldsymbol{\Delta}$ satisfying $\boldsymbol{\Delta}^T\boldsymbol{\Delta} \leq \boldsymbol{I}$, if and only if there exists a constant $\varphi > 0$ such that

$$\boldsymbol{G} + \varphi\boldsymbol{M}\boldsymbol{M}^T + \frac{1}{\varphi}\boldsymbol{N}^T\boldsymbol{N} \leq 0.$$

III. DISTURBANCE REJECTION OF LINEAR SWITCHED SYSTEMS

In this section, let $P \geq 0$, an ellipsoid $\varepsilon := \{x|\ x^T P x \leq 1\}$. We present some criteria to determine whether ε is an inescapable set of switched system (1) with unit-amplitude disturbance input.

If ε is an inescapable set of a subsystem of switched system (1), then we can always activate this subsystem so that ε is an inescapable set of switched system (1). Therefore, to make the switching problem non-trivial, we make the following assumption:

Assumption 1: ε is not an inescapable set of any subsystem of switched system (1). By Theorem 2.1 in [16], this assumption holds if and only if there doesn't exist a real number $\alpha_i \geq 0$ such that

$$\begin{bmatrix} A_i^T P + PA_i + \alpha_i P & PB_i \\ B_i^T P & -\alpha_i I \end{bmatrix} \leq 0, \forall\ i \in \mathcal{I}. \quad (3)$$

Proposition 1. *The following two conditions are equivalent:*

(a) ε is an inescapable set of system (1);

(b) There exists a switching signal $r^*(t)$ such that under this switching signal, the derivative of $V(x(t)) = x^T(t)Px(t)$ along the solution of (1) satisfies $\frac{d}{dt}[x^T(t)Px(t)] \leq 0$ for all $\omega(t) \in \Omega, x^T(t)Px(t) \geq 1$.

Proof: Suppose that (b) holds, but ε is not an inescapable set of system (1). By Definition 1, there doesn't exist a switching signal $r(t)$ such that $x(0) \in \varepsilon, \omega(t) \in \Omega$ implies $x(t) \in \varepsilon$ for $t \geq 0$, i.e., for arbitrary switching signal $r_0(t) \in \mathcal{S}$, there exist $x(0) = x_0 \in \varepsilon, \omega_0(t) \in \Omega$, $T_0 > 0$ such that $x_f^T P x_f > 1$, where $x_f = x(T_0)$. Let $f(t) := x^T(t)Px(t)$. Obviously, $f(t)$ is differentiable. Moreover, $f(0) = x^T(0)Px(0) \in \varepsilon, f(T_0) > 1$. By Lemma 1, there exists $\bar{t} \in (0, T_0)$ such that $f(\bar{t}) > 1, f'(\bar{t}) > 0$, which contradicts (b). Hence, (a) holds.

Suppose that (a) holds, but (b) doesn't hold. Then for arbitrary switching signal, there exist $\omega_0(t) \in \Omega$ and $t_0 \geq 0$ such that $x^T(t_0)Px(t_0) \geq 1$, but $\frac{d}{dt}[x^T(t)Px(t)]|_{t=t_0} > 0$. Then by the definition of derivative, for sufficiently small $\upsilon > 0$, $x^T(t_0 + \upsilon)Px(t_0 + \upsilon) > 1$, which contradicts (a). Hence (b) holds.

For system (2), we have the following necessary and sufficient condition which can be regarded as a special case of switched system (1) by assuming that $L = 1$.

Remark 2. *This proposition presents necessary and sufficient condition to determine whether ε is an inescapable set of a switched system, which will play an important role in our later discussions. But It doesn't provide a concrete* *method to design a switching law. We will design the convex-based switching law.*

Corollary 1. *ε is an inescapable set of system (2) if and only if the derivative of $V(x(t)) = x^T(t)Px(t)$ along the solution of system (2) satisfies $\frac{d}{dt}[x^T(t)Px(t)] \leq 0$ for all $\omega(t) \in \Omega, x^T(t)Px(t) \geq 1$.*

Theorem 1. *$\varepsilon = \{x|\ x^TPx \leq 1\}$ is an inescapable set of switched system (1), if there exist $\tau_i \geq 0$ satisfying $\sum_{i=1}^L \tau_i > 0$, and $\alpha \geq 0$ such that*

$$\left[\begin{array}{cc} (\sum_{i=1}^L \tau_i A_i)^T P + P(\sum_{i=1}^L \tau_i A_i) + \alpha P & P(\sum_{i=1}^L \tau_i B_i) \\ (\sum_{i=1}^L \tau_i B_i)^T P & -\alpha I \end{array}\right] \leq 0. \quad (4)$$

Proof: Suppose that there exist $\tau_i \geq 0$ satisfying $\sum_{i=1}^L \tau_i > 0$, and $\alpha \geq 0$ such that (4) holds, then for any $x(t) \in \mathbb{R}^n$ and $\omega(t) \in \mathbb{R}^m$, we have

$$\begin{aligned} & x^T(t)[(\sum_{i=1}^L \tau_i A_i)^T P + P(\sum_{i=1}^L \tau_i A_i)]x(t) \\ + \quad & x^T(t)P(\sum_{i=1}^L \tau_i B_i)\omega(t) \\ + \quad & w^T(t)(\sum_{i=1}^L \tau_i B_i)^T Px(t) \\ - \quad & \alpha[w^T(t)w(t) - x^T(t)Px(t)] \leq 0, \end{aligned}$$

i.e.,

$$\begin{aligned} & \sum_{i=1}^L \tau_i \{x^T(t)(A_i^T P + PA_i)x(t) + x^T(t)PB_i\omega(t) \\ + \quad & \omega^T(t)B_i^T Px(t) - \alpha_i[w^T(t)w(t) - x^T(t)Px(t)]\} \\ \leq \quad & 0, \end{aligned}$$

where

$$\alpha_1 = \alpha_2 = \cdots = \alpha_L = \frac{\alpha}{\sum_{i=1}^L \tau_i}.$$

Since $\tau_i \geq 0, \sum_{i=1}^L \tau_i > 0$, we get

$$\begin{aligned} & \sum_{i=1}^L \tau_i \min_{i \in \mathcal{I}} \{x^T(t)(A_i^T P + PA_i)x(t) \\ + \quad & x^T(t)PB_i\omega(t) + \omega^T(t)B_i^T Px(t) \\ - \quad & \alpha_i[w^T(t)w(t) - x^T(t)Px(t)]\} \\ \leq \quad & 0. \end{aligned}$$

Define the following switching signal

$$\begin{aligned} r^*(t) \quad = \quad & argmin_{i \in \mathcal{I}} \{x^T(t)(A_i^T P + PA_i)x(t) \\ + \quad & x^T(t)PB_i\omega(t) + \omega^T(t)B_i^T Px(t) \\ - \quad & \alpha_i[w^T(t)w(t) - x^T(t)Px(t)]\}. \end{aligned}$$

Then, under the switching signal $r^*(t)$, from previous discussions, we immediately get that for all $\omega(t) \in \Omega$, $x^T(t)Px(t) \geq 1$,

$$\begin{aligned} & \frac{d}{dt}x^T(t)Px(t) = x^T(t)(A_{r^*(t)}^T P + PA_{r^*(t)})x(t) \\ + \quad & x^T(t)PB_{r^*(t)}\omega(t) + \omega^T(t)B_{r^*(t)}^T Px(t) \\ \leq \quad & \alpha_{r^*(t)}[\omega^T(t)\omega(t) - x^T(t)Px(t)] \\ \leq \quad & 0, \end{aligned}$$

where α_{r^*} is defined as above. By Proposition 1, ε is an inescapable set of switched system (1).

Based on Lemma 2, we can get a necessary and sufficient condition for the case that $L = 2$.

Theorem 2. *Assume that $L = 2$. $\varepsilon = \{x|\ x^TPx \leq 1\}$ is an inescapable set of the switched system (1), if and only if there exist $\tau_i \geq 0$ satisfying $\sum_{i=1}^2 \tau_i > 0$, and $\alpha \geq 0$ such that*

$$\left[\begin{array}{cc} (\sum_{i=1}^2 \tau_i A_i)^T P + P(\sum_{i=1}^2 \tau_i A_i) + \alpha P & P(\sum_{i=1}^2 \tau_i B_i) \\ (\sum_{i=1}^2 \tau_i B_i)^T P & -\alpha I \end{array}\right] \leq 0. \quad (5)$$

Proof: We only need to prove the necessity since the sufficiency is proved in Theorem 1.

Suppose that $\varepsilon = \{x|\ x^TPx \leq 1\}$ is an inescapable set of switched system (1), then by Proposition 1, there exists a switching signal $r^*(t)$ such that the derivative of $V(x(t)) = x^T(t)Px(t)$ along the solution of system (1) satisfies

$$\frac{d}{dt}[x^T(t)Px(t)] \leq 0 \text{ for all } \omega(t) \in \Omega, x^T(t)Px(t) \geq 1.$$

Let

$$H_i := \left[\begin{array}{cc} A_i^T P + PA_i & PB_i \\ B_i^T P & 0 \end{array}\right], i = 1, 2.$$

Then, for all $\omega(t) \in \Omega, x^T(t)Px(t) \geq 1$, we have either $\left[\begin{array}{cc} x^T(t) & \omega^T(t) \end{array}\right] H_1 \left[\begin{array}{c} x(t) \\ \omega(t) \end{array}\right] \leq 0$ or $\left[\begin{array}{cc} x^T(t) & \omega^T(t) \end{array}\right] H_2 \left[\begin{array}{c} x(t) \\ \omega(t) \end{array}\right] \leq 0$. By Lemma 2, there exist $\tau_i, i = 1, 2$, nonnegative and not all zero such that for all $\omega(t) \in \Omega, x^TPx \geq 1$, $\left[\begin{array}{cc} x^T(t) & \omega^T(t) \end{array}\right] (\sum_{i=1}^2 \tau_i H_i) \left[\begin{array}{c} x(t) \\ \omega(t) \end{array}\right] \leq 0$, i.e.,

$$\left[\begin{array}{cc} x^T(t) & \omega^T(t) \end{array}\right] \left[\begin{array}{cc} (\sum_{i=1}^2 \tau_i A_i)^T P + P(\sum_{i=1}^2 \tau_i A_i) & P(\sum_{i=1}^2 \tau_i B_i) \\ (\sum_{i=1}^2 \tau_i B_i)^T P & 0 \end{array}\right] \left[\begin{array}{c} x(t) \\ \omega(t) \end{array}\right] \leq 0,$$

for all $\omega(t) \in \Omega, x^T(t)Px(t) \geq 1$. By Corollary 1, ε is an inescapable set of the LTI system $(\sum_{i=1}^2 \tau_i A_i, \sum_{i=1}^2 \tau_i B_i)$. By Theorem 2.1 in [16], this is equivalent to (5).

Remark 3. *By Theorem 2.1 in [16], Theorem 1 can be restated as follows:* ε *is an inescapable set of switched system (1) if there exist* $\tau_i \geq 0$ *satisfying* $\sum_{i=1}^{L} \tau_i > 0$ *such that* ε *is an inescapable set of the linear time-invariant (LTI) system* $(\sum_{i=1}^{L} \tau_i A_i, \sum_{i=1}^{L} \tau_i B_i)$*. For Theorem 2, we can get similar results.*

Remark 4. *Although we made Assumption 1 so as to make our switching problem non-trivial. In fact, Both Theorem 1 and Theorem 2 have covered the case that Assumption 1 is not true. That is, if* ε *is an inescapable set of the* i*-th subsystem,* $i \in \mathcal{I}$*, then we can choose* $\tau_i = 1$ *and* $\tau_j = 0, \forall\, j \neq i$ *in (4).*

Remark 5. *All the criteria in Theorems 1-2 are expressed in terms of LMIs, which can be easily tested with efficient LMI algorithm [18]-[19].*

IV. EXTENSIONS TO UNCERTAIN SWITCHED SYSTEMS

Consider the following uncertain switched system:

$$\dot{x}(t) = (A_{r(t)} + \Delta A_{r(t)})x(t) + (B_{r(t)} + \Delta B_{r(t)})\omega(t), \quad (6)$$

where $x(t), \omega(t), r(t)$ are defined as before. $[\Delta A_i, \Delta B_i] = E_i \Delta(t)[H_{A_i}, H_{B_i}]$, where H_{A_i}, H_{B_i}, E_i are constant matrices of compatible dimensions. $\Delta(t) \in \Gamma := \{\Delta(t) : \Delta^T(t)\Delta(t) \leq I\}$.

In this section, we give the following assumptions:

Assumption 2: There exists $i \in \mathcal{I}$ such that $E_i \neq 0$.

Assumption 3: For any $i \in \mathcal{I}$, $\begin{bmatrix} H_{A_i} & H_{B_i} \end{bmatrix} \neq 0$.

Remark 6. *Assumptions 2-3 will be used in the proof of Theorem 3 below.*

Similarly, a set $\mathcal{F}$ is said to be a robustly inescapable set of the uncertain switched system (6) with unit-amplitude disturbance input if the following two conditions hold:

(I) $0 \in \mathcal{F}, x(0) \in \mathcal{F}, \omega(t) \in \Omega$;

(II) There exists a switching signal $r^*(t)$ such that $\omega(t) \in \Omega, x(0) \in \mathcal{F}$ implies that $x(t) \in \mathcal{F}$ for all $\Delta(t) \in \Gamma$, where $t \geq 0$.

Given the ellipsoid $\varepsilon = \{x|\ x^T P x \leq 1\}$, where $P \geq 0$, we have the following results to determine whether ε is an inescapable set of switched system (6) with unit-amplitude disturbance input.

Proposition 2. ε *is an inescapable set of system (6) if and only if there exists a switching signal* $r^*(t)$ *such that for all* $\Delta(t) \in \Gamma, \omega(t) \in \Omega, x^T(t)Px(t) \geq 1$*, the derivative of* $V(x(t)) = x^T(t)Px(t)$ *along the solution of (6) satisfies*

$$\frac{d}{dt}[x^T(t)Px(t)] \leq 0.$$

Proof: The proof is similar to that of Proposition 1

Theorem 3. $\varepsilon = \{x|\ x^T P x \leq 1\}$ *is an inescapable set of the uncertain switched system (6) with* $\omega(t) \in \Omega$ *if there exist* $\tau_i \geq 0$ *satisfying* $\sum_{i=1}^{L} \tau_i > 0$*,* $\alpha \geq 0$ *and* $\varphi > 0$ *such that*

$$\begin{bmatrix} (\sum_{i=1}^{L} \tau_i A_i)^T P + P(\sum_{i=1}^{L} \tau_i A_i) + \alpha P + \varphi(\sum_{i=1}^{L} P E_i E_i^T P) & P(\sum_{i=1}^{L} \tau_i B_i) & \tau_1 H_{A_1}^T & \cdots & \tau_L H_{A_L}^T \\ (\sum_{i=1}^{L} \tau_i B_i)^T P & -\alpha I & \tau_1 H_{B_1}^T & \cdots & \tau_L H_{B_L}^T \\ \tau_1 H_{A_1} & \tau_1 H_{B_1} & -\varphi I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tau_L H_{A_L} & \tau_L H_{B_L} & 0 & \cdots & -\varphi I \end{bmatrix} \leq 0. \quad (7)$$

Proof: By Theorem 1, ε is an inescapable set of system (6) if there exist $\tau_i \geq 0, \sum_{i=1}^{L} \tau_i > 0$, and $\alpha \geq 0$ such that

$$\begin{bmatrix} (\sum_{i=1}^{L} \tau_i \bar{A}_i)^T P + P(\sum_{i=1}^{L} \tau_i \bar{A}_i) + \alpha P & P(\sum_{i=1}^{L} \tau_i \bar{B}_i) \\ (\sum_{i=1}^{L} \tau_i \bar{B}_i)^T P & -\alpha I \end{bmatrix} \leq 0, \quad (8)$$

where $\bar{A}_i = A_i + E_i \Delta(t) H_{A_i}, \bar{B}_i = B_i + E_i \Delta(t) H_{B_i}$.

(8) can be rewritten as

$$G + \sum_{i=1}^{L} \tau_i [M_i \Delta(t) N_i + N_i^T \Delta^T(t) M_i^T] \leq 0,$$

i.e.,

$$G + \begin{bmatrix} M_1 & \cdots & M_L \end{bmatrix} \Delta(t) \begin{bmatrix} \tau_1 N_1 \\ \vdots \\ \tau_L N_L \end{bmatrix} + \begin{bmatrix} \tau_1 N_1^T & \cdots & \tau_L N_L^T \end{bmatrix} \Delta^T(t) \begin{bmatrix} M_1^T \\ \vdots \\ M_L^T \end{bmatrix} \leq 0, \quad (9)$$

where

$$G = \begin{bmatrix} (\sum_{i=1}^{L} \tau_i A_i)^T P + P(\sum_{i=1}^{L} \tau_i A_i) + \alpha P & P(\sum_{i=1}^{L} \tau_i B_i) \\ (\sum_{i=1}^{L} \tau_i B_i)^T P & -\alpha I \end{bmatrix},$$

$$M_i = \begin{bmatrix} PE_i \\ 0 \end{bmatrix},$$

$$N_i = \begin{bmatrix} H_{A_i} & H_{B_i} \end{bmatrix}.$$

By Lemma 3, (9) holds if and only if there exists $\varphi > 0$ such that

$$G + \varphi MM^T + \frac{1}{\varphi}N^T N \le 0,$$

where

$$M = \left[\begin{array}{ccc} M_1, & \cdots, & M_L \end{array}\right],$$

$$N = \left[\begin{array}{c} \tau_1 N_1 \\ \vdots \\ \tau_L N_L \end{array}\right].$$

Then by Schur complement formula [14], we get

$$\left[\begin{array}{cc} G + \varphi MM^T & N^T \\ N & -\varphi I \end{array}\right] \le 0,$$

i.e., (7) holds. Hence, this theorem holds.

Theorem 4. *Assume that $L = 2$. $\varepsilon = \{x|\ x^T P x \le 1\}$ is an inescapable set of uncertain switched system (6) with $\omega \in \Omega$ if and only if there exist $\tau_i \ge 0$ satisfying $\sum_{i=1}^{2} \tau_i > 0$, $\alpha \ge 0$ and $\varphi > 0$ such that*

$$\left[\begin{array}{cccc} (\sum_{i=1}^{2} \tau_i A_i)^T P + P(\sum_{i=1}^{2} A_i) + \alpha P + \varphi(\sum_{i=1}^{2} PE_i E_i^T P) & P(\sum_{i=1}^{2} B_i) & \tau_1 H_{A_1}^T & \tau_2 H_{A_2}^T \\ (\sum_{i=1}^{2} B_i)^T P & -\alpha I & \tau_1 H_{B_1}^T & \tau_2 H_{B_2}^T \\ \tau_1 H_{A_1} & \tau_1 H_{B_1} & -\varphi I & 0 \\ \tau_2 H_{A_2} & \tau_2 H_{B_2} & 0 & -\varphi I \end{array}\right] \le 0. \qquad (10)$$

Proof: By Lemmas 2-3 and Theorem 3, this theorem can be proved easily.

V. EXAMPLES

Example 1.

Consider system (1) with $N = 2$ and

$$P = \left[\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array}\right], A_1 = \left[\begin{array}{cc} -1 & 2 \\ 2 & -1 \end{array}\right],$$

$$B_1 = \left[\begin{array}{cc} -1/2 & 1/2 \\ 1/2 & -1/2 \end{array}\right], A_2 = \left[\begin{array}{cc} -1 & -2 \\ -2 & -1 \end{array}\right],$$

$$B_2 = \left[\begin{array}{cc} -1/2 & -1/2 \\ -1/2 & -1/2 \end{array}\right].$$

Since both A_1 and A_2 are unstable, there doesn't exist α_i such that (3) holds. Using the LMI Control Toolbox in Matlab [19] to solve the LMI (4), we get $\tau_1 = \tau_2 = \alpha = 1/2$. Hence, $\varepsilon = \{x|x^T x \le 1/2\}$ is an inescapable set of this switched system.

Example 2. *Consider system (6) with $N = 2$ and*

$$P = \left[\begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array}\right], A_1 = \left[\begin{array}{cc} -1 & 2 \\ 2 & -1 \end{array}\right],$$

$$B_1 = \left[\begin{array}{cc} -0.1000 & 0.1000 \\ 0.1000 & -0.1000 \end{array}\right], E_1 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right],$$

$$H_{A1} = \left[\begin{array}{cc} 0.1000 & 0 \\ 0 & 0 \end{array}\right],$$

$$H_{B1} = \left[\begin{array}{cc} 0 & 0.1000 \\ 0 & 0 \end{array}\right], A_2 = \left[\begin{array}{cc} -1 & -2 \\ -2 & -1 \end{array}\right],$$

$$B_2 = \left[\begin{array}{cc} -0.1000 & -0.1000 \\ -0.1000 & -0.1000 \end{array}\right], E_2 = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right],$$

$$H_{A2} = \left[\begin{array}{cc} 0 & 0.1000 \\ 0.1000 & 0 \end{array}\right], H_{B2} = \left[\begin{array}{cc} 0.1000 & 0 \\ 0 & 0.1000 \end{array}\right].$$

Using the LMI Control Toolbox in Matlab [19] to solve the LMI (10), we get

$$\left\{\begin{array}{l} \tau_1 = 13.9716 \\ \tau_2 = 13.9695 \\ \tau_3 = 22.4856 \\ \tau_4 = 7.2740 \end{array}\right.$$

Hence, $\varepsilon = \{x|x^T x \le 1/2\}$ is an inescapable set of this uncertain switched system.

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