Exponential representations of two-input nonlinear discrete-time dynamics

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***Abstract*— A differential/difference representation of nonlinear multi-input discrete-time dynamics is introduced. On these bases, the exponential representation of the associated flow is explicitely characterized in the two-input case. The specific case of sampled dynamics is discussed and illustrated by a chained dynamics.**

***Index Terms*— Nonlinear discrete-time systems, multi-input dynamics, sampled systems, exponential series.**

I. INTRODUCTION

Exponential representations of flows associated with the solution to differential equations are well known tools in the continuous-time case when considering autonomous or controlled equations (see for example ([1], [3], [10]), while quite unexplored in the discrete-time context. Given a nonlinear *first-order difference equation* of the form

$$x(k+1) = F(x(k), u(k))$$

it has been shown in [6] that, under some suitable conditions, such a difference equation can be rewritten as a differential equation with respect to the control variable with state initialization specified by the drift term. It results that its behaviour over several steps can be revisited as continuous with respect to the control with instantaneous jumps piloted by the drift. In this context, the notion of flow characterizing the evolution along the control variable or equivalently the solution of the so built differential equation, makes sense. In [7], an exponential representation of this flow has been given and completely specified for the single-input case. The present paper extends this result to the multi-input case. The main difficulty comes out from the nonlinear dependency on several input variables resulting in a system of partial-derivative equations. The specific case of sampled dynamics is discussed and illustrated by a chained dynamics.

The paper is organized as follows. Section 2 introduces the differential representation of a multi-input difference equation. Section 3 gives the exponential form representation of the associated flow. The sampled case is discussed in Section 4 with an example. Some notations are given below.

- Throughout the paper $x \in \mathcal{X}$, an open set of R^n - which can be all R^n -, $u \in \mathcal{U}^m$, a neighborhood of 0 in R^m; the mapping $x \to F(x,u)$, describing forced discrete-time dynamics is an R^n-valued function, analytic on its domain of definition which contains a neighborhood of $u = 0$, the mapping $x \to F_0(x) := F(x, 0)$ describes the free evolution or drift term, all the vector fields are assumed analytic on their domains of definitions - infinitely differentiable admitting convergent Taylor series expansions in a neighborhood of each point of $\mathcal{X}$ - and complete when necessary - the associated flow is defined at any time and for any initial condition -.

- Given a generic map on $\mathcal{X}$, its evaluation at a point x is denoted either by "(x)" or "$\big|_x$". Given a function $\lambda : \mathcal{X} \to R$ and a vector field θ on $\mathcal{X}$ - $\theta(x) \in T_x\mathcal{X}$ - , the differential operator L_θ acts on λ as $L_\theta\lambda := \frac{\partial \lambda}{\partial x}\theta$ so giving the Lie derivative of λ along θ. The repeated use of this Lie derivative gives for $k > 0$, $L^k_\theta\lambda := \frac{\partial L^{k-1}_\theta \lambda}{\partial x}\theta$ with $L^k_\theta := L_\theta \circ \cdots \circ L_\theta$; $k-$times and $L^0_\theta = I$, the identity operator. Given another vector field σ on $\mathcal{X}$, $[\theta, \sigma]$ denotes the usual Lie bracket of vector fields; $L_{[\theta,\sigma]}I_n = (L_\theta \circ L_\sigma - L_\sigma \circ L_\theta)I_n$, where I_n indicates the identity function on $\mathcal{X}$; $ad_\theta(\sigma) := [\theta, \sigma], ad^0_\theta(\sigma) := \sigma$, and for $k > 0$, $ad^{k+1}_\theta(\sigma) := [\theta, ad^k_\theta(\sigma)]$.

- Given a formal indeterminate ψ, the notations "e^ψ" and "$Log(I + \psi)$" stand for the usual exponential and logarithmic Taylor series expansions of the indeterminate into parentheses; i.e.

$$\begin{aligned} e^\psi &= I + \sum_{i \geq 1} \frac{\psi^i}{i!} \\ Log(I + \psi) &= \sum_{i \geq 1} \frac{(-1)^{i-1}\psi^i}{i}. \end{aligned}$$

As an example, if $\psi = L_\theta$, $e^{L_\theta} = I + \sum_{k \geq 1} \frac{L^k_\theta}{k!}$. The following property holds

$$e^{L_\theta}\lambda\big|_x := \lambda(x) + \sum_{k \geq 1} \frac{L^k_\theta \lambda}{k!}(x) = \lambda(e^{L_\theta} I_n\big|_x).$$

For notational convenience, e^θ can also be used instead of e^{L_θ}.

- In the sequel, a Lie monomial of a given set of indeterminates - vector fields - indicates a Lie bracket of them, a Lie polynomial indicates a finite sum with real coefficients of Lie monomials and a Lie series indicates an infinite sum of Lie polynomials. When a degree is assigned to each vector field, the notions of degree for monomials and homogeneity for polynomials are the usual ones.

- When not explicitely specified, convergence issues of the series manipulated will not be addressed in this formal context.

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II. THE CONTEXT AND PROBLEM STATEMENT

This section extends to the multi-input case the representations introduced in [6] for single-input dynamics.

A. Differential/difference representations of multi-input nonlinear discrete-time dynamics

Let a multi-input nonlinear difference equation be

$$x(k+1) = F(x(k), u(k)) \tag{1}$$

where $x \in R^n$, $u = (u_1, \ldots, u_m)^T \in \mathcal{U}^m$ with $F_0(x) := F(x;0)$ and assume rank$\left(\frac{\partial F}{\partial u}\right) = m$.

The following assumption is made.

H1: There exist m analytic vector fields, $({}_iG(.,u); i = (1,...,m))$, satisfying

$$\frac{\partial F(\cdot,u)}{\partial u_i} = {}_iG(F(\cdot,u),u). \tag{2}$$

Remark. The invertibility of $F_0(x)$ is sufficient to guarantee **H1** so that, in such a case, ${}_iG(x,u)$ is locally uniquely defined as

$${}_iG(x,u) := \left.\frac{\partial F(x,u)}{\partial u_i}\right|_{x=F^{-1}(x,u)}. \tag{3}$$

Sampled dynamics enter in such a class. ◁

The following relations are directly deduced from **H1** due to commutativity of the partial-derivatives operators. Indicating by $J_x[.]$ the Jacobian, we deduce from (2),

$$\begin{aligned}\frac{\partial^2 F(x;u)}{\partial u_i \partial u_j} &= L_{{}_iG(.,u)} \circ L_{{}_jG(.,u)} I_n\Big|_{F(x;u)} \\ &= \left(J_x[{}_jG(.,u)]{}_iG(.,u) + \frac{\partial_j G(.,u)}{\partial u_i}\right)\Bigg|_{F(x;u)}\end{aligned}$$

equal to

$$\begin{aligned}\frac{\partial^2 F(x;u)}{\partial u_j \partial u_i} &= L_{{}_jG(.,u)} \circ L_{{}_iG(.,u)} I_n\Big|_{(F(x;u),u)} \\ &= \left(J_x[{}_iG(.,u)]{}_jG(\cdot,u) + \frac{\partial_i G(.,u)}{\partial u_j}\right)\Bigg|_{F(x;u)}\end{aligned}$$

which can be rearranged to state what we will refer to as the *compatibility conditions*.

Compatibility conditions - A family of u-dependent vector fields $({}_iG(.,u); i = (1,...,m))$ satisfy the *compatibility conditions* if, for any $(i,j) \in (1,...,m)$ with $i \neq j$ and $u \in \mathcal{U}^m$, the equalities below are satisfied on $\mathcal{X}$

$$[{}_iG(.,u), {}_jG(.,u)]|_x = \frac{\partial_i G(x,u)}{\partial u_j} - \frac{\partial_j G(x,u)}{\partial u_i}. \tag{4}$$

Provided the compatibility conditions are satisfied and adopting the notation $x^+(u)$ to represent a curve in R^n parameterized by $u \in \mathcal{U}^m$, it makes sense now to represent a discrete-time dynamics of the form (1) satisfying H1, as a system of coupled differential/difference equations. We set

Definition 2.1: Analytically parameterized discrete-time dynamics- **APDTD**
Let $(x,u) \in \mathcal{X} \times \mathcal{U}^m$ and assume maps and vector fields analytic in their arguments. Given a map F_0 on R^n and m vector fields $({}_iG(.,u); u \in \mathcal{U}^m)$ on $\mathcal{X}$, complete and satisfying the compatibility conditions, an APDTD is defined by the system of differential/difference equations

$$x^+ = F_0(x) \tag{5}$$

$$\frac{\partial x^+(u)}{\partial u_i} = {}_iG(x^+(u),u);\, i = (1,\cdots,m);\, x^+(0) = x^+. \tag{6}$$

For a given $(x,u) \in \mathcal{X} \times \mathcal{U}^m$, integrating (6) with respect to each u_i and evaluating the result at (5), we get a mapping $F(x,u) = x^+(u)$ so illustrating how a discrete-time dynamics can be revisited as a trajectory in R^n, parameterized by u accordingly to (6) and passing through $x^+(0) = F_0(x)$.
In this differential geometric context, the aim of this paper is to specify the exponential representation of the flow which characterizes the solution to (6). Two main difficulties occur, due to the nonlinearity in u of each equation in (6) and due to the coupled interaction between them as partial-derivatives.

B. Recalls about the single-input case

In this paragraph, we refer to the previous notations setting $m = 1$ and denoting without ambiguity ${}_1G(.,u)$ by $G(.,u)$ and by G_i the vector fields characterizing the series expansion around u of $G(.,u)$; i.e.

$$G(.,u) = G_1(.) + uG_2(.) + \sum_{i \geq 2} \frac{u^i}{i!} G_{i+1}(.). \tag{7}$$

Given a single-input difference equation of the form (1) satisfying **H1** and referring to the literature about differential geometry, formal calculus and combinatorics (see for example [2], [4], [9], [11]), it has been shown in [7] that the solution to (6) admits an exponential representation with exponent described by a Lie element in the vector fields G_i. More precisely, let $\phi(u,0,\cdot)$ be the flow associated with $G(.,u)$, the unique solution of

$$\frac{\partial \phi(u,0,\cdot)}{\partial u} = G(\phi(u,0,\cdot),u); \quad \phi(0,0,\cdot) = I_n \tag{8}$$

and let $\overrightarrow{\exp} \int_0^u L_{G(.,v)} dv$, the *right chronological exponential* defined by its asymptotic expansion (see [1] [11])

$$\overrightarrow{\exp} \int_0^u L_{G(.,v)} dv := I + \sum_{p \geq 1} \int_0^u \int_0^{v_1} \ldots \int_0^{v_{p-1}} L_{G(.,v_p)} \circ \ldots \circ L_{G(.,v_1)} dv_p \ldots dv_1$$

we have

$$\phi(u,0,\cdot) = \overrightarrow{\exp} \int_0^u L_{G(.,v)} dv I_n = e^{u\mathcal{G}(.,u)} I_n \tag{9}$$

where the exponent $u\mathcal{G}(.,u)$ is described by its expansion

$$u\mathcal{G}(.,u) = \sum_{p \geq 1} u^p \mathcal{B}_p(G_1, ..., G_p) \tag{10}$$

where $\mathcal{B}_p(G_1,\ldots,G_p)$ stands for a homogeneous Lie polynomial of degree p in its arguments. The decomposition of each $\mathcal{B}_p$ as a Lie polynomial in the G_i's can be iteratively deduced from the formal equality

$$\frac{\partial}{\partial u}u\mathcal{G}(.,u) \quad = \quad Z(-ad_{u\mathcal{G}(.,u)})G(.,u) \qquad (11)$$

where the function $Z(.)$ is defined for any formal indeterminate ψ by its Taylor expansion

$$Z(-\psi)=\frac{\psi}{1-e^{-\psi}}=\sum_{i\geq 0}(-1)^i b_i\frac{\psi^i}{i!}.$$

The coefficients b_i are the Bernoulli numbers. For the first ones: $b_0=1$, $b_1=-1/2$, $b_2=1/6$, $b_{2k+1}=0$ for $k>0$, $b_4=-1/30$, $b_6 = 1/42$. We get for $p\geq 1, \Sigma_{q=1}^{j}l_q+k=p$

$$\mathcal{B}_p=\frac{G_p}{p!}+\sum_{k=1}^{p-1}\sum_{j\geq 1}\sum_{l_1,\ldots,l_j\geq 1}\frac{(-1)^j b_j}{j!}ad_{\mathcal{B}_{l_1}}\circ\ldots\circ ad_{\mathcal{B}_{l_j}}\frac{G_k}{(k-1)!}$$

and for the first terms we get

$$\begin{aligned}\mathcal{B}_1 &= G_1, \quad \mathcal{B}_2=\frac{1}{2!}G_2,\\ \mathcal{B}_3 &= \frac{1}{3!}(G_3+1/2[G_1,G_2]), \quad \mathcal{B}_4=\frac{1}{4!}(G_4+[G_1,G_3]).\end{aligned}$$

C. Some notations in the multi-input case

In the sequel, indicating as ${}_{ii_1\cdots i_p}G_{p+1}(.,u)$, any partial derivative of order p of ${}_iG(.,u)$ with respect to $u_{i_1}...u_{i_p}$; i.e.

$${}_{ii_1\cdots i_p}G_{p+1}(.,u):=\frac{\partial^p {}_iG(.,u)}{\partial u_{i_p}\cdots\partial u_{i_1}}$$

the series expansion of ${}_iG(.,u)$ around $u=0$ is given by

$${}_iG(.,u)={}_iG_1+\sum_{p\geq 1}\sum_{i_1,\ldots,i_p=1}^{m}\frac{u_{i_1}...u_{i_p}}{p!}({}_{ii_1\cdots i_p}G_{p+1})$$

with ${}_iG_1 := {}_iG(.,u)|_{u=0}$ and ${}_{ii_1\cdots i_p}G_{p+1} := {}_{ii_1\cdots i_p}G_{p+1}(.,u)|_{u=0}$.
Denoting by $i^{p+1}:=(i,...,i)$ a multi-index of length $p+1$, with all elements equal to i, we analogously set ${}_{i^{p+1}}G(\cdot,u)|_{u_i=0}:=\left.\frac{\partial^p {}_iG(\cdot,u)}{\partial u_i^p}\right|_{u_i=0}$ so that the series expansion of ${}_iG(.,u)$ around $u_i=0$, for $u_j=Cst$ when $j\neq i$, is given by

$${}_iG(\cdot,u) \quad = \quad {}_iG(\cdot,u)|_{u_i=0}+\sum_{p\geq 1}\frac{u_i^p}{p!}({}_{i^{p+1}}G(\cdot,u)|_{u_i=0}).$$

By convention, any ${}_{ii_1\cdots i_p}G_{p+1}$ will be said of degree $p+1$ and by construction, for any permutation σ of a multi-index $(i_1,...,i_p)$, we have ${}_{ii_1i_2\ldots i_p}G_{p+1}={}_{i\sigma(i_1)\sigma(i_2)\ldots\sigma(i_p)}G_{p+1}$.
As previously noted, compatibility conditions of the vector fields ${}_iG(.,u)$ make sense to (6) and completeness ensure integrability of (6). It is interesting to rewrite compatibility conditions as equalities independent on the input variables so enlightening how these conditions specify the involutivity of the Lie algebra generated by all the ${}_{ii_1i_2\ldots i_p}G'_{p+1}s$. Under successive derivatives with respect to u and evaluation at $u=0$, (4) can be equivalently rewritten as the successive equalities below iteratively deduced by applying the formal rule of derivatives of products and sums.

Lemma 2.1: For any multi-index $(i_1,...,i_l)\in(1,...,m)$ with $i_1\neq i_2$, (4) are equivalent to

$$\begin{aligned}[{}_{i_1}G_1,{}_{i_2}G_1] &= {}_{i_1i_2}G_2-{}_{i_2i_1}G_2\\ [{}_{i_1i_3}G_2,{}_{i_2}G_1] &+ [{}_{i_1}G_1,{}_{i_2i_3}G_2]={}_{i_1i_2i_3}G_3-{}_{i_2i_1i_3}G_3\\ [{}_{i_1i_3i_4}G_4,{}_{i_2}G_1] &+ [{}_{i_1i_3}G_2,{}_{i_2i_4}G_2]+[{}_{i_1i_4}G_2,{}_{i_2i_3}G_2]\\ &+ [{}_{i_1}G_1,{}_{i_2i_3i_4}G_3]\\ &= {}_{i_1i_2i_3i_4}G_4-{}_{i_2i_1i_3i_4}G_4, \quad \ldots \quad .\end{aligned}$$

III. EXPONENTIAL REPRESENTATION OF THE FLOW

To simplify the notations, we treat the case $m=2$, but the method extends according to the same lines to more than two independent control variables. To simplify nonlinearity in u, we propose to rewrite equations (5 - 6) for $m=2$ in an extended state space, as it is usual when dealing with non-autonomous differential equations. Setting $\zeta=(x^T,z^T)^T\in\mathcal{X}\times\mathcal{U}^2$, (5 - 6) can be rewritten as

$$\begin{aligned}x^+ &= F_0(x); \quad z^+=0\\ \frac{\partial x^+(u)}{\partial u_i} &= {}_iG(x^+(u);z^+(u));\ i=(1,2);\ x^+(0)=x^+\\ \frac{\partial z_i^+(u)}{\partial u_i} &= 1; \quad \frac{\partial z_i^+(u)}{\partial u_j}=0; \quad j\neq i; \quad z^+(0)=z^+\end{aligned}$$

or in a more compact form as

$$\begin{aligned}\zeta^+ &= \bar{F}_0(\zeta)\\ \frac{\partial\zeta^+(u_1,u_2)}{\partial u_1} &= {}_1\bar{G}(\zeta^+(u)); && (12)\\ \frac{\partial\zeta^+(u_1,u_2)}{\partial u_2} &= {}_2\bar{G}(\zeta^+(u)); \quad \zeta^+(0)=(x^+,0) && (13)\end{aligned}$$

with $\bar{F}_0(\zeta)^T=(F_0(x)^T,0)^T$ and ${}_i\bar{G}(\zeta)=({}_iG(x,u)^T,e_i^T)^T$ where e_i is the i-th unit vector in R^2.
Translating the compatibility conditions (4), set over the ${}_iG(.,u)'s$, into conditions over $({}_1\bar{G},{}_2\bar{G})$, we get the condition of nilpotency; i.e.

$$[{}_1\bar{G}(\zeta),{}_2\bar{G}(\zeta)]=0 \qquad (14)$$

which is necessary and sufficient for the existence of a solution $\zeta^+(u)$ to (12-13). In fact, the equality

$$\frac{\partial^2\zeta^+(u_1,u_2)}{\partial u_1\partial u_2}=\frac{\partial^2\zeta^+(u_1,u_2)}{\partial u_2\partial u_1}$$

is immediately translated into commutativity of the extended Lie derivative operators $(L_{{}_1\bar{G}},L_{{}_2\bar{G}})$.
According to these notations, the formal integration of (12-13) is greatly simplified so getting directly the exponential form of the flow

$$\zeta^+(u)=e^{{}_1\bar{G}u_1+{}_2\bar{G}u_2}I_{n+2}\Big|_{\zeta^+(0)}. \qquad (15)$$

Denoting now by π the projection on R^n; i.e. $\pi(x,u)=x$, by $\phi_i(u_i,0,.)$ the flow with exponent $u_i\mathcal{G}_i(.,u)$ associated with ${}_iG(.,u)$ accordingly to (8), (9) and (10), we easily deduce the equivalent representations of the solution to (5 -6) below.

Proposition 3.1: Consider the two-input difference equation (1) verifying **H1** - or equivalently, consider an APDTD of the form (5 -6) -, then (1) - or equivalently the solution to (6) at (5) - is given by

$$x^+(u) = e^{{}_1\bar{G}u_1 + {}_2\bar{G}u_2}\pi\Big|_{(F_0(x),0)} \qquad (16)$$

which can be rewritten as the composition of single-input flows either

$$\begin{aligned} x^+(u) &= e^{{}_1\bar{G}u_1}\circ e^{{}_2\bar{G}u_2}\pi\Big|_{(F_0(x),0)} && (17)\\ &= \phi_2(u_2,0,\phi_1(u_1,0,x^+(0))) && (18)\end{aligned}$$

or

$$\begin{aligned} x^+(u) &= e^{{}_2\bar{G}u_2}\circ e^{{}_1\bar{G}u_1}\pi\Big|_{(F_0(x),0)} && (19)\\ &= \phi_1(u_1,0,\phi_2(u_2,0,x^+(0))) && (20)\end{aligned}$$

with asymptotic behaviour described either by

$$\begin{aligned} x^+(u) &= \overrightarrow{\exp}\int_0^{u_1} L_{{}_1G(x,v_1,0)}dv_1\circ \\ &\circ\overrightarrow{\exp}\int_0^{u_2} L_{{}_2G(x,u_1,v_2)}dv_2 I_n\Big|_{x^+(0)} \end{aligned} \qquad (21)$$

or

$$\begin{aligned} x^+(u) &= \overrightarrow{\exp}\int_0^{u_2} L_{{}_2G(x,0,v_2)}dv_2\circ \\ &\circ\overrightarrow{\exp}\int_0^{u_1} L_{{}_1G(x,v_1,u_2)}dv_1 I_n\Big|_{x^+(0)} \end{aligned} \qquad (22)$$

and exponential representation described either by

$$x^+(u) = e^{u_1\mathcal{G}_1(.,u_1,0)}\circ e^{u_2\mathcal{G}_2(.,u_1,u_2)}I_n\Big|_{x^+(0)} \qquad (23)$$

or

$$x^+(u) = e^{u_2\mathcal{G}_2(.,0,u_2)}\circ e^{u_1\mathcal{G}_1(.,u_1,u_2)}I_n\Big|_{x^+(0)}. \qquad (24)$$

Proof: The proof is easily performed interchanging the role of u_1 and u_2. The solution can in fact be obtained either by chaining the integration of $x^+(u_1,v_2)$ along v_2, between 0 and u_2, for a fixed u_1 and the integration of $x^+(v_1,0)$ along v_1, between 0 and u_1 - equations (17 -18 - 21 - 23) - or by chaining the integration of $x^+(v_1,u_2)$ along v_1, between 0 and u_1, for a fixed u_2 and the integration of $x^+(0,v_2)$ along v_2, between 0 and u_2 - equations (19 -20 - 22 - 24) -. It is immediate to deduce (16) from (15) and then the equality between (17) and (19) due to the commutativity of the operators or equivalently the compatibility conditions. The equality between (17) and (18) and equivalently between (19) and (20) are just a matter of computations. The chronological series expansions (21) or (22) with exponential representations (23) or (24) are easily deduced by applying the results previously recalled in the single-input case. ■

Denoting now by $\phi(u,0,.) : R^n \to R^n$, the unique multi-input flow associated with the solution to the set of partial-derivative equations (6), we have in conclusion the exponential representation below.

Theorem 3.1: Exponential representation of discrete-time dynamics

Consider the two-input difference equation (1) satisfying **H1** - or equivalently, consider an APDTD of the form (5 -6) -, then

$$x^+(u) = \phi(u,0,x^+(0)) = e^{u\mathcal{G}(.,u)}I_n\Big|_{x^+(0)} \qquad (25)$$

with $u\mathcal{G}(.,u)$ a vector field on R^n, parameterized by (u_1,u_2) which is a Lie element in the ${}_{ii_1\ldots i_p}G'_{p+1}s$, given by

$$\begin{aligned} u\mathcal{G}(.,u) &= \sum_{p_2\ge 1} u_2^{p_2}\mathcal{B}_{0,p_2}({}_2G_1,\ldots,{}_{2^{p_2}}G_{p_2}) \qquad (26)\\ &+ \sum_{p_1\ge1,p_2\ge0} u_1^{p_1}u_2^{p_2}\mathcal{B}_{p_1,p_2}({}_{i_1}G_1,\ldots,{}_{i_1,\ldots,i_{p_1+p_2}}G_{p_1+p_2}) \end{aligned}$$

where $\mathcal{B}_{p_1,p_2}(.)$ stands for a homogeneous Lie polynomial of degree p_1+p_2 in its arguments. The expansion (26) is deduced from the equalities below

$$\frac{\partial}{\partial u_i}u\mathcal{G}(.,u) = Z(-ad_{u\mathcal{G}(.,u)})_iG(.,u); \quad i=(1,2). \qquad (27)$$

The decomposition of the $\mathcal{B}'_{p_1,p_2}s$ as Lie polynomials can be iteratively computed according to $\mathcal{B}_{0,0}=0$ and for $p_2\ge 1$ and $\sum_{q=1}^{j} m_q + k_2 = p_2$

$$\begin{aligned} \mathcal{B}_{0,p_2} &= \frac{{}_{2^{p_2}}G_{p_2}}{p_2!} + \frac{1}{p_2}\sum_{1\le k_2}\sum_{j\ge1} \\ &\sum_{m_1,\ldots,m_j\ge0}\frac{(-1)^j b_j}{j!}ad_{\mathcal{B}_{0,m_1}}\circ\ldots\circ ad_{\mathcal{B}_{0,m_j}}\frac{{}_{2^{k_2}}G_{k_2}}{(k_2-1)!} \end{aligned} \qquad (28)$$

for $p_1\ge1, p_2\ge0$ and $\sum_{q=1}^{j} l_q + k_1 = p_1$

$$\begin{aligned} \mathcal{B}_{p_1,p_2} &= \frac{{}_{1^{p_1}2^{p_2}}G_{p_1+p_2}}{p_1!p_2!} + \frac{1}{p_1}\sum_{1\le k_1,0\le k_2}\sum_{j\ge1}\sum_{l_1,\ldots,l_j\ge0} \\ &\sum_{m_1,\ldots,m_j\ge0}\frac{(-1)^j b_j}{j!}ad_{\mathcal{B}_{l_1,m_1}}\circ\ldots\circ ad_{\mathcal{B}_{l_j,m_j}}\frac{{}_{1^{k_1}2^{k_2}}G_{k_1+k_2}}{(k_1-1)!k_2!} \end{aligned} \qquad (29)$$

Proof: From Proposition 3.1 we immediately deduce the existence of an exponential representation of the multi-input flow of the form (25). The proof of the equalities (27) follows exactly the same arguments as in the single-input case [8]. By expanding the two members of (27) and by identifying the coefficients we obtain (28) and (29). ■

Because the role of u_1 and u_2 can be interchanged, we equivalently get

$$u\mathcal{G}(.,u) = \sum_{p_1\ge1} u_1^{p_1}\mathcal{B}_{p_1,0} + \sum_{p_1\ge0,p_2\ge1} u_1^{p_1}u_2^{p_2}\mathcal{B}_{p_1,p_2}$$

with

$$\mathcal{B}_{p_1,0} = \frac{{}_{1^{p_1}}G_{p_1}}{p_1!} + \frac{1}{p_1}\sum_{1\le k_1}\sum_{j\ge 1}\sum_{l_1,\dots,l_j\ge 0}\frac{(-1)^j b_j}{j!} ad_{\mathcal{B}_{l_1,0}}\circ\dots\circ ad_{\mathcal{B}_{l_j,0}}\frac{{}_{1^{k_1}}G_{k_1}}{(k_1-1)!}$$

and for $p_1 \ge 0, p_2 \ge 1$, $\sum_{q=1}^{j} l_q + k_1 = p_1$, $\sum_{q=1}^{j} m_q + k_2 = p_2$,

$$\mathcal{B}_{p_1,p_2} = \frac{{}_{2^{p_2}1^{p_1}}G_{p_1+p_2}}{p_2!p_1!} + \frac{1}{p_2}\sum_{0\le k_1,1\le k_2}\sum_{j\ge 1}\sum_{l_1,\dots,l_j\ge 0}\sum_{m_1,\dots,m_j\ge 0}\frac{(-1)^j b_j}{j!} ad_{B_{l_1,m_1}}\circ\dots\circ ad_{B_{l_j,m_j}}\frac{{}_{2^{k_2}1^{k_1}}G_{k_1+k_2}}{(k_2-1)!k_1!}.$$

A. Some specific cases

Assuming that ${}_{i=(1,2)}G(x,u)$ depends on the corresponding u_i only; i.e. ${}_iG(x,u) = {}_iG(x,u_i)$, compatibility conditions (4) reduce to nilpotency

$$[{}_1G(.,u_1), {}_2G(.,u_2)]|_x = 0$$

and the solution reduces to

$$x^+(u) = e^{u_1\mathcal{G}_1(.,u_1)+u_2\mathcal{G}_2(.,u_2)}I_n|_{F_0(x)}$$

with each $u_i\mathcal{G}_i(.,u_i)$ described by (10).

Assuming that ${}_{i=(1,2)}G(x,u)$ does not depend on the control (autonomous vector fields); i.e. ${}_iG(x,u) = {}_iG_1(x)$, then compatibility conditions (4) reduce to

$$[{}_1G_1, {}_2G_1]\Big|_x = 0$$

so that the solution reduces to

$$x^+(u) = e^{u_1({}_1G_1)+u_2({}_2G_1)}I_n\Big|_{F_0(x)}.$$

B. Some computations

Let us give an insight on the first terms of the exponent (26)

$$u\mathcal{G}(.,u) = u_1B_{1,0} + u_1^2B_{2,0} + u_1^3B_{3,0} + u_2B_{0,1} + u_2^2B_{0,2} + u_2^3B_{0,3} + u_1u_2B_{1,1} + u_1u_2^2B_{1,2} + u_2u_1^2B_{2,1} + O(u^4)$$

with

$$B_{1,0} = {}_1G_1,\ 2B_{2,0} = {}_{11}G_2,\ 3!B_{3,0} = {}_{111}G_3 + \frac{1}{2}[{}_1G_1, {}_{11}G_2]$$

$$B_{0,1} = {}_2G_1,\ 2B_{0,2} = {}_{22}G_2,\ 3!B_{0,3} = {}_{222}G_3 + \frac{1}{2}[{}_2G_1, {}_{22}G_2]$$

and from (29)

$$\begin{aligned} B_{1,1} &= {}_{12}G_2 + \frac{1}{2}[{}_2G_1, {}_1G_1] \\ 2B_{1,2} &= {}_{122}G_3 + [{}_2G_1, {}_{12}G_2] + \frac{1}{2}[{}_{22}G_2, {}_1G_1] \\ &\quad + \frac{1}{3!}[{}_2G_1[{}_2G_1, {}_1G_1]] \\ 2B_{2,1} &= {}_{112}G_3 + \frac{1}{2}[{}_2G_1, {}_{11}G_2] + \frac{1}{2}[[{}_2G_1, {}_1G_1], {}_1G_1] \end{aligned}$$

or equivalently

$$\begin{aligned} B_{1,1} &= {}_{21}G_2 + \frac{1}{2}[{}_1G_1, {}_2G_1] \\ 2B_{1,2} &= {}_{221}G_3 + \frac{1}{2}[{}_1G_1, {}_{22}G_2] + \frac{1}{2}[[{}_1G_1, {}_2G_1], {}_2G_1] \\ 2B_{2,1} &= {}_{211}G_3 + [{}_1G_1, {}_{21}G_2] + \frac{1}{2}[{}_{11}G_2, {}_2G_1] \\ &\quad + \frac{1}{3!}[{}_1G_1[{}_1G_1, {}_2G_1]] \end{aligned}$$

so that

$$B_{1,1} = \frac{1}{2}({}_{12}G_2 + {}_{21}G_2)$$

$$4B_{1,2} = {}_{122}G_3 + {}_{221}G_3 + [{}_2G_1, {}_{12}G_2] + \frac{4}{3!}[{}_2G_1[{}_2G_1, {}_1G_1]]$$

$$4B_{2,1} = {}_{112}G_3 + {}_{211}G_3 + [{}_1G_1, {}_{21}G_2] + \frac{4}{3!}[{}_1G_1[{}_1G_1, {}_2G_1]])$$

Replacing these expressions into the exponential form, we get in conclusion, up to an error in $O(u^3)$, in the exponent

$$\phi(u,0,x^+(0)) = e^{u\mathcal{G}(.,u)}I_n\Big|_{F_0(x)} = e^{u_1({}_1G_1)+u_2({}_2G_1)+\frac{u_1^2}{2}{}_{11}G_2+\frac{u_2^2}{2}{}_{22}G_2+\frac{u_1u_2}{2}({}_{12}G_2+{}_{21}G_2)}I_n\Big|_{F_0(x)}.$$

IV. THE CASE OF SAMPLED DYNAMICS

Let the *two input-affine continuous-time dynamics*

$$\dot{x}(t) = f(x(t)) + u_1(t)g_1(x(t)) + u_2(t)g_2(x(t)) \quad (30)$$

with f and g_i analytic vector fields on R^n. Given a sampling period $\delta \ge 0$, setting $(t = k\delta; k \ge 0)$, the sampling instants, assume the input signal $u(t)$ constant over time intervals of amplitude δ and let $u(k)$ be its constant value over the interval $[k\delta, (k+1)\delta[$ and $x(k)$ the value of $x(t)$ at time $t = k\delta$. It is well known that the solution at time $t = (k+1)\delta$, for an initialization at $x(k)$ describes a nonlinear difference equation - the sampled equivalent to (30) - (i.e. the state evolutions coincide at each sampling time), as

$$\begin{aligned} x(k\delta+\delta) := x(k+1) &= e^{\delta f + u_1(k)\delta g_1 + u_2(k)\delta g_2}I_n\Big|_{x(k)} \\ &= F^\delta(x(k), \delta u(k)). \end{aligned}$$

It follows that the results of Theorem 3.1 still apply so getting

Theorem 4.1: For a fixed sampling period δ, the zero–order sampled equivalent $F^\delta(x, \delta u)$ to (30) admits the differential representation

$$\begin{aligned} x^+ &= e^{\delta f}I_n\Big|_x = F_0^\delta(x) \\ \frac{d}{d\delta u_i}(x^+(\delta u)) &= {}_iG^\delta(x^+(\delta u), \delta u); \quad x^+(0) = x^+ \end{aligned}$$

with

$${}_iG^\delta(., \delta u) = Z^{-1}(-ad_{\delta f + \delta u g})g_i. \quad (31)$$

when $Z^{-1}(.)$ denotes the formal inverse of $Z(.)$;i.e.

$$Z^{-1}(-ad_\zeta) = \int_0^1 e^{-sad_\zeta} ds = \frac{1-e^{-ad_\zeta}}{ad_\zeta} = I + \sum_{i\geq 1} \frac{(-1)^i}{(i+1)!} ad_\zeta^i.$$

Proof: The proof requires to verify that **H1** is always verified in this sampled case being $F_0^\delta(x) := e^{\delta f} I_n|_x$ always invertible for sufficiently small values of δ ensuring the series convergence; i.e. $(F_0^\delta)^{-1}(x) := e^{-\delta f} I_n\big|_x$. The expression (31) of ${}_iG^\delta(.,\delta u)$ follows from (27) because, in this sampled case, $\delta u \mathcal{G}(.,\delta u) = \delta f + \delta u_1 g_1 + \delta u_2 g_2$, so that

$$g_i = Z(-ad_{\delta f + \delta u g})\, {}_iG^\delta(.,\delta u)$$

and thus (31) holds true. ∎

In this sampled case, the compatibility conditions reduce to combinatoric identities deduced from (31).

A. The example of chained dynamics

Let the one-chain system on R^4 be

$$\dot{x_1} = u_1, \quad \dot{x_2} = u_2, \quad \dot{x_3} = x_2 u_1, \quad \dot{x_4} = x_3 u_1 \qquad (32)$$

with sampled equivalent easily computed as

$$\begin{aligned} x_1(k+1) &= x_1(k) + \delta u_1(k); \quad x_2(k+1) = x_2(k) + \delta u_2(k) \\ x_3(k+1) &= x_3(k) + \delta x_2(k) u_1(k) + \frac{\delta^2}{2} u_2(k) u_1(k) \\ x_4(k+1) &= x_4(k) + \delta x_3(k) u_1(k) + \frac{\delta^2}{2} x_2(k) u_1^2(k) \\ &\quad + \frac{\delta^3}{3!} u_2(k) u_1^2(k). \end{aligned}$$

$F^\delta(x,\delta u)$ is thus polynomial with $F_0^\delta(x) = x$. The vector fields $({}_1G^\delta(.,\delta u), {}_2G^\delta(.,\delta u))$ exist, are unique and can be computed according to (3) or (31) so getting

$$\begin{aligned} {}_1G^\delta(\cdot,\delta u) &= (1, 0, x_2 - \frac{\delta}{2} u_2, x_3 - \frac{\delta^2}{3!} u_1 u_2)^T \\ &= {}_1G_1^\delta + {}_{12}G_2^\delta \delta u_2 + {}_{112}G_3^\delta \delta^2 u_1 u_2 \\ {}_2G^\delta(\cdot,\delta u) &= (0, 1, \frac{\delta}{2} u_1, \frac{\delta^2}{3!} u_1^2)^T \\ &= {}_2G_1^\delta + {}_{21}G_2^\delta \delta u_1 + {}_{211}G_3^\delta \frac{\delta^2}{2} u_1^2 \end{aligned}$$

and the other terms equal to zero. The compatibility conditions reduce to

$$\begin{aligned} &[{}_1G^\delta(\cdot,\delta u_1,\delta u_2), {}_2G^\delta(\cdot,\delta u_1,\delta u_2)] = \\ &{}_{12}G_2^\delta(\cdot,\delta u_1,\delta u_2) - {}_{21}G_2^\delta(\cdot,\delta u_1,\delta u_2) \end{aligned}$$

easily verified. The sampled equivalent model exhibits the differential/difference representation

$$\begin{aligned} & x^+ = x; \quad x^+(0) = x^+ \\ & \frac{\partial x_1^+(\delta u)}{\partial \delta u_1} = 1, \quad \frac{\partial x_2^+(\delta u)}{\partial \delta u_1} = 0 \\ & \frac{\partial x_3^+(\delta u)}{\partial \delta u_1} = x_2^+(\delta u) - \frac{\delta u_2}{2}, \quad \frac{\partial x_4^+(\delta u)}{\partial \delta u_1} = x_3^+(\delta u) - \frac{\delta^2 u_1 u_2}{3!} \\ & \frac{\partial x_1^+(\delta u)}{\partial \delta u_2} = 0, \quad \frac{\partial x_2^+(\delta u)}{\partial \delta u_2} = 1 \\ & \frac{\partial x_3^+(\delta u)}{\partial \delta u_2} = \frac{\delta u_1}{2}, \quad \frac{\partial x_4^+(\delta u)}{\partial \delta u_2} = \frac{\delta^2 u_1^2}{3!}. \end{aligned}$$

Computing $B_{1,0} = {}_1G_1^\delta$, $B_{0,1} = {}_2G_1^\delta$, $B_{1,1} = {}_{12}G_2^\delta + \frac{1}{2}[{}_2G_1^\delta, {}_1G_1^\delta] = 0$, $B_{1,2} = \frac{1}{12}[{}_2G_1^\delta[{}_2G_1^\delta, {}_1G_1^\delta]] = 0$, $B_{2,1} = \frac{1}{2}{}_{112}G_3^\delta + \frac{1}{12}[{}_1G_1^\delta[{}_1G_1^\delta, {}_2G_1^\delta]] = 0$ and the other terms equal to zero, we recover the finite exponent

$$\begin{aligned} \delta u \mathcal{G}(.,\delta u) &= \delta u_1 B_{1,0} + \delta u_2 B_{0,1} = \delta u_1 {}_1G_1^\delta + \delta u_2 {}_2G_1^\delta \\ &= (\delta u_1, \delta u_2, \delta u_1 x_2, \delta u_1 x_3)^T \end{aligned}$$

and thus by direct integration of (32) with respect to t

$$\begin{aligned} F^\delta(x,\delta u) &= e^{\delta u \mathcal{G}(.,\delta u)} I_n\Big|_x \\ &= e^{\delta u_1 \frac{\partial}{\partial x_1} + \delta u_2 \frac{\partial}{\partial x_2} + \delta u_1 x_2 \frac{\partial}{\partial x_3} + \delta u_1 x_3 \frac{\partial}{\partial x_4}} I_n\Big|_x. \end{aligned}$$

In this case, the compatibility conditions reduce to combinatorics equalities deduced from (31).

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