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Guaranteed Cost Control for Uncertain Nonlinear time-delay Systems

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Abstract-- This paper deals with the problem of guaranteed cost control for a class of uncertain nonlinear time-delay systems. The nonlinearities are assumed to satisfy global Lipschitz conditions and the uncertain parameters are supposed to reside in a polytope. The aim is to find a state-feedback controller, such that the closed-loop system is not only asymptotically stable, but also has better performance. This problem is solved through a parameter-dependent approach, which has the potential to yield less conservative results than the quadratic framework. A numerical example shows the applicability of the proposed controller design method.

I. INTRODUCTION

The quadratic optimization problem has drawn much attention from researchers worldwide due to its wide applications in engineering systems. The optimal control strategy is based on the exact mathematical model and does not take uncertainties into consideration. Therefore, the frequently encountered parameter and nonlinear uncertainties often make the system performance deteriorate, or even destabilize a practical system. In such cases, Chang and Peng proposed the guaranteed cost control (GCC) for uncertain systems [1]. Since then, the problem of guaranteed cost control has been widely studied and many important results have been reported [2-11]. See, for instance, the GCC for continuous time system is considered in [2-5] and [6] investigates the GCC problem for discrete time systems. Theses works are also extended to time-delay systems in [7-9] and nonlinear uncertain systems in [10,11]. It is worth noting that most of the aforementioned GCC results for uncertain systems are within the well-known quadratic framework, which entails a fixed Lyapunov matrix for the whole uncertain domain. The quadratic framework has been generally regarded as being conservative, and recently many researchers try to propose parameter-dependent Lyapunov functions for systems with different types of uncertainties.

In this paper, we make an attempt to solve the problem of guaranteed cost control for a class of nonlinear uncertain time-delay systems. The nonlinearities are assumed to satisfy global Lipschitz conditions and the uncertain parameters are supposed to reside in a polytope. Our objective is to design a state-feedback controller, such that the closed-loop system is asymptotically stable and has a better performance. The problem is solved through a parameter-dependent approach, that is, we use different Lyapunov matrices for each vertex of the uncertain polytope which has the potential to yield less conservative results than the quadratic framework. We eliminate the product terms between the positive definitive matrix and system matrices by introducing a sufficiently small positive constant and slack matrix variable. A numerical example is provided to show the applicability of the proposed controller design method.

The notations used throughout the paper are fairly standard. The superscript "T" stands for matrix transposition; R^n denotes the n dimensional Euclidean space, $R^{m\times n}$ is the set of all $m\times n$ real matrices and the notation $P>0$ means that P is symmetric and positive definite. In addition, in symmetric block matrices or long matrix expressions, we use asterisk (*) as an ellipsis for the terms that are induced by symmetry and $diag\{\cdots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. PROBLEM DESCRIPTION

Consider the following nonlinear continuous system with state-delay:

$$\begin{aligned}&\dot{x}(t)=A_0x(t)+A_dx(t-h(t))+Ff(x(t),x(t-h(t)))+Bu(t)\\&x(t)=\phi(t),\ t\in[-h(t),0]\end{aligned}\tag{1}$$

where $x(t)\in R^n$ is the state vector, $u(t)\in R^m$ is the control input, $h(t)$ is the time-varying delay satisfying $0\le h(t)\le \bar{h}<\infty, \dot{h}(t)\le\tau<1$, where $\bar{h}$ and τ are real constant scalars, $\{\phi(t),\ t\in[-\bar{h},0]\}$ is a real-valued initial function, and $f(x(t),x(t-h(t)))$ represents nonlinear uncertainties.

The cost function is

$$J=\int_0^\infty x^T(t)Qx(t)+u^T(t)Ru(t)\tag{2}$$

where Q,R are given weighting matrices which are positive-definite symmetric matrices.

The system matrices are assumed to be unknown but belong to a given convex polyhedral domain, i.e.

$$\begin{aligned}&M:=[A_0,A_d,F,B]\in\Re\\&\Re:=\Big\{[A_0(\lambda),A_d(\lambda),F(\lambda),B(\lambda)]\Big|[A_0(\lambda),A_d(\lambda),F(\lambda),B(\lambda)]\\&=\sum_{i=1}^{s}\lambda_i[A_{0i}(\lambda),A_{di}(\lambda),F_i(\lambda),B_i(\lambda)];\sum_{i=1}^{s}\lambda_i=1,\lambda_i\ge0\Big\}\end{aligned}\tag{3}$$

The polytopic uncertainty has been widely used in the problems of robust control and filtering for uncertain systems [12,13].

Throughout out the paper, we make the following assumption:

Assumption 1 The nonlinear functions satisfy

(1) $f(0,0)=0$;

(2)(Lipschitz condition): For all $x_1,x_2,y_1,y_2\in R^n$, the nonlinear function sastisfies

$$\|f(x_1,x_2)-f(y_1,y_2)\|\le\|M_1(x_1-y_1)\|+\|M_2(x_2-y_2)\|$$

where M_1,M_2 are real constant matrices.

Construct a state-feedback controller

$$u(t)=Kx(t)\tag{4}$$

where K is the feedback gain to be determined.

By connecting the controller (4) to the original system (1), we have the closed-loop system as follows

$$\dot{x}(t) = \bar{A}_0 x(t) + \bar{A}_d x(t-h(t)) + \bar{F} f(x(t), x(t-h(t))) \tag{5}$$

where $\bar{A}_0 = A_0 + BK$, $\bar{A}_d = A_d$, $\bar{F} = F$. Then $\left[\bar{A}_0, \bar{A}_d, \bar{F}\right]$ can also be described by convex polyhedral domain, with the corresponding vertex matrices.

$$\bar{A}_{0i} = A_{0i} + B_i K,\ \bar{A}_{di} = A_{di},\ \bar{F}_i = F_i \quad i = 1, \cdots, s \tag{6}$$

Our objective is to develop a state-feedback controller of the form (4) such that for all admissible uncertainties and time-delay, the closed-loop system (5) is asymptotically stable and the cost function (2) has an upper bound.

We first introduce the following lemma, which will be used in our derivation.

Lemma 1 [14] Suppose M, N are two vectors, then for any scalar $\varepsilon > 0$ there holds

$$MN + N^T M^T \le \varepsilon M M^T + \varepsilon^{-1} N^T N \tag{7}$$

III. Main Results

To facilitate the presentation, we first solve the GCC problem for precisely known systems, that is, we assume that $M \in \Re$ is arbitrary but has a constant value. Then we have the following theorem for the closed-loop system (5).

Theorem 1 Consider system (1), assume that $M \in \Re$ is arbitrary but has a constant value, then the closed-loop system (5) is asymptotically stable if there exist matrices $0 < P = P^T \in R^{n \times n}$, $S \in R^{n \times n}$, $Q \in R^{n \times n}$ satisfying

$$\begin{bmatrix} \Delta & P\bar{A}_d \\ \bar{A}_d^T P & -(1-\tau)S + 2\varepsilon^{-1} M_2^T M_2 \end{bmatrix} < 0 \tag{8}$$

where $\Delta = \bar{A}_0^T P + P\bar{A}_0 + S + 2\varepsilon^{-1} M_1^T M_1 + \varepsilon P \bar{F} \bar{F}^T P + Q$.

Then, the cost function satisfies

$$J < x^T(0) P x(0) + \int_{-\bar{h}}^{0} x^T(t) S x(t) \tag{9}$$

Proof: Define a Lyapunov functional as

$$V(x(t)) = x^T(t) P x(t) + \int_{t-h(t)}^{t} x^T(\tau) S x(\tau) d\tau \tag{10}$$

where $P \in R^{n \times n}, S \in R^{n \times n}$ are positive-definite symmetric matrices. Then, along any trajectory of (5), we have

$$\begin{aligned} &\dot{V}(x(t)) = \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t) + x^T(t) S x(t) \\ &-(1-\dot{h}(t)) x^T(t-h(t)) S x(t-h(t)) \\ &= x^T(t)\left[\bar{A}_0^T P + P\bar{A}_0\right] x(t) + f^T(x(t), x(t-h(t))) \bar{F}^T P x(t) \\ &+ x^T(t) P \bar{F} f(x(t), x(t-h(t))) \\ &+ x^T(t) P A_d x(t-h(t)) + x^T(t-h(t)) A_d^T P x(t) + x^T(t) S x(t) \\ &-(1-\dot{h}(t)) x^T(t-h(t)) S x(t-h(t)) \end{aligned} \tag{11}$$

From assumption 1, we obtain

$$\left\| f(x(t), x(t-h(t))) \right\| \le \left\| M_1 x(t) \right\| + \left\| M_2 x(t-h(t)) \right\|$$

where M_1, M_2 are known matrices. Therefore, we have

$$\left\| f(x(t), x(t-h(t))) \right\|^2 \le 2\left\| M_1 x(t) \right\|^2 + 2\left\| M_2 x(t-h(t)) \right\|^2 \tag{12}$$

Note that Assumption 1 and Lemma 1 lead to

$$\begin{aligned} &x^T(t) P \bar{F} f(x(t), x(t-h(t))) \\ &+ f^T(x(t), x(t-h(t))) \bar{F}^T P x(t) \\ &\le \varepsilon x^T(t) P \bar{F} \bar{F}^T P x(t) \\ &+ \varepsilon^{-1} f^T(x(t), x(t-h(t))) f(x(t), x(t-h(t))) \\ &\le \varepsilon x^T(t) P \bar{F} \bar{F}^T P x(t) \\ &+ \varepsilon^{-1} 2(\left\| M_1 x(t) \right\|^2 + \left\| M_2 x(t-h(t)) \right\|^2) \\ &= \varepsilon x^T(t) P \bar{F} \bar{F}^T P x(t) \\ &+ \varepsilon^{-1} 2(x^T(t) M_1^T M_1 x(t) \\ &+ x^T(t-h(t)) M_2^T M_2 x(t-h(t))) \end{aligned} \tag{13}$$

where ε is any positive constant.

From (11)-(13), it follows that

$$\begin{aligned} &\dot{V}(x(t)) \le x^T(t)\left[\bar{A}_0^T P + P\bar{A}_0\right] x(t) + x^T(t) P\bar{A}_d x(t-h(t)) \\ &+ x^T(t-h(t)) \bar{A}_d^T P x(t) + 2\varepsilon^{-1}(x^T(t) M_1^T M_1 x(t) \\ &+ x^T(t-h(t)) M_2^T M_2 x(t-h(t))) + \varepsilon x^T(t) P \bar{F} \bar{F}^T P x(t) \\ &+ x^T(t) S x(t) - (1-\tau) x^T(t-h(t)) S x(t-h(t)) \\ &= [x(t)\ \ x(t-h(t))] \begin{bmatrix} \Omega & P\bar{A}_d \\ \bar{A}_d^T P & -(1-\tau)S + 2\varepsilon^{-1} M_2^T M_2 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h(t)) \end{bmatrix} \end{aligned} \tag{14}$$

where $\Omega = \bar{A}_0^T P + P\bar{A}_0 + S + 2\varepsilon^{-1} M_1^T M_1 + \varepsilon P \bar{F} \bar{F}^T P$

By Schur complement (8) guarantees

$$\dot{V}(x(t)) < -x^T(t) Q x(t) < 0 \tag{15}$$

By the standard Lyapunov theorem, the closed-loop system (5) is asymptotically stable.

By integrating both sides of (14) from 0 to ∞, we have

$$J < V(x(0)) = x^T(0) P x(0) + \int_{-h(0)}^{0} x^T(t) Q x(t) < x^T(0) P x(0) + \int_{-\bar{h}}^{0} x^T(t) Q x(t)$$

which completes the proof.

Then, the following theorem provides sufficient conditions for the GCC problem.

Theorem 2 Consider system (1), assume that $M \in \Re$ is arbitrary but has a constant value. Then the GCC problem can be solved by the following LMI-based optimization problem:

$$\min(r_1 + Tr(R_2)) \quad \text{subject to}$$

$$\begin{bmatrix} -\sigma^{-2}R^{-1} & -\sigma^{-1}H & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -\sigma^{-1}(V_1 + V_1^T - L) & -\sigma^{-1}V_1^T + V_1^T A_0^T + H^T B^T & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\sigma^{-1}L & A_d V & \varepsilon F & L & L & L M_1^T & 0 \\ * & * & * & -(1-\tau)(V_2 + V_2^T - Z) & 0 & 0 & 0 & 0 & V_2^T M_2^T \\ * & * & * & * & -\varepsilon I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Z & 0 & 0 & 0 \\ * & * & * & * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & * & * & * & -0.5\varepsilon I & 0 \\ * & * & * & * & * & * & * & * & -0.5\varepsilon I \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} r_1 & x^T(0) \\ * & L \end{bmatrix} > 0 \tag{17}$$

$$\begin{bmatrix} R_2 & U^T \\ * & Z \end{bmatrix} > 0 \tag{18}$$

where σ is sufficiently small positive constant, $H \in R^{m\times n}, V_1 \in R^{m\times n}, V_2 \in R^{m\times n}$, $0 < L \in R^{n\times n}$, $R_2 \in R^{n\times n}$, $0 < Z \in R^{n\times n}$, scalar $r_1 > 0$ are the matrix variables to be determined.

Under the above conditions, the corresponding controller is $K = HV_1^{-1}$.

Proof: First, since the (2,2) block of (16) implies $V_1 + V_1^T - L > 0$, by noting $L > 0$ we know that V_1 is invertible. Therefore, $K = HV_1^{-1}$ exists if conditions (16)-(18) are satisfied. Substituting $H = KV_1$ into (16) yields

$$\begin{bmatrix} -\sigma^{-2}R^{-1} & -\sigma^{-1}KV_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -\sigma^{-1}(V_1+V_1^T-L) & -\sigma^{-1}V_1^T + V_1^T A_0^T + V_1^T K^T B^T & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\sigma^{-1}L & A_d V_2 & \varepsilon F & L & L & LM_1^T & 0 \\ * & * & * & -(1-\tau)(V_2+V_2^T-Z) & 0 & 0 & 0 & 0 & V_2^T M_2^T \\ * & * & * & * & -\varepsilon I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Z & 0 & 0 & 0 \\ * & * & * & * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & * & * & * & -0.5\varepsilon I & 0 \\ * & * & * & * & * & * & * & * & -0.5\varepsilon I \end{bmatrix} < 0 \tag{19}$$

By following similar lines as in the proof of Theorem 1 in [16], (19) is equivalent to

$$\begin{bmatrix} -\sigma^{-2}R^{-1} & \sigma^{-1}KL & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -\sigma^{-1}L & -\sigma^{-1}L + LA_0^T + LK^T B^T & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\sigma^{-1}L & A_d Z & \varepsilon F & L & L & LM_1^T & 0 \\ * & * & * & -(1-\tau)Z & 0 & 0 & 0 & 0 & ZM_2^T \\ * & * & * & * & -\varepsilon I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Z & 0 & 0 & 0 \\ * & * & * & * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & * & * & * & -0.5\varepsilon I & 0 \\ * & * & * & * & * & * & * & * & -0.5\varepsilon I \end{bmatrix} < 0 \tag{20}$$

Define the following matrices $P = L^{-1}, S = Z^{-1}$ and substituting them into (20) yields

$$\begin{bmatrix} -\sigma^{-2}R^{-1} & \sigma^{-1}KP^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -\sigma^{-1}P^{-1} & \Sigma & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\sigma^{-1}P^{-1} & A_d S^{-1} & \varepsilon F & P^{-1} & P^{-1} & P^{-1}M_1^T & 0 \\ * & * & * & -(1-\tau)S^{-1} & 0 & 0 & 0 & 0 & S^{-1}M_2^T \\ * & * & * & * & -\varepsilon I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -S^{-1} & 0 & 0 & 0 \\ * & * & * & * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & * & * & * & -0.5\varepsilon I & 0 \\ * & * & * & * & * & * & * & * & -0.5\varepsilon I \end{bmatrix} < 0 \tag{21}$$

where $\Sigma = -\sigma^{-1}P^{-1} + P^{-1}A_0^T + P^{-1}K^T B^T$.

Considering the matrix definition of $\bar{A}_0, \bar{A}_d, \bar{F}$ in (5), and by Schur complement, (21) is equivalent to

$$\begin{bmatrix} -\sigma^{-1}P^{-1} + P^{-1}K^T RKP^{-1} & -\sigma^{-1}P^{-1}(I+\sigma\bar{A}_0)^T & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -\sigma^{-1}P^{-1} & A_d S^{-1} & \varepsilon\bar{F} & P^{-1} & P^{-1} & P^{-1}M_1^T & 0 \\ * & * & -(1-\tau)S^{-1} & 0 & 0 & 0 & 0 & S^{-1}M_2^T \\ * & * & * & -\varepsilon I & 0 & 0 & 0 & 0 \\ * & * & * & * & -S^{-1} & 0 & 0 & 0 \\ * & * & * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & * & * & -0.5\varepsilon I & 0 \\ * & * & * & * & * & * & * & -0.5\varepsilon I \end{bmatrix} < 0 \tag{22}$$

Performing a congruence transformation to (22) by $T = diag\{P, I, I, I, I, I, I\}$ yields

$$\begin{bmatrix} -\sigma^{-1}P+K^TRK & -\sigma^{-1}(I+\sigma\bar{A}_0)^T & 0 & 0 & 0 & 0 & 0 & 0 \\ * & -\sigma^{-1}P^{-1} & \bar{A}_dS^{-1} & \varepsilon\bar{F} & P^{-1} & P^{-1} & P^{-1}M_1^T & 0 \\ * & * & -(1-\tau)S^{-1} & 0 & 0 & 0 & 0 & S^{-1}M_2^T \\ * & * & * & -\varepsilon I & 0 & 0 & 0 & 0 \\ * & * & * & * & -S^{-1} & 0 & 0 & 0 \\ * & * & * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & * & * & -0.5\varepsilon I & 0 \\ * & * & * & * & * & * & * & -0.5\varepsilon I \end{bmatrix} < 0 \tag{23}$$

then, by Schur complement, (23) is equivalent to

$$\begin{bmatrix} \psi & \bar{A}_dS^{-1} & \varepsilon\bar{F} & P^{-1} & P^{-1} & P^{-1}M_1^T & 0 \\ * & -(1-\tau)S^{-1} & 0 & 0 & 0 & 0 & S^{-1}M_2^T \\ * & * & -\varepsilon I & 0 & 0 & 0 & 0 \\ * & * & * & -S^{-1} & 0 & 0 & 0 \\ * & * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & * & -0.5\varepsilon I & 0 \\ * & * & * & * & * & * & -0.5\varepsilon I \end{bmatrix} < 0 \tag{24}$$

where $\psi = -\sigma^{-1}P^{-1} + (\sigma^{-1}I + \bar{A}_0)(\sigma^{-1}P - K^TRK)^{-1}(\sigma^{-1}I + \bar{A}_0)^T$.

Now by using the matrix inverse lemma, we have

$$\begin{aligned} &(\sigma^{-1}P - K^TRK)^{-1} \\ &= \sigma P^{-1} + \sigma P^{-1}K^T(R^{-1} - K\sigma P^{-1}K^T)^{-1}K\sigma P^{-1} \\ &= \sigma P^{-1} + \sigma P^{-1}K^T[R + \sigma RK(P - \sigma K^TRK)^{-1}K^TR]\cdot K\sigma P^{-1} \\ &= \sigma P^{-1} + \sigma^2P^{-1}K^TRKP^{-1} + \sigma^3P^{-1}K^TRKP^{-1}\cdot \\ &(P - \sigma K^TRK)^{-1}K^TRKP^{-1} \\ &= \sigma P^{-1} + \sigma^2P^{-1}K^TRKP^{-1} + O(\sigma^3) \end{aligned} \tag{25}$$

$$\begin{aligned} &\psi = -\sigma^{-1}P^{-1} + (\sigma^{-1}I + \bar{A}_0)(\sigma P^{-1} - K^TRK)^{-1}\cdot \\ &(\sigma^{-1}I + \bar{A}_0)^T \\ &= -\sigma^{-1}P^{-1} + (\sigma^{-1}I + \bar{A}_0)\cdot \\ &(\sigma P^{-1} + \sigma^2P^{-1}K^TRKP^{-1} + O(\sigma^3))(\sigma^{-1}I + \bar{A}_0)^T \\ &= \bar{A}_0P^{-1} + P^{-1}\bar{A}_0^T + P^{-1}K^TRKP^{-1} + O(\sigma) \end{aligned} \tag{26}$$

Obviously, since σ is a sufficiently small positive constant, (24) is equivalent to

$$\begin{bmatrix} \Upsilon & \bar{A}_dS^{-1} & \varepsilon\bar{F} & P^{-1} & P^{-1} & P^{-1}M_1^T & 0 \\ * & -(1-\tau)S^{-1} & 0 & 0 & 0 & 0 & S^{-1}M_2^T \\ * & * & -\varepsilon I & 0 & 0 & 0 & 0 \\ * & * & * & -S^{-1} & 0 & 0 & 0 \\ * & * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & * & -0.5\varepsilon I & 0 \\ * & * & * & * & * & * & -0.5\varepsilon I \end{bmatrix} < 0 \tag{27}$$

where $\Upsilon = \bar{A}_0P^{-1} + P^{-1}A_0^T + P^{-1}K^TRKP^{-1}$.

Performing congruence transformations to (27) by $T = diag\{P, S, I, I, I, I\}$ yield

$$\begin{bmatrix} \Xi & P\bar{A}_d & \varepsilon P\bar{F} & I & I & M_1^T & 0 \\ * & -(1-\tau)S & 0 & 0 & 0 & 0 & M_2^T \\ * & * & -\varepsilon I & 0 & 0 & 0 & 0 \\ * & * & * & -S^{-1} & 0 & 0 & 0 \\ * & * & * & * & -Q^{-1} & 0 & 0 \\ * & * & * & * & * & -0.5\varepsilon I & 0 \\ * & * & * & * & * & * & -0.5\varepsilon I \end{bmatrix} < 0 \tag{28}$$

where $\Xi = P\bar{A}_0 + \bar{A}_0^TP + K^TRK$.

By Schur complement, (28) is equivalent to

$$\begin{bmatrix} \mathrm{E} & P\bar{A}_{di} \\ \bar{A}_{di}^TP & -(1-\tau)S + 2\varepsilon^{-1}M_2^TM_2 \end{bmatrix} < 0 \tag{29}$$

where $\mathrm{E} = \bar{A}_{0i}^TP + P\bar{A}_{0i} + S + 2\varepsilon^{-1}M_1^TM_1 + \varepsilon P\bar{F}\bar{F}^TP + Q + K^TRK$.

Then, by following similar lines as in the proof of Theorem 1, (29) guarantees

$$\dot{V}(x(t)) < -x^T(t)(Q + K^TRK)x(t) < 0 \tag{30}$$

Integrating both sides of (30) from 0 to ∞ , we have the performance (9), from which we can conclude that

$$J < x^T(0)L^{-1}x(0) + \int_{-\bar{h}}^{0} x^T(t)Z^{-1}x(t) \tag{31}$$

Define

$$x^T(0)L^{-1}x(0) < r_1 \tag{32}$$

then, by Schur complement, (32) is equivalent to (17).

In addition, since $Tr(AB) = Tr(BA)$, we have

$$\int_{-\bar{h}}^{0} x^T(t)Z^{-1}x(t) = \int_{-\bar{h}}^{0} Tr(x(t)x^T(t)Z^{-1}) \tag{33}$$

Define $\int_{-h}^{0} x(t)x^T(t) = UU^T$, then, by Schur complement, (18) implies

$$\int_{-\bar{h}}^{0} x^T(t)Z^{-1}x(t) = Tr(UU^TZ^{-1}) = Tr(U^TZ^{-1}U) < Tr(R_2) \tag{34}$$

From the above, we can obtain that system (5) satisfies cost $r_1 + Tr(R_2)$ which is larger than minimum value J^*. So, the solution strategy is an approximating process. Finally, we could obtain $r_1 + Tr(R_2)$ which is larger than J^* but very close to it. Then the proof is completed.

Remark 1 It is worth noting that the derivation of Theorem 2 does not follow the standard procedures. As can be seen above, by introducing the sufficiently small positive constant σ and the slack matrix variable V_1, V_2, we eventually eliminate the product terms between the positive definitive matrix and system matrices. This idea, which originates from the work in [15], will enable us to obtain parameter-dependent results for the GCC problem addressed in the paper, yielding the following corollary.

Corollary 1 Consider system (1), assume $M \in \Re$ represents an uncertain system. Then the GCC problem can be solved by the following LMI-based optimization problem:

$\min(r_1 + Tr(R_2))$ subject to (35),(17),(18).

$$
\begin{bmatrix}
-\sigma^{-2}R^{-1} & H & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & -\sigma^{-1}(L-V_1-V_1^T) & -\sigma^{-1}V_1^T+V_1^T A_{0i}^T+H^T B_i^T & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\sigma^{-1}L & A_{di}Z & \varepsilon F_i & L & L & LM_1^T & 0 \\
* & * & * & -(1-\tau)(Z-V_2-V_2^T) & 0 & 0 & 0 & 0 & V_2^T M_2^T \\
* & * & * & * & -\varepsilon I & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -Z & 0 & 0 & 0 \\
* & * & * & * & * & * & -Q^{-1} & 0 & 0 \\
* & * & * & * & * & * & * & -0.5\varepsilon I & 0 \\
* & * & * & * & * & * & * & * & -0.5\varepsilon I
\end{bmatrix} < 0 \tag{35}
$$

where σ is a sufficiently small positive constant, $H \in R^{m\times n}, V_1 \in R^{m\times n}, V_2 \in R^{m\times n}$, $0 < L \in R^{n\times n}, R_2 \in R^{n\times n}, 0 < Z \in R^{n\times n}$, scalar $r_1 > 0$ are the matrix variables to be determined.

Moreover, under the above conditions, the corresponding controller is $K = HV_1^{-1}$.

Remark 2 In the case when there is no nonlinear perturbation in system (1), that is $f(x(t), x(t-h(t)))$ =0, Theorem 2 is specialized as follows.

Corollary 2 Consider system (1), assume that $M \in \Re$ is arbitrary but has a constant value, and $f(x(t), x(t-h(t)))$ =0. Then the GCC problem can be solved by the following LMI-based optimization problem:

$$\min(r_1 + Tr(R_2)) \quad \text{subject to (36),(17),(18).}$$

$$
\begin{bmatrix}
-\sigma^{-2}R^{-1} & -\sigma^{-1}H & 0 & 0 & 0 & 0 \\
* & -\sigma^{-1}(V_1+V_1^T-L) & -\sigma^{-1}V_1^T+V_1^T A_0^T+H^T B^T & 0 & 0 & 0 \\
* & * & -\sigma^{-1}L & A_d V & L & L \\
* & * & * & -(1-\tau)(V_2+V_2^T-Z) & 0 & 0 \\
* & * & * & * & -Z & 0 \\
* & * & * & * & * & -Q^{-1}
\end{bmatrix} < 0 \tag{36}
$$

where σ is a sufficiently small positive constant, $H \in R^{m\times n}, V_1 \in R^{m\times n}, V_2 \in R^{m\times n}, 0 < L \in R^{n\times n}$, $R_2 \in R^{n\times n}, 0 < Z \in R^{n\times n}$, scalar $r_1 > 0$ are the matrix variables to be determined.

Remark 3 In the case when there is no time-delay in system (1), that is A_d =0, $M_2 = 0$, Theorem 2 is specialized as follows.

Corollary 3 Consider system (1), assume that $M \in \Re$ is arbitrary but has a constant value, and A_d =0, $M_2 = 0$. Then the GCC problem can be solved by the following LMI-based optimization problem:

$$\min(r_1 + Tr(R_2)) \quad \text{subject to (37),(17),(18).}$$

$$
\begin{bmatrix}
-\sigma^{-2}R^{-1} & -\sigma^{-1}H & 0 & 0 & 0 & 0 \\
* & -\sigma^{-1}(V_1+V_1^T-L) & -\sigma^{-1}V_1+V_1A_0^T+H^TB^T & 0 & 0 & 0 \\
* & * & -\sigma^{-1}L & \varepsilon F & L & LM_1^T \\
* & * & * & -\varepsilon I & 0 & 0 \\
* & * & * & * & -Q^{-1} & 0 \\
* & * & * & * & * & -0.5\varepsilon I
\end{bmatrix} < 0 \tag{37}
$$

where σ is a sufficiently small positive constant, $H \in R^{m\times n}, V_1 \in R^{m\times n}, 0 < L \in R^{n\times n}$, $R_2 \in R^{n\times n}$, scalar $r_1 > 0$ are the matrix variables to be determined.

Remark 4 Since LMIs (36) and (37) do not contain product terms between the Lyapunov matrices and system matrices, Corollaries 2 and 3 can be readily extended to polytopic uncertain cases, yielding parameter-dependent robust GCC results.

IV. Illustrative Examples

Consider the following uncertain nonlinear time-delay system:

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 1 & -1+0.1\alpha \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t-0.2\sin(t)) \\
&+ \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} f(x(t), x(t-0.2\sin(t)) + \begin{bmatrix} -1 \\ 2 \end{bmatrix} u(t)
\end{aligned} \tag{38}
$$

where α denotes an uncertain parameter, which satisfy $|\alpha| \le 1$, $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\bar{h} = 1$. Then, the system matrices could be described as a convex polyhedron with two vertices. Assume $\tau = 0.2$ and the nonlinear uncertainties satisfy $\|f(x(t), x(h-(t))\| \le \|M_1 x\| + \|M_2 x\|$, where $M_1 = M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The performance cost of system (1) is as Eq.(2). Weighting matrices are $Q = I_{2\times 2}$, $R = I_{1\times 1}$.

Using software Matlab Toolbox, setup the optimization problem and solve it. The designed optimal guaranteed cost controller is given by $K = HV_1^{-1}$.

$$H = [\text{-}1.3021 \quad 1.6634],$$

$$V_1 = \begin{bmatrix} 3.5183 & 0.4454 \\ 0.4451 & 1.6629 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 3.5782 & \text{-}0.4717 \\ \text{-}0.4716 & 1.7789 \end{bmatrix}.$$

So, the desired state feedback control law is

$$u(t) = [\text{-}0.5140 \quad 1.1380] x(t).$$

corresponding cost is J^* =1.6328, have less conservatives than non-parameter dependent method, in which case, the cost is J^* =1.6547.

V. CONCLUDING REMARKS

In this paper, we investigate the problem of guaranteed cost control for a class of uncertain nonlinear systems with time-delay. The problem is solved through a parameter-dependent approach, which can lead to potentially less conservative results. The obtained results are expressed as LMI-based optimization problems that can be easily implemented by using standard numerical software.

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