Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

A New Stability Analysis of Time Delay Control for Input/Output Linearizable Plants

Je Hyung Jung, Pyung-Hun Chang and Oh-Seok Kwon

***Abstract*—Time Delay Control(TDC) is a robust nonlinear control scheme that uses Time Delay Estimation(TDE). The TDC has a very simple structure, and its application on a real system needs its stability be guaranteed. The existing work has proposed sufficient stability condition of TDC for input/output linearizable plants in that it has been assumed that time delay was infinitesimal. However, it is impossible to implement infinitesimal time delay in a real system. Therefore, in this research we propose a new sufficient stability condition of TDC for input/output linearizable plants with finite a time delay. It can be verified by simulation results that the existing sufficient stability condition fails even under small time delays while the proposed condition performs well.**

I. INTRODUCTION

Time Delay Control(TDC) is a control technique that estimates and compensates system uncertainty, i.e., unmodeled dynamics, parameter variations and disturbances by utilizing time-delayed signal of some system variables [1]. Owing to the effectiveness and efficiency due to the Time -Delayed Estimation (TDE), TDC displays particularly robust performance despite its relatively simple gain selection procedure, which is attributed to the unusually compact structure of TDC. For this reason, we are convinced that TDC deserves serious research work to investigate further improvements.

The problem of *stability analysis* is an important piece of work that stands out of many aspects of TDC that demand further research. Hence this paper attempts to present a refined stability analysis together with a practical stability condition useful for control design. Provided below are the background and context associated with this research.

The stability analysis of TDC becomes complicated and difficult due to time delay terms that inevitably appear in the closed-loop dynamics. And, it becomes even more complicated when the plant to be controlled happens to be a nonlinear multivariable system. In [2], necessary and sufficient stability condition for Linear Time Invariant (LTI) Single Input Single Output (SISO) plants has been presented based on Nyquist stability criterion, whereas in [3], sufficient condition has been derived from Nyquist stability criterion and Kharitonov method. These analyses though complete, limited to LTI SISO plants and their application to real systems tend to be complicated.

For nonlinear multivariable plants, on the other hand, stability analysis has been presented by Youcef-Toumi [4]. The analysis is based on a set of assumptions: time delay $L \to 0$; the plant has exponentially stable zero dynamics [5]; and the desired trajectory and its derivatives are bounded. The analysis results in a sufficient condition $\| \mathbf{I} - \mathbf{B}(\mathbf{x})\bar{\mathbf{B}}^{-1} \| < 1$ with $\mathbf{B}(\mathbf{x})$ being an input distribution matrix obtained during the input/output linearization procedure and $\bar{\mathbf{B}}$ being a constant matrix of TDC, which is relatively easy to determine for controller design. $\|\bullet\|$ denotes matrix norm.

Although the assumption of $L \to 0$ is obviously unrealizable, it is expected that with a sufficiently small L, $\bar{\mathbf{B}}$ leads to $\| \mathbf{I} - \mathbf{B}(\mathbf{x})\bar{\mathbf{B}}^{-1} \| < 1$, that satisfies the condition for stability. However, we have observed that this is not the case: the closed-loop systems based on $\bar{\mathbf{B}}$ could actually drive the system unstable even with L as small as 0.001s. As a result, we have realized the importance and necessity of addressing this issue for the practical implementation of TDC in real systems. In this paper, therefore, we are going to present stability analysis for the case of *finite* L as well as a corresponding stability condition having a form similar to $\| \mathbf{I} - \mathbf{B}(\mathbf{x})\bar{\mathbf{B}}^{-1} \| < 1$ that enables a selection of $\bar{\mathbf{B}}$ to guarantee stability.

This paper is organized as follows. In Section II, we briefly review the Input/Output Linearization(IOL) and TDC law and deal with the problem of the previous stability criterion. Section III describes our stability analysis and derivation of the stability condition. In Section IV, we examine the stability criterion through simulation. Finally, in section V the results are summarized and conclusions are drawn.

This work was supported in part by the Korea Science and Engineering Foundation under HWRS-ERC project.

J. H. Jung is with the Department of Mechanical Engineering, Korea Advanced Institute of Science and Technology (KAIST), 373-1, Kusong-dong, Yusong-gu, Daejon, South KOREA (corresponding author to provide phone: +82-42-869-3266; fax: +82-42-869-5226; e-mail: gory@mecha.kaist.ac.kr).

P.-H. Chang is with the Department of Mechanical Engineering, KAIST (e-mail: phchang@mecha.kaist.ac.kr).

O.-S. Kwon is with the Intelligent Manufacturing Group, LG Production engineering Research Center, 451-713, Jinwuy-Myun, Pyungtaek-Si, Kyunggi-Do, South KOREA (e-mail: mesign@lge.com).

II. TDC AND PROBLEM WITH THE EXISTING STABILITY CONDITION

A. Input/Output Linearization

In this subsection, we briefly describe input/output linearization technique [4]-[5]. Consider a general system with m inputs, m outputs, and n states, as described by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u} \\ \mathbf{y} &= \mathbf{c}(\mathbf{x}) \end{aligned} \quad (1)$$

where, $\mathbf{x} \in \Re^n$ is the state vector, $\mathbf{u} \in \Re^m$ is the input vector and $\mathbf{y} \in \Re^m$ is the output vector. $\mathbf{f} : \Re^n \to \Re^n$, $\mathbf{G} : \Re^n \to \Re^{n \times m}$, $\mathbf{c} : \Re^n \to \Re^m$ are assumed smooth functions of state vector $\mathbf{x}$. And the following $\mathbf{g}_j$ denote the jth column of matrix $\mathbf{G}$.

In the input/output linearization procedure, (1) can be written in matrix form as

$$\mathbf{Dy} = \mathbf{a}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u} \quad (2)$$

where, $\mathbf{D} \equiv diag(d^{r_i} / dt^{r_i})$, $r_i\,(i = 1, \cdots, m)$ denotes relative degree defined in [4]-[5], d^{r_i} / dt^{r_i} denotes r_i th derivative with respect to time, $\mathbf{a(x)}$ and $\mathbf{B(x)}$ are as follows,

$$\mathbf{a}(\mathbf{x}) = [L_{\mathbf{f}}^{r_1}(c_1(\mathbf{x})) \cdots L_{\mathbf{f}}^{r_m}(c_m(\mathbf{x}))]^T$$
$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} L_{\mathbf{g}_1}(L_{\mathbf{f}}^{r_1-1}(c_1(\mathbf{x}))) \cdots L_{\mathbf{g}_m}(L_{\mathbf{f}}^{r_1-1}(c_1(\mathbf{x}))) \\ \vdots \qquad\qquad \vdots \\ L_{\mathbf{g}_1}(L_{\mathbf{f}}^{r_m-1}(c_m(\mathbf{x}))) \cdots L_{\mathbf{g}_m}(L_{\mathbf{f}}^{r_m-1}(c_m(\mathbf{x}))) \end{bmatrix} \quad (3)$$

where, $L_{\mathbf{f}}(\varphi(\mathbf{x})) : \Re^n \to \Re$ and $L_{\mathbf{g}_j}(\varphi(\mathbf{x})) : \Re^n \to \Re$ stand for the Lie derivative of $\varphi(\mathbf{x})$ with respect to $\mathbf{f(x)}$ and $\mathbf{g}_j\mathbf{(x)}$ respectively.

If $\mathbf{B(x)}$ is nonsingular, then the following control law is generated.

$$\mathbf{u} = \mathbf{B}^{-1}(\mathbf{x})[-\mathbf{a}(\mathbf{x}) + \mathbf{v}] \quad (4)$$

where, $\mathbf{v} \in \Re^m$ is the new input vector and ith component of $\mathbf{v}$ is given by

$$v_i = y_{d_i}^{(r_i)} + \gamma_{1i} e_i^{(r_i - 1)} + \cdots + \gamma_{r_i i} e_i \quad (5)$$

where, the ith component of error vector is defined as $e_i \triangleq y_{d_i} - y_i$.

And, substituting (4) into (2), we obtain the following m decoupled linear SISO system.

$$\mathbf{Dy} = \mathbf{v} \quad (6)$$

The ith component of (6) is the error dynamics described as follows:

$$e_i^{(r_i)} + \gamma_{1i} e_i^{(r_i - 1)} + \cdots + \gamma_{r_i i} e_i = 0 \quad (7)$$

in that, if the parameters $\gamma_{1i}, \cdots, \gamma_{r_i i}$ are chosen so that the roots of the following characteristic equation lie in the left half plane of s,

$$s^{r_i} + \gamma_{1i} s^{r_i - 1} + \cdots + \gamma_{r_i i} s = 0 \quad (8)$$

then e_i asymptotically approaches zero. If, in addition, the zero dynamics of (1) is exponentially stable and the desired trajectory and its derivatives are bounded, then the closed system is stable.

The control law of (4) works only when $\mathbf{a(x)}$ and $\mathbf{B(x)}$ are known accurately. If, however, there are uncertainties in the model, the system can no longer be linearized as in (6), and consequently, closed loop stability of the system can no longer be guaranteed either. Therefore, in the following subsection, we describe Time Delay Control law [1]-[4],[11] which possesses excellent robustness in view of uncertain system dynamics, unpredicted disturbances, and parameter variations.

B. Time Delay Control

The TDC law associated with the input/output linearization discussed above [4] is provided here.

The relationship in (2) can be rearranged to the following form,

$$\mathbf{Dy} = \mathbf{H}(\mathbf{x}) + \bar{\mathbf{B}}\mathbf{u} \quad (9)$$

where, $\mathbf{H(x)}$ is given by

$$\mathbf{H}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) + (\mathbf{B}(\mathbf{x}) - \bar{\mathbf{B}})\mathbf{u} \quad (10)$$

and $\bar{\mathbf{B}}$ is a constant matrix, which is chosen on the basis of $\mathbf{B(x)}$.

Let control input $\mathbf{u}$ be

$$\mathbf{u} = \bar{\mathbf{B}}^{-1}(-\hat{\mathbf{H}} + \mathbf{v}) \quad (11)$$

where, the ith component of $\mathbf{v}$ is identical to (5) and $\hat{\mathbf{H}}$ is the estimation of $\mathbf{H}$. If $\hat{\mathbf{H}} = \mathbf{H}$, then (6) is satisfied.

TDC uses the following estimation method to determine $\hat{\mathbf{H}}$. If time delay L is sufficiently small, then the following approximation holds.

$$\mathbf{H}(t) \cong \hat{\mathbf{H}}(t) = \mathbf{H}(t - L) = \mathbf{Dy}(t - L) - \bar{\mathbf{B}}\mathbf{u}(t - L) \quad (12)$$

Substituting (12) into (11), we could obtain TDC law as follows.

$$\mathbf{u}(t) = \mathbf{u}(t - L) + \bar{\mathbf{B}}^{-1}(-\mathbf{Dy}(t - L) + \mathbf{v}(t)) \quad (13)$$

According to (13), the advantage of TDC is that it does not require any real time computation of nonlinear dynamics and the uncertainties, thus, it only needs small computation power [1]-[4],[11].

C. The existing stability criterion of TDC and its drawbacks

The existing sufficient stability condition of TDC for input/output linearizable plants is proposed as follows: under the condition that zero dynamics of the plant is exponentially stable and the desired trajectory and its derivatives are bounded [4], following inequality holds.

$$\left\| \mathbf{I} - \mathbf{B}(\mathbf{x})\bar{\mathbf{B}}^{-1} \right\| < 1 \tag{14}$$

where, $\mathbf{I}$ is an $m \times m$ identity matrix and $\bar{\mathbf{B}}$ is an $m \times m$ constant matrix chosen on the basis of $\mathbf{B(x)}$. Yet, we have observed in simulations and experiments that TDC is unstable in spite of satisfying (14). One of such simple tests is TDC of a DC motor. Fig. 1 shows the closed loop system of DC motor with TDC[1], whereas Fig. 2 shows the real part of its dominant pole[2]. Observing Fig. 2, in spite of satisfying (14) (in this case, $\mathrm{B(x)=1/J}$, $\bar{\mathrm{B}}^{-1}=\bar{\mathrm{J}}$), the system is unstable near 1 and the unstable region increases with the time delay. This phenomenon occurs as it has been assumed in the derivation of (14) that the time delay L is sufficient small ($L \to 0$). In general, when TDC is applied for a real system, we use sampling time interval as time delay L so it is impossible to implement an infinitesimal time delay in a real system. Therefore, in the following section , we treat stability analysis of TDC for the real system of which time delay L is finite.

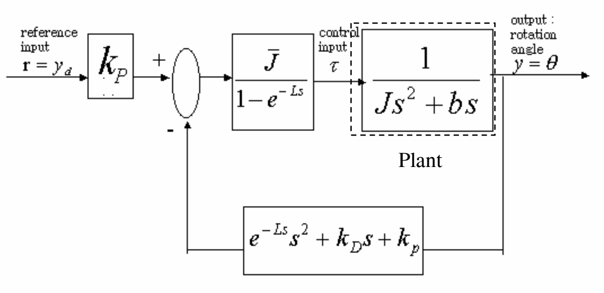

Fig. 1. Block Diagram of a DC motor with TDC

III. A NEW TDC STABILITY ANALYSIS

In this section, under the condition that the plant is input/output linearizable [6] and that the zero dynamics of the plant is exponentially stable, we will show TDC stability analysis for the real system of which time delay L is finite. And using this result, we will derive stability condition and

[1] The closed loop system in Fig. 1 may be considered as neutral type system if reference input $r = 0$. In this paper, open loop systems(plants) that we deal with have no time delay and are input/output linearizable.

[2] In order to obtain the dominant pole, e^{-Ls} in Fig. 1 was approximated by (3,3) Padeʹ approximant [12].

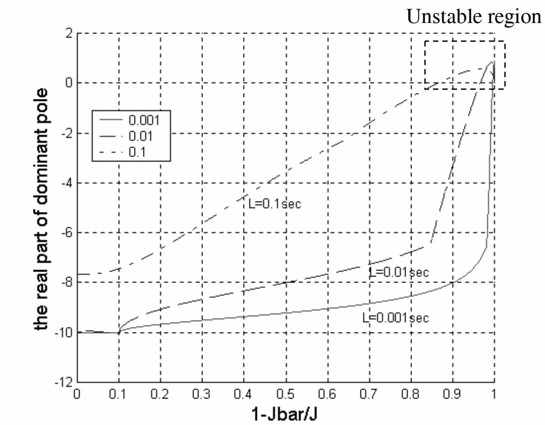

Fig. 2. Real part of the dominant pole of Fig. 1

criterion i.e., stable range of $\bar{\mathbf{B}}$.

A. TDC stability Analysis

In the subsection II.B, if $\hat{\mathbf{H}} = \mathbf{H}$, then the ith relation of error dynamics is identical to (7). But, in a real system, $\hat{\mathbf{H}} \neq \mathbf{H}$ due to time delay $L \neq 0$. Therefore, substituting (11), and (12) into (9), the error dynamics of a real system is obtained as follows

$$\mathbf{v}(t) - \mathbf{D}\mathbf{y}(t) = \mathbf{H}(t-L) - \mathbf{H}(t) \tag{15}$$

If we define error vector $\boldsymbol{\varepsilon}(t)$ as $\boldsymbol{\varepsilon}(t) = \mathbf{v}(t) - \mathbf{D}\mathbf{y}(t)$, then the relation of $\varepsilon_i(t)$ and $e_\mathrm{i}(t)$ would be

$$\varepsilon_\mathrm{i} = e_i^{(r_i)} + \gamma_{1i} e_i^{(r_i - 1)} + \ldots + \gamma_{r_i i} e_i \tag{16}$$

(16) can be presented in vector form as follows.

$$\boldsymbol{\varepsilon} = \mathbf{De} + \mathbf{K}_1 \mathbf{D}_1 \mathbf{e} + \ldots + \mathbf{K}_\mathrm{p} \mathbf{D}_\mathrm{p} \mathbf{e} + \ldots + \mathbf{K}_\mathrm{r} \mathbf{D}_\mathrm{r} \mathbf{e} \tag{17}$$

where, $\mathbf{D}_\mathrm{p} \equiv diag(d^{r_i - \mathrm{p}} / dt^{r_i - \mathrm{p}})$ for $\mathrm{p} = 1, \ldots, \mathrm{r}$, $\mathrm{r} \equiv \max(r_i)$ for $i = 1, \ldots, m$ and $\mathbf{K}_1, \cdots, \mathbf{K}_\mathrm{p}, \cdots, \mathbf{K}_\mathrm{r}$ are constant diagonal matrices defined as

$$\mathbf{K}_1 = \begin{bmatrix} \gamma_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \gamma_{1m} \end{bmatrix}, \cdots, \mathbf{K}_\mathrm{p} = \begin{bmatrix} \gamma_{p1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \gamma_{pm} \end{bmatrix}, \cdots, \mathbf{K}_\mathrm{r} = \begin{bmatrix} \gamma_{r_1 1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \gamma_{r_m m} \end{bmatrix}.$$

If r_i-p<0 for $\mathrm{p} = 1, \ldots, \mathrm{r}$ and $i = 1, \ldots, m$, then set $e_i^{(r_i - \mathrm{p})} = 0$ in (17) to express (16). Using (10), (2) and (13), it is possible to rearrange (15) as follows.

$$\begin{aligned} \boldsymbol{\varepsilon}(t) = & (\mathbf{I} - \mathbf{B}(t)\bar{\mathbf{B}}^{-1})\boldsymbol{\varepsilon}(t-L) + (\mathbf{I} - \mathbf{B}(t)\bar{\mathbf{B}}^{-1})(\mathbf{v}(t) - \mathbf{v}(t-L)) \\ & + \mathbf{B}(t)[(\mathbf{B}^{-1}(t) - \mathbf{B}^{-1}(t-L))\mathbf{D}\mathbf{y}(t-L) \\ & - (\mathbf{B}^{-1}(t)\mathbf{a}(t) - \mathbf{B}^{-1}(t-L)\mathbf{a}(t-L))] \end{aligned} \tag{18}$$

where, $\mathbf{a(x)}, \mathbf{B(x)}$ are expressed as functions of time t instead of state vector $\mathbf{x}$.

When TDC is applied to input/output linearizable plants, the error dynamics is expressed as in (18). For this error dynamics, we show stability analysis using input/output stability in L_2 space [6]-[8].

Lemma 1: When TDC is applied to input/output linearizable plants, the error dynamics is identical to (17) and the following relation is satisfied [6]-[7].

$$\left\|\mathbf{D}_{\rm p}\mathbf{e}\right\|_{T2} \le \beta_{\rm p}\left\|\boldsymbol{\varepsilon}\right\|_{T2} + \beta_{\rm cp}$$

where, $\beta_{\rm p} = \| \mathbf{G}_{\rm p} \|_2$ and $\beta_{\rm cp}$ are constants deciding by initial conditions(i.e., $\| \mathbf{e}(t=t_0) \|_2, \| \mathbf{e}^{(\rm p)}(t=t_0) \|_2$) for $\rm p = 1,\ldots,r$ and $\mathbf{G}_{\rm p}$ is the operator $\mathbf{G}_{\rm p}: \boldsymbol{\varepsilon} \mapsto \mathbf{D}_{\rm p}\mathbf{e}$. $\left\|\bullet\right\|_{T2}$ denotes the L_2^m norm of $\bullet(t)$ truncated at T.

Proof: The relationship in (17) is regarded as a linear decoupled differential equation with input $\boldsymbol{\varepsilon}(t)$. We will obtain L_2 gain of the operator $\mathbf{G}_{\rm p}: \boldsymbol{\varepsilon} \mapsto \mathbf{D}_{\rm p}\mathbf{e}$ for $\rm p = 1,\ldots,r$. From (17), if we consider operator $G_{{\rm p},i}: \varepsilon_i \mapsto e_i^{(r_i - \rm p)}$ for each component in the case of $r_i - \rm p \ge 0$, then the transfer function from input ε_i to output $e_i^{(r_i - \rm p)}$ can be written as

$$\frac{e_i^{(r_i-\rm p)}(s)}{\varepsilon_i(s)} = g_{{\rm p},i}(s) = \frac{s^{r_i - \rm p}}{s^{r_i} + \gamma_{1i}s^{r_i-1} + \cdots + \gamma_{r_i i}} \tag{19}$$

The L_2 gain of transfer function is defined by [6]-[7].

$$\left\|\mathbf{G}_{{\rm p},i}\right\|_2 = \max_{\omega}\left|g_{{\rm p},i}(j\omega)\right| \tag{20}$$

and $\| \mathbf{G}_{\rm p} \|_2 = \| \mathbf{M}_{\mathbf{G}_{\rm p}} \|_{i2}$ [6]. Subscript $i2$ denotes induced matrix 2 norm and ith diagonal term of diagonal matrix $\mathbf{M}_{\mathbf{G}_{\rm p}}$ is $(\mathbf{M}_{\mathbf{G}_{\rm p}})_{ii} = \max_{\omega}\left|g_{{\rm p},i}(j\omega)\right|$. Therefore, the L_2 gain $\beta_{\rm p}$ of the multiple input/multiple output system is obtained as follows [6]-[7].

$$\beta_{\rm p} = \left\|\mathbf{G}_{\rm p}\right\|_2 \quad ({\rm p} = 1,\cdots,{\rm r}) \tag{21}$$

Hence, we have [6]-[7]

$$\left\|\mathbf{D}_{\rm p}\mathbf{e}\right\|_{T2} \le \beta_{\rm p}\left\|\boldsymbol{\varepsilon}\right\|_{T2} + \beta_{\rm cp} \quad ({\rm p} = 1,\cdots,{\rm r}) \tag{22}$$

This completes the proof of Lemma 1. ■

Definition : We define as follows.

$$\begin{array}{ll} \mathbf{I} - \mathbf{B}(t)\bar{\mathbf{B}}^{-1} \triangleq \boldsymbol{\Delta} \;, & \mathbf{B}^{-1}(t) - \mathbf{B}^{-1}(t-L) \triangleq \tilde{\mathbf{B}}^{-1} \\ \mathbf{B}^{-1}(t)\mathbf{a}(t) \triangleq \mathbf{Q}(t) \;, & \bullet(t) - \bullet(t-L) \triangleq \tilde{\bullet} \end{array} \tag{23}$$

Lemma 2: When TDC is applied to input/output linearizable plants, if we assume $\mathbf{D}\mathbf{y}_{\rm d} \in L_2^m$ and $\mathbf{D}_{\rm p}\mathbf{y}_{\rm d} \in L_2^m$ for $\rm p = 1,\cdots,r$ for the desired trajectory and its derivatives and $\mathbf{w} \in L_2^m$ for the disturbance, we have

$$(1-\mu)\left\|\boldsymbol{\varepsilon}\right\|_{T2} \le \delta_1\left\|\mathbf{D}_1\mathbf{e}\right\|_{T2} + \cdots + \delta_{\rm r}\left\|\mathbf{D}_{\rm r}\mathbf{e}\right\|_{T2} + \phi$$

where,

$$\begin{aligned} &\mu \triangleq \| \boldsymbol{\Delta} + L\boldsymbol{\Delta}\mathbf{K}_1 - \mathbf{B}\tilde{\mathbf{B}}^{-1} - L\mathbf{B}q_1 \|_{i2} \\ &\delta_{\rm j} \triangleq \| L\boldsymbol{\Delta}(\mathbf{K}_{\rm j+1} - \mathbf{K}_1\mathbf{K}_{\rm j}) + \mathbf{B}\tilde{\mathbf{B}}^{-1}\mathbf{K}_{\rm j} + L\mathbf{B}q_1\mathbf{K}_{\rm j} - L\mathbf{B}q_{\rm j+1} - \mathbf{B}\tilde{q}_{\rm j} \|_{i2} \quad ({\rm j}=1,\cdots,{\rm r-1}) \\ &\delta_{\rm r} \triangleq \| -L\boldsymbol{\Delta}\mathbf{K}_1\mathbf{K}_{\rm r} + \mathbf{B}\tilde{\mathbf{B}}^{-1}\mathbf{K}_{\rm r} + L\mathbf{B}q_1\mathbf{K}_{\rm r} - \mathbf{B}\tilde{q}_{\rm r} \|_{i2} \\ &\psi \triangleq \| \boldsymbol{\Delta}\mathbf{D}\tilde{\mathbf{y}}_{\rm d} + \mathbf{B}\tilde{\mathbf{B}}^{-1}\mathbf{D}\mathbf{y}_{\rm d}(t-L) - \mathbf{B}\tilde{\mathbf{Q}}_{\rm d} - \mathbf{B}\tilde{\mathbf{w}} \|_2 \end{aligned} \tag{24}$$

and $\mathbf{B}$ denotes $\mathbf{B}(t)$, $q_1(t),\cdots,q_r(t)$ are bounded scalar functions of time and $\mathbf{Q}_{\rm d}$ is the function of $\mathbf{D}_{\rm p}\mathbf{y}_{\rm d}$ for $\rm p = 1,\cdots,r$.

Proof: $\boldsymbol{\Delta}$ and $\mathbf{B}^{-1}(t)$ is bounded from the prerequisite condition that the plant is input/output linearizable. Therefore $\tilde{\mathbf{B}}^{-1}(t)$ is bounded because $\tilde{\mathbf{B}}^{-1}$ is the difference of $\mathbf{B}^{-1}(t)$ between time t and time $t-L$.

We express $\mathbf{Q}(t)$ in (23) as follows.

$$\mathbf{Q}(t) = \mathbf{Q}_{\rm d}(t) + \mathbf{O}_q(\mathbf{D}_1\mathbf{e}(t),\cdots,\mathbf{D}_{\rm r}\mathbf{e}(t)) + \mathbf{w}(t) \tag{25}$$

where

$$\mathbf{Q}_{\rm d}(t) = \mathbf{Q}_{\rm d}(\mathbf{D}_1\mathbf{y}_{\rm d}(t),\cdots,\mathbf{D}_{\rm r}\mathbf{y}_{\rm d}(t)) \tag{26}$$

$\mathbf{Q}_{\rm d}$ is the function of the desired trajectory and its derivative i.e., the function of $\mathbf{D}_{\rm p}\mathbf{y}_{\rm d}$ for $\rm p = 1,\cdots,r$ and $\mathbf{w}(t)$ is disturbance. And, assume $\mathbf{O}_{\rm q}$ as follows [8]

$$\mathbf{O}_q(\mathbf{D}_1\mathbf{e}(t),\cdots,\mathbf{D}_{\rm r}\mathbf{e}(t)) \approx q_1(t)\mathbf{D}_1\mathbf{e}(t) + \cdots + q_{\rm r}(t)\mathbf{D}_{\rm r}\mathbf{e}(t) \tag{27}$$

where, $q_1(t),\cdots,q_r(t)$ are bounded scalar functions of time. We consider (27) for two cases as follows.

i) $\mathbf{D}_1\mathbf{e}(t),\cdots,\mathbf{D}_{\rm r}\mathbf{e}(t)$ are converging : Naturally, (27) is valid.

ii) $\mathbf{D}_1\mathbf{e}(t),\cdots,\mathbf{D}_{\rm r}\mathbf{e}(t)$ are diverging: Mathematically, (27) does not express $\mathbf{O}_{\rm q}$ accurately. But in the context of stability, if (25) is diverging, it is possible for $\mathbf{D}_1\mathbf{e}(t),\cdots,\mathbf{D}_{\rm r}\mathbf{e}(t)$ to express $\mathbf{O}_{\rm q}$ by (27) as (27) is also diverging. Therefore, in the diverging case (27) is valid.

$\mathbf{v}(t)$ and $\mathbf{D}\mathbf{y}(t-L)$ in (18) can be expressed as

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{D}\mathbf{y}_{\rm d}(t) + \mathbf{K}_1\mathbf{D}_1\mathbf{e}(t) + \cdots + \mathbf{K}_{\rm r}\mathbf{D}_{\rm r}\mathbf{e}(t) \\ \mathbf{D}\mathbf{y}(t-L) &= \mathbf{v}(t-L) - \boldsymbol{\varepsilon}(t-L) \\ &= \mathbf{D}\mathbf{y}_{\rm d}(t-L) + \mathbf{K}_1\mathbf{D}_1\mathbf{e}(t-L) + \cdots \\ &\quad + \mathbf{K}_{\rm r}\mathbf{D}_{\rm r}\mathbf{e}(t-L) - \boldsymbol{\varepsilon}(t-L) \end{aligned} \tag{28}$$

Substituting (23), (25), and (28) into (18) and rearranging,

$$\begin{aligned}\boldsymbol{\varepsilon}(t) &= \boldsymbol{\Delta\varepsilon}(t-L) + \boldsymbol{\Delta}\mathbf{D}\tilde{\mathbf{y}}_{\mathrm{d}} + \boldsymbol{\Delta}\mathbf{K}_1\left(\mathbf{D}_1\mathbf{e}(t) - \mathbf{D}_1\mathbf{e}(t-L)\right) \\ &\quad + \cdots + \boldsymbol{\Delta}\mathbf{K}_{\mathrm{r}}\left(\mathbf{D}_{\mathrm{r}}\mathbf{e}(t) - \mathbf{D}_{\mathrm{r}}\mathbf{e}(t-L)\right) \\ &\quad + \mathbf{B}(t)\tilde{\mathbf{B}}^{-1}[\mathbf{D}\mathbf{y}_{\mathrm{d}}(t-L) + \mathbf{K}_1\mathbf{D}_1\mathrm{e}(t-L) + \cdots \\ &\quad + \mathbf{K}_{\mathrm{r}}\mathbf{D}_{\mathrm{r}}\mathbf{e}(t-L) - \boldsymbol{\varepsilon}(t-L)] - \mathbf{B}(t)\left[\tilde{\mathbf{Q}}_{\mathrm{d}} + \tilde{\mathbf{O}}_q + \tilde{\mathbf{w}}\right]\end{aligned} \tag{29}$$

Using (27), $\tilde{\mathbf{O}}_q$ in (29) can be expressed as follows.

$$\begin{aligned}\tilde{\mathbf{O}}_q &= \mathbf{O}_q(t) - \mathbf{O}_q(t-L) \\ &= q_1(t)(\mathbf{D}_1\mathbf{e}(t) - \mathbf{D}_1\mathbf{e}(t-L)) + (q_1(t) - q_1(t-L))\mathbf{D}_1\mathbf{e}(t-L) + \cdots \\ &\quad + q_{\mathrm{r}}(t)(\mathbf{D}_{\mathrm{r}}\mathbf{e}(t) - \mathbf{D}_{\mathrm{r}}\mathbf{e}(t-L)) + (q_{\mathrm{r}}(t) - q_{\mathrm{r}}(t-L))\mathbf{D}_{\mathrm{r}}\mathbf{e}(t-L)\end{aligned} \tag{30}$$

Substituting (30) into (29) and using the following Euler approximation for (29),

$$\begin{aligned}&\mathbf{D}_{\mathrm{r}}\mathbf{e}(t) - \mathbf{D}_{\mathrm{r}}\mathbf{e}(t-L) \approx L\mathbf{D}_{\mathrm{r-1}}\,\mathbf{e}(t-L)\ ,\ \cdots, \\ &\mathbf{D}_1\mathbf{e}(t) - \mathbf{D}_1\mathbf{e}(t-L) \approx L\,\mathbf{D}\mathbf{e}(t-L) \\ &\qquad = L\left[\boldsymbol{\varepsilon}(t-L) - \mathbf{K}_1\mathbf{D}_1\mathbf{e}(t-L) - \cdots - \mathbf{K}_{\mathrm{r}}\mathbf{D}_{\mathrm{r}}\mathbf{e}(t-L)\right]\end{aligned} \tag{31}$$

the relationship in (29) can be expressed as follows.

$$\begin{aligned}\boldsymbol{\varepsilon}(t) &= (\boldsymbol{\Delta} + L\boldsymbol{\Delta}\mathbf{K}_1 - \mathbf{B}\tilde{\mathbf{B}}^{-1} - L\mathbf{B}q_1)\boldsymbol{\varepsilon}(t-L) \\ &\quad + \{L\boldsymbol{\Delta}(\mathbf{K}_2 - \mathbf{K}_1\mathbf{K}_1) + \mathbf{B}\tilde{\mathbf{B}}^{-1}\mathbf{K}_1 + L\mathbf{B}q_1\mathbf{K}_1 \\ &\quad - L\mathbf{B}q_2 - \mathbf{B}\tilde{q}_1\}\mathbf{D}_1\mathbf{e}(t-L) + \cdots + \{L\boldsymbol{\Delta}(\mathbf{K}_{\mathrm{r}} - \mathbf{K}_1\mathbf{K}_{\mathrm{r-1}}) \\ &\quad + \mathbf{B}\tilde{\mathbf{B}}^{-1}\mathbf{K}_{\mathrm{r-1}} + L\mathbf{B}q_1\mathbf{K}_{\mathrm{r-1}} - L\mathbf{B}q_{\mathrm{r}} - \mathbf{B}\tilde{q}_{\mathrm{r-1}}\}\mathbf{D}_{\mathrm{r-1}}\mathbf{e}(t-L) \\ &\quad + (-L\boldsymbol{\Delta}\mathbf{K}_1\mathbf{K}_{\mathrm{r}} + \mathbf{B}\tilde{\mathbf{B}}^{-1}\mathbf{K}_{\mathrm{r}} + L\mathbf{B}q_1\mathbf{K}_{\mathrm{r}} - \mathbf{B}\tilde{q}_{\mathrm{r}})\mathbf{D}_{\mathrm{r}}\mathbf{e}(t-L) \\ &\quad + (\boldsymbol{\Delta}\mathbf{D}\tilde{\mathbf{y}}_{\mathrm{d}} + \mathbf{B}\tilde{\mathbf{B}}^{-1}\mathbf{D}\mathbf{y}_{\mathrm{d}}(t-L) - \mathbf{B}\tilde{\mathbf{Q}}_{\mathrm{d}} - \mathbf{B}\tilde{\mathbf{w}})\end{aligned} \tag{32}$$

We define the norm of each term of (32) as (24), then, from (32), we can obtain the following inequality.

$$\begin{aligned}\|\boldsymbol{\varepsilon}(t)\|_{T2} &\le \mu\|\boldsymbol{\varepsilon}(t-L)\|_{T2} + \delta_1\|\mathbf{D}_1\mathbf{e}(t-L)\|_{T2} + \cdots \\ &\quad + \delta_{\mathrm{r}}\|\mathbf{D}_{\mathrm{r}}\mathbf{e}(t-L)\|_{T2} + \psi\end{aligned} \tag{33}$$

where, ψ can be bounded as

$$\begin{aligned}\psi &\le \|\boldsymbol{\Delta}\|_{i2}\|\mathbf{D}\tilde{\mathbf{y}}_{\mathrm{d}}\|_2 + \|\mathbf{B}\tilde{\mathbf{B}}^{-1}\|_{i2}\|\mathbf{D}\mathbf{y}_{\mathrm{d}}(t-L)\|_2 + \|\mathbf{B}\|_{i2}(\|\tilde{\mathbf{Q}}_{\mathrm{d}}\|_2 + \|\tilde{\mathbf{w}}\|_2) \\ &\triangleq \phi\end{aligned} \tag{34}$$

in that if we assume $\mathbf{D}\mathbf{y}_{\mathrm{d}} \in L_2^m$, $\mathbf{D}_{\mathrm{p}}\mathbf{y}_{\mathrm{d}} \in L_2^m$ for $\mathrm{p} = 1,\cdots,\mathrm{r}$ and $\mathbf{w} \in L_2^m$, then $\mathbf{D}\tilde{\mathbf{y}}_{\mathrm{d}} \in L_2^m$, $\mathbf{D}\mathbf{y}_{\mathrm{d}}(t-L) \in L_2^m$, $\tilde{\mathbf{Q}}_{\mathrm{d}} \in L_2^m$ and $\tilde{\mathbf{w}} \in L_2^m$. Therefore, ϕ is bounded.

Because $\|\bullet\|_{T2}$ is defined in the L_2 space as follows [6],

$$\|\bullet\|_{T2} = \left(\int_0^T \|\bullet(t)\|_2^2\, dt\right)^{1/2} \tag{35}$$

it satisfies that $\|\bullet(t-L)\|_{T2} \le \|\bullet(t)\|_{T2}$.

Hence, $\|\boldsymbol{\varepsilon}\|_{T2}$ is bounded as follows.

$$(1-\mu)\|\boldsymbol{\varepsilon}\|_{T2} \le \delta_1\|\mathbf{D}_1\mathbf{e}\|_{T2} + \cdots + \delta_{\mathrm{r}}\|\mathbf{D}_{\mathrm{r}}\mathbf{e}\|_{T2} + \phi \tag{36}$$

This completes the proof of Lemma 2. ■

Lemma 3: When TDC is applied to input/output linearizable plants, under the condition that the zero dynamics of the plant is exponentially stable, we can obtain the sufficient stability condition of TDC for an input/output linearizable plant as follows.

$$\mu + \beta_1\delta_1 + \cdots + \beta_{\mathrm{r}}\delta_{\mathrm{r}} < 1$$

Proof : From Lemma 2 , if we assume that $\mu < 1$, then we obtain the inequality as

$$\|\boldsymbol{\varepsilon}\|_{T2} \le \frac{1}{1-\mu}\left[\delta_1\|\mathbf{D}_1\mathbf{e}\|_{T2} + \cdots + \delta_{\mathrm{r}}\|\mathbf{D}_{\mathrm{r}}\mathbf{e}\|_{T2} + \phi\right] \tag{37}$$

and assuming that

$$\beta_1\delta_1 < 1-\mu\ ,\ \cdots,\ \beta_{\mathrm{r}}\delta_{\mathrm{r}} < 1-\mu \tag{38}$$

and combining (22) and (37), then we have r inequalities as follows.

$$\begin{aligned}&\|\mathbf{D}_1\mathbf{e}\|_{T2} \le \frac{\beta_1}{1-\mu-\beta_1\delta_1}[\delta_2\|\mathbf{D}_2\mathbf{e}\|_{T2} + \cdots + \delta_{\mathrm{r}}\|\mathbf{D}_{\mathrm{r}}\mathbf{e}\|_{T2} + \phi] + \frac{1-\mu}{1-\mu-\beta_1\delta_1}\beta_{\mathrm{c1}} \\ &\quad\vdots \\ &\|\mathbf{D}_{\mathrm{r}}\mathbf{e}\|_{T2} \le \frac{\beta_{\mathrm{r}}}{1-\mu-\beta_{\mathrm{r}}\delta_{\mathrm{r}}}[\delta_1\|\mathbf{D}_1\mathbf{e}\|_{T2} + \cdots + \delta_{\mathrm{r-1}}\|\mathbf{D}_{\mathrm{r-1}}\mathbf{e}\|_{T2} + \phi] + \frac{1-\mu}{1-\mu-\beta_{\mathrm{r}}\delta_{\mathrm{r}}}\beta_{\mathrm{cr}}\end{aligned} \tag{39}$$

We express (39) in matrix form as follows.

$$\mathbf{E}_T \le \mathbf{R}\mathbf{E}_T + \mathbf{V} \tag{40}$$

where, $\mathbf{E}_T, \mathbf{R}, \mathbf{V}$ are as follows.

$$\begin{aligned}&\mathbf{E}_T = \left(\|\mathbf{D}_1\mathbf{e}\|_{T2} \ \cdots\ \|\mathbf{D}_{\mathrm{r}}\mathbf{e}\|_{T2}\right)^T \\ &\mathbf{R} = \begin{pmatrix} 0 & \frac{\beta_1\delta_2}{1-\mu-\beta_1\delta_1} & \cdots & \frac{\beta_1\delta_{\mathrm{r}}}{1-\mu-\beta_1\delta_1} \\ \frac{\beta_2\delta_1}{1-\mu-\beta_2\delta_2} & 0 & \cdots & \frac{\beta_2\delta_{\mathrm{r}}}{1-\mu-\beta_2\delta_2} \\ \vdots & & \ddots & \vdots \\ \frac{\beta_{\mathrm{r}}\delta_1}{1-\mu-\beta_{\mathrm{r}}\delta_{\mathrm{r}}} & \cdots & \frac{\beta_{\mathrm{r}}\delta_{\mathrm{r-1}}}{1-\mu-\beta_{\mathrm{r}}\delta_{\mathrm{r}}} & 0 \end{pmatrix} \\ &\mathbf{V} = \left(\frac{\beta_1\phi + (1-\mu)\beta_{\mathrm{c1}}}{1-\mu-\beta_1\delta_1} \ \cdots\ \frac{\beta_{\mathrm{r}}\phi + (1-\mu)\beta_{\mathrm{cr}}}{1-\mu-\beta_{\mathrm{r}}\delta_{\mathrm{r}}}\right)^T\end{aligned} \tag{41}$$

Since the entries of matrix $\mathbf{R}$ are all negative, the following statements hold [8]:

1) The spectral radius of $\mathbf{R}$ is less than one;
2) The inverse matrix $(\mathbf{I}-\mathbf{R})^{-1}$ is a nonnegative matrix.
3) The leading principal minors of the matrix $\mathbf{I}-\mathbf{R}$ are all

positive.

Hence, from the condition of $\det(\mathbf{I}-\mathbf{R})>0$, we have the following equation.

$$\mu+\beta_1\delta_1+\cdots+\beta_r\delta_r<1 \qquad (42)$$

Note that (42) is based on the assumptions of $\mu<1$ and (38). Then, by satisfying (42), $(\mathbf{I}-\mathbf{R})^{-1}\geq 0$ and letting $T\to\infty$, then

$$\mathbf{E}\leq(\mathbf{I}-\mathbf{R})^{-1}\mathbf{V} \qquad (43)$$

where, $\mathbf{E}=\lim_{T\to\infty}\mathbf{E}_T$.

Therefore, satisfying (42), under the condition that zero dynamics of the plant is exponentially stable, if $\mathbf{D}\mathbf{y}_d\in L_2^m$, $\mathbf{D}_p\mathbf{y}_d\in L_2^m$ for $p=1,\cdots,r$ and $\mathbf{w}\in L_2^m$, then $\mathbf{D}_p\mathbf{e}\in L_2^m$ for $p=1,\cdots,r$.

Hence, (42) is the sufficient stability condition of TDC for input/output linearizable plants under the condition of that zero dynamics of the plant is exponentially stable. This completes the proof of Lemma 3. ■

Lets state the TDC stability theorem.

Theorem: When TDC is applied to input/output linearizable plant, if

1. $\mathbf{D}\mathbf{y}_d\in L_2^m$ and $\mathbf{D}_p\mathbf{y}_d\in L_2^m$ for $p=1,\cdots,r$.
2. disturbance $\mathbf{w}\in L_2^m$.
3. $\mu+\beta_1\delta_1+\cdots+\beta_r\delta_r<1$,

then, under the condition that zero dynamics of the plant is exponentially stable, overall closed system using TDC is L_2 stable.

Proof: Through Lemma 3, overall closed system using TDC becomes L_2 stable. This completes the proof of Theorem. ■

B. Stability Criterion

From sufficient stability condition in the previous section, we can derive the stability criterion through following procedures.

The symbols μ and δ_i defined in (24) are bounded as

$$\begin{aligned}&\mu\leq\|\boldsymbol{\Delta}\|_{i2}(1+L\|\mathbf{K}_1\|_{i2})+c_0\\&\delta_j\leq L\|\boldsymbol{\Delta}\|_{i2}\|\mathbf{K}_{j+1}-\mathbf{K}_1\mathbf{K}_j\|_{i2}+c_j\ (\mathrm{j=1,\cdots,r\text{-}1})\\&\delta_r\leq L\|\boldsymbol{\Delta}\|_{i2}\|\mathbf{K}_1\|_{i2}\|\mathbf{K}_r\|_{i2}+c_r\end{aligned} \qquad (44)$$

where, $\boldsymbol{\Delta}=\mathbf{I}-\mathbf{B}\bar{\mathbf{B}}^{-1}$ which we defined in (23) and $c_0,\cdots,c_r$ are as follows.

$$\begin{aligned}&c_0=\|-\mathbf{B}\tilde{\mathbf{B}}^{-1}-L\mathbf{B}q_1\|_{i2}\\&c_j=\|\mathbf{B}\tilde{\mathbf{B}}^{-1}\mathbf{K}_j+L\mathbf{B}q_1\mathbf{K}_j-L\mathbf{B}q_{j+1}-\mathbf{B}\tilde{q}_j\|_{i2}\ (\mathrm{j=1,\cdots,r\text{-}1})\\&c_r=\|\mathbf{B}\tilde{\mathbf{B}}^{-1}\mathrm{K}_r+L\mathbf{B}q_1\mathbf{K}_r-\mathbf{B}\tilde{q}_r\|_{i2}\end{aligned} \qquad (45)$$

Substituting (44) into (42) and rearranging it leads to,

$$\begin{aligned}\mu+\beta_1\delta_1+\cdots+\beta_r\delta_r\leq\|\boldsymbol{\Delta}\|_{i2}(1+L\lambda_1+L\beta_1\lambda_{11}+\cdots\\+L\beta_{r-1}\lambda_{1r-1}+L\beta_r\lambda_1\lambda_r)+c_0+\beta_1c_1+\cdots+\beta_rc_r\end{aligned} \qquad (46)$$

where, $\lambda_p=\|\mathbf{K}_p\|_{i2}$ $(p=1,\cdots,r)$ and $\lambda_{1j}=\|\mathbf{K}_{j+1}-\mathbf{K}_1\mathbf{K}_j\|_{i2}$ $(j=1,\cdots,r\text{-}1)$.

If right-hand side of (46) is less than 1, the stability can be confirmed. Rearranging the condition that the right side of (46) is less than 1, we have the range of $\bar{\mathbf{B}}$ as follows.

$$\|\mathbf{I}-\mathbf{B}\bar{\mathbf{B}}^{-1}\|_{i2}<\frac{1-c}{1+L(\lambda_1+\beta_1\lambda_{11}+\cdots+\beta_{r-1}\lambda_{1r-1}+\beta_r\lambda_1\lambda_r)} \qquad (47)$$

and c is given by

$$c=c_0+\beta_1c_1+\cdots+\beta_rc_r \qquad (48)$$

where, $c_0,\cdots,c_r$ are expressed by (45).

C. Application Example of the Proposed Stability Criterion

In subsection III.B, we proposed the stable range of $\bar{\mathbf{B}}$ of the stability criterion. In this subsection, we demonstrate its application by examples.

For simplicity, we focus on the particular case where

1) $r_i=r$ for $i=1,\cdots,m$ i.e., each relative degree equals to the maximum relative degree.

2) The diagonal constant matrices $\mathbf{K}_1,\cdots,\mathbf{K}_r$ are chosen such a way that the system is critically damped and all the diagonal components of each $\mathbf{K}_1,\cdots,\mathbf{K}_r$ are chosen equally. Then, from Lemma 1, $\beta_p=1/(\eta_pk_p)$ for $p=1,\cdots,r$ where, k_p denotes the diagonal component of $\mathbf{K}_p$ and η_p is a coefficient, which is determined by the maximum relative degree r. And c of (48) is expressed as $c=c_0+c_1/(\eta_1k_1)+\cdots+c_r/(\eta_rk_r)$. In (45), because $\tilde{\bullet}$ denotes $\tilde{\bullet}=\bullet(t)-\bullet(t-L)$, if $\dot{\bullet}\ \left(=d\bullet/dt\right)$ is bounded, then we can express $\tilde{\bullet}\approx L\dot{\bullet}$. Using this we can restrict $c_0,\cdots,c_r$ of (45) as follows.

$$\begin{aligned}&c_0\leq L\|\mathbf{B}\|_{i2}(\|\dot{\mathbf{B}}^{-1}\|_{i2}+\|q_1\|_{i2})\\&c_j\leq k_jL\|\mathbf{B}\|_{i2}(\|\dot{\mathbf{B}}^{-1}\|_{i2}+\|q_1\|_{i2})+L\|\mathbf{B}\|_{i2}(\|q_{j+1}\|_{i2}+\|\dot{q}_j\|_{i2})\\&c_r\leq k_rL\|\mathbf{B}\|_{i2}(\|\dot{\mathbf{B}}^{-1}\|_{i2}+\|q_1\|_{i2})+L\|\mathbf{B}\|_{i2}\|\dot{q}_r\|_{i2}\end{aligned} \qquad (49)$$

where, $j=1,\cdots,r\text{-}1$ and $k_j(j=1,\cdots,r\text{-}1)$, k_r are chosen as

$$k_j\gg\frac{1}{\eta_j}L\|\mathbf{B}\|_{i2}(\|q_{j+1}\|_{i2}+\|\dot{q}_j\|_{i2})\ ,\ k_r\gg\frac{1}{\eta_r}L\|\mathbf{B}\|_{i2}\|\dot{q}_r\|_{i2} \qquad (50)$$

then c is bounded approximately as follows.

$$c \le (1+\sum_{i=1}^{\mathrm{r}}\frac{1}{\eta_i})L\|\mathbf{B}\|_{i2}(\|\dot{\mathbf{B}}^{-1}\|_{i2}+\|q_1\|_{i2}) \qquad (51)$$

From (47), to satisfy the following equation,

$$\begin{aligned}&1+L(\lambda_1+\beta_1\lambda_{11}+\cdots+\beta_{\mathrm{r}-1}\lambda_{1\mathrm{r}-1}+\beta_{\mathrm{r}}\lambda_1\lambda_{\mathrm{r}})\\&\gg(1+\sum_{i=1}^{\mathrm{r}}\frac{1}{\eta_i})L\|\mathbf{B}\|_{i2}(\|\dot{\mathbf{B}}^{-1}\|_{i2}+\|q_1\|_{i2})\end{aligned} \qquad (52)$$

λ,β i.e., $k_1,\cdots,k_{\mathrm{r}}$ can be chosen so that the range of $\bar{\mathbf{B}}$ in the stability criterion can be approximated by

$$\|\mathbf{I}-\mathbf{B}\bar{\mathbf{B}}^{-1}\|_{i2}<\frac{1}{1+L(\lambda_1+\beta_1\lambda_{11}+\cdots+\beta_{\mathrm{r}-1}\lambda_{1\mathrm{r}-1}+\beta_{\mathrm{r}}\lambda_1\lambda_{\mathrm{r}})} \qquad (53)$$

In other words, if we choose $k_1,\cdots,k_r$ to satisfy (50) and (52), then we can express the stability criterion as in (53).

For the case of $\mathrm{r}=2$, $\eta_1=\eta_2=1$ and $\lambda_1=k_1$, $\lambda_2=k_2$, $\lambda_{11}=0.75k_1^2$. If k_1,k_2 are chosen so as to satisfy (50) and (52), then the stability criterion can be expressed as follows.

$$\|\mathbf{I}-\mathbf{B}\bar{\mathbf{B}}^{-1}\|_{i2}<\frac{1}{1+2.75k_1L} \qquad (54)$$

For the case of $\mathrm{r}=4$, $\eta_1=4/3\sqrt{3}$, $\eta_2=2/3$, $\eta_3=4/3\sqrt{3}$, $\eta_4=1$ and $\lambda_1=k_1$, $\lambda_2=k_2$, $\lambda_3=k_3$, $\lambda_4=k_4$ and $\lambda_{11}=(5k_1^2)/8$, $\lambda_{12}=(5k_1k_2)/6$, $\lambda_{13}=(15k_1k_3)/16$, if we choose $k_1,\cdots,k_4$ to satisfy (50) and (52), then the stability criterion can be expressed as follows.

$$\|\mathbf{I}-\mathbf{B}\bar{\mathbf{B}}^{-1}\|_{i2}<\frac{1}{1+5.2797k_1L} \qquad (55)$$

Remark 1: If time delay L is zero, (53) is equal to $\|\mathbf{I}-\mathbf{B}\bar{\mathbf{B}}^{-1}\|_{i2}<1$, which was suggested by Youcef-Toumi [4].

Remark 2: If time delay L is large, in order to satisfy (53), $\bar{\mathbf{B}}$ need to be similar to $\mathbf{B}$.

IV. SIMULATION

In this section, from the following a simulation, we verify the stability criterion proposed in section III. In the following simulation, we consider for the case, which was exampled in subsection III.C and choose components $k_1,\cdots,k_{\mathrm{r}}$ of matrices $\mathbf{K}_1,\cdots,\mathbf{K}_{\mathrm{r}}$ to satisfy (50) and (52) in order to apply (53) as the stability criterion.

The plant for simulation is a six degrees of freedom(d.o.f) manipulator(PUMA560) [9]. Joint torque vector $\boldsymbol{\tau}\in\Re^6$ of this 6 d.o.f manipulator is the input of the plant and if we adopt the joint variable vector $\boldsymbol{\theta}\in\Re^6$ as the output of the plant and do input/output linearization, then no zero dynamics of the plant exists and $r_i=\mathrm{r}=2$ for $i=1,\cdots,6$, so we use (54) as the stability criterion.

When we set $\bar{\mathbf{B}}^{-1}=\alpha\mathbf{I}_{6\times6}$, we could plot $\|\mathbf{I}-\mathbf{B(x)}\bar{\mathbf{B}}^{-1}\|_{i2}$ against α as shown by the solid line in Fig. 3 for the all state vectors $\mathbf{x}=(\boldsymbol{\theta}\quad\dot{\boldsymbol{\theta}})^T$ within the workspace of the plant. In Fig. 3, $\mathbf{B(x)}$ is the inverse matrix of the inertial matrix $\mathbf{M}(\boldsymbol{\theta})$ of the manipulator referred in [9]. The dotted line of Fig. 3 denotes the boundary of the stability criterion for the case of time delay $L=0.001$ s and $k_1=10$ in (54) i.e., $\|\mathbf{I}-\mathbf{B(x)}\bar{\mathbf{B}}^{-1}\|_{i2}<0.9732$.

For simulation, we consider time delay $L=0.001$ s, $k_1=10$, $k_2=25$ and $\mathbf{y}_{\mathrm{d}},\dot{\mathbf{y}}_{\mathrm{d}},\ddot{\mathbf{y}}_{\mathrm{d}}$ as $y_{di}=e^{-(\pi t/2)}\sin(\pi t/2)$ (i=1,$\cdots$,6). And, $\mathbf{y}_{\mathrm{d}},\dot{\mathbf{y}}_{\mathrm{d}},\ddot{\mathbf{y}}_{\mathrm{d}}$ are in the L_2 space i.e., $\mathbf{y}_{\mathrm{d}},\dot{\mathbf{y}}_{\mathrm{d}},\ddot{\mathbf{y}}_{\mathrm{d}}\in L_2^6$.

Fig. 4 shows the simulation results. In a previous research, Youcef-Toumi has proposed $\|\mathbf{I}-\mathbf{B(x)}\bar{\mathbf{B}}^{-1}\|_{i2}<1$ as the sufficient stability condition. However, we observe the unstable response for the case of $\bar{\mathbf{B}}^{-1}=\alpha\mathbf{I}_{6\times6}$ $(\alpha=0.01)$ satisfying $\|\mathbf{I}-\mathbf{B(x)}\bar{\mathbf{B}}^{-1}\|_{i2}<1$ in as shown in Fig. 4(a). Abovementioned, although time delay L is sufficiently small (not zero) and $\|\mathbf{I}-\mathbf{B(x)}\bar{\mathbf{B}}^{-1}\|_{i2}<1$, the system could still be unstable. Therefore, a more general stable range for $\bar{\mathbf{B}}$ is required for small and finite L.

The L_2 stable responses can be observed in Fig. 4(b), 4(c) and 4(d), for $\bar{\mathbf{B}}$ being selected within the suggested range $\|\mathbf{I}-\mathbf{B(x)}\bar{\mathbf{B}}^{-1}\|_{i2}<0.9732$. Fig. 4(b) and 4(c) are the simulation results when $\bar{\mathbf{B}}^{-1}=\alpha\mathbf{I}_{6\times6}$ $(\alpha=0.2,\alpha=0.35)$ are chosen at the boundaries of the stability criterion. Fig. 4(d) is the simulation result when $\bar{\mathbf{B}}^{-1}=\alpha\mathbf{I}_{6\times6}$ $(\alpha=0.3)$ is chosen arbitrarily to satisfy $\|\mathbf{I}-\mathbf{B(x)}\bar{\mathbf{B}}^{-1}\|_{i2}<0.9732$.

Hence, from the simulation results, we observe that suggested stability criterion for TDC works well for the real situation where time delay L is finite.

V. CONCLUSION

In this paper, we have derived the sufficient stability condition of the Time Delay Control for input/output linearizable plants considering small and finite time delays. The performance of the proposed method has been compared with the existing sufficient stability criterion $\|\mathbf{I}-\mathbf{B(x)}\bar{\mathbf{B}}^{-1}\|<1$ proposed by Youcef-Toumi, where he assumed infinitesimally small time delay. Through simulation example with small and finite time delay, we observed that the existing stability condition fails to stabilize the closed loop systems, whereas the proposed method works well. In real control systems where a finite time delay is introduced by the digital device such as computers, DSPs, etc., the proposed stability criterion could be extremely important to guarantee stable operation.

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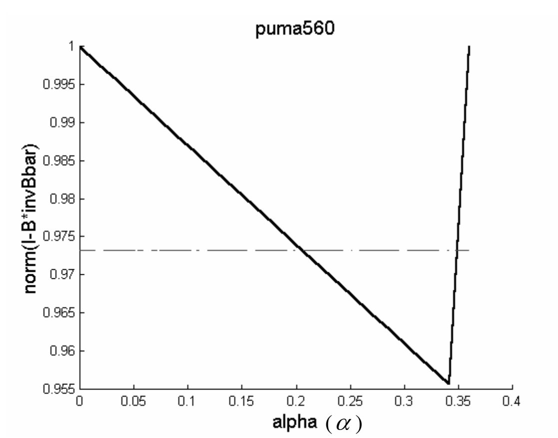
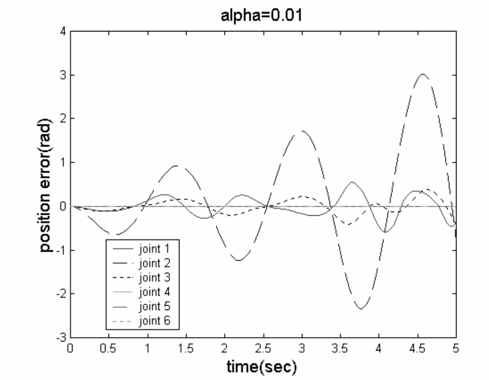
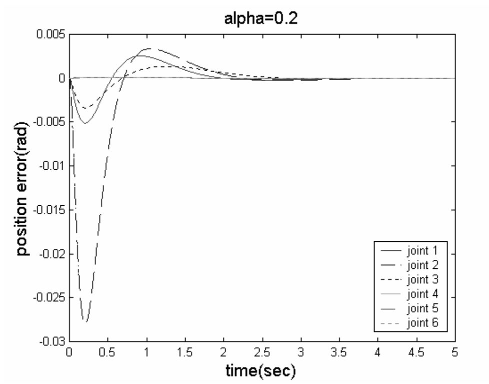


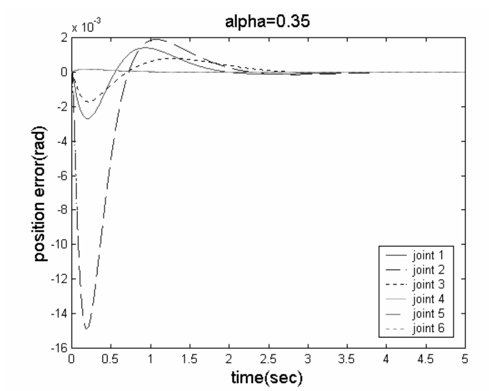
Fig. 3. $\| \mathbf{I} - \mathbf{B(x)}\bar{\mathbf{B}}^{-1} \|_{i2}$ plot for PUMA560 manipulator



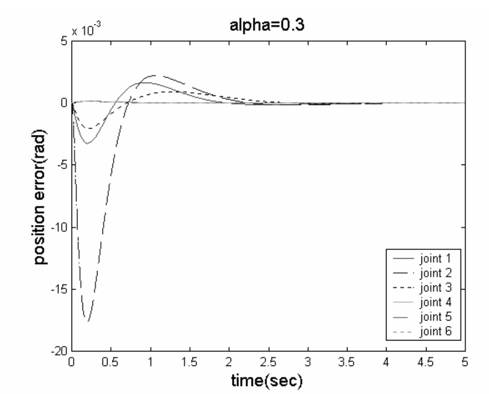
(a) $\bar{\mathbf{B}}^{-1} = \alpha \mathbf{I}_{6\times6}$ ($\alpha = 0.01$ case)



(b) $\bar{\mathbf{B}}^{-1} = \alpha \mathbf{I}_{6\times6}$ ($\alpha = 0.2$ case)



(c) $\bar{\mathbf{B}}^{-1} = \alpha \mathbf{I}_{6\times6}$ ($\alpha = 0.35$ case)



(d) $\bar{\mathbf{B}}^{-1} = \alpha \mathbf{I}_{6\times6}$ ($\alpha = 0.3$ case)

Fig. 4. Simulation Results for PUMA560 manipulator