Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Consensus of Information Under Dynamically Changing Interaction Topologies

Wei Ren and Randal W. Beard

Abstract— **This paper considers the problem of information consensus among multiple agents in the presence of limited and unreliable information exchange with dynamically changing interaction topologies. Both discrete and continuous update schemes are proposed for information consensus. The paper shows that information consensus under dynamically changing interaction topologies can be achieved asymptotically if the union of the directed interaction graphs across some time intervals has a spanning tree frequently enough as the system evolves. Simulation results show the effectiveness of our update schemes.**

I. Introduction

The study of information flow and interaction among multiple agents in a group plays an important role in understanding the coordinated movements of these agents. Research efforts in this direction are reported in [1], [2], [3], [4], [5], to name a few. Some applications of coordinated control require information to be shared among multiple agents in a group (c.f. [6], [3], [7], [8], [9], [5]), which in turn requires information consensus. In this paper, we extend the results of [3] to the case of directed graphs and present conditions for consensus of information under dynamically changing interaction topologies.

In contrast to [3], directed graphs will be used to represent the interaction (information exchange) topology between agents, where information can be exchanged via communication or direct sensing. A preliminary result for information consensus is presented in [5], where a linear update scheme is proposed for directed graphs. However the analysis in [5] was not able to utilize all available communication links. A solution to this issue was presented in [10] for time-invariant communication topologies. Information consensus for dynamically evolving information was addressed in [8] in the context of spacecraft formation flying where the exchanged information is the configuration of the virtual structure associated with the (dynamically evolving) formation.

In many applications, the interaction topology between agents may change dynamically. For example, communication links between agents may be unreliable due to disturbances and/or subject to communication range limitations. If information is being exchanged by direct sensing, the locally visible neighbors of a vehicle will likely change over time. In the ground breaking work by Jadbabaie et al. [3], a theoretical explanation is provided for the observed behavior of the Vicsek model [6]. Possible changes over time in each agent's nearest neighbors is explicitly taken into account, and is an example of information consensus under dynamically changing interaction topologies. Furthermore, it is shown in [3] that consensus can be achieved if the union of the interaction graphs for the team are connected frequently enough as the system evolves. However, the approach in [3] is based on bidirectional information exchange, modelled by undirected graphs.

There are a variety of practical applications where information only flows in one direction. For example, in leader-following scenarios, the leader may be the only vehicle equipped with a communication transmitter. For heterogeneous teams, some vehicles may have transceivers, while other less capable members only have receivers. There is a need to extend the results reported in [3] to interaction topologies with directional information exchange.

In addition, in [3] certain constraints are imposed on the weighting factors in the information update schemes, which may be extended to more general cases. For example, it may be desirable to weigh the information from different agents differently to represent the relative confidence of each agent's information or relative reliabilities of different communication or sensing links.

The objective of this paper is to extend the work of Jadbabaie et al. [3] to the case of directed graphs and explore the minimum requirements to reach consensus. As a comparison, Ref. [4] solves the *average-consensus* problem with directed graphs, which requires the graph to be strongly connected and balanced. We show that under certain assumptions *consensus*[1] can be achieved asymptotically under dynamically changing interaction topologies if the union of the collection of interaction graphs across some time intervals has a spanning tree frequently enough. The spanning tree requirement is a milder condition than connectedness and is therefore more suitable for practical applications. We also allow the relative weighting factors to be time-varying, which provides additional flexibility. As a result, the convergence conditions and update schemes in [3] are shown to be a special case of a more general result.

An additional contribution of this paper is that we show that a nonnegative matrix with the same positive row sums has its spectral radius (its row sum in this case) as a simple eigenvalue if and only if the directed graph of this matrix has a spanning tree. In contrast, the Perron-Frobenius Theorem [11] for nonnegative matrices only deals with irreducible matrices, that is, matrices with strongly connected graphs. Besides having a spanning tree, if this matrix also has positive diagonal entries, we show that its row sum is the unique eigenvalue of maximum modulus.

II. Problem Statement

Let $\mathcal{A} = \{A_i | i \in \mathcal{I}\}$ be a set of n agents, where $\mathcal{I} = \{1, 2, \cdots, n\}$. A directed graph $\mathcal{G}$ will be used to model

W. Ren and R. W. Beard are with the Department of Electrical and Computer Engineering, Brigham Young University, Provo, Utah 84602, USA {weiren,beard}@ee.byu.edu

[1] not necessarily average-consensus

the interaction topology among these agents. In $\mathcal{G}$, the ith vertex represents the ith agent A_i and a directed edge from A_i to A_j denoted as (A_i, A_j) represents a unidirectional information exchange link from A_i to A_j, that is, agent j can receive or obtain information from agent i, $(i,j) \in \mathcal{I}$. The interaction topology may be dynamically changing, therefore let $\bar{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_M\}$ denote the set of all possible directed interaction graphs defined for $\mathcal{A}$. In applications, the possible interaction topologies will likely be a subset of $\bar{\mathcal{G}}$. Obviously, $\bar{\mathcal{G}}$ has finite elements. The union of a group of directed graphs $\{\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, \cdots, \mathcal{G}_{i_m}\} \subset \bar{\mathcal{G}}$ is a directed graph with vertices given by A_i, $i \in \mathcal{I}$ and edge set given by the union of the edge sets of $\mathcal{G}_{i_j}$, $j = 1, \cdots, m$.

A directed path in graph $\mathcal{G}$ is a sequence of edges $(A_{i_1}, A_{i_2}), (A_{i_2}, A_{i_3}), (A_{i_3}, A_{i_4}), \cdots$ in that graph. Graph $\mathcal{G}$ is called strongly connected if there is a directed path from A_i to A_j and A_j to A_i between any pair of distinct vertices A_i and A_j, $\forall (i,j) \in \mathcal{I}$. A directed tree is a directed graph, where every node, except the root, has exactly one parent. A spanning tree of a directed graph is a directed tree formed by graph edges that connect all the vertices of the graph (c.f. [12]). Let $M_n(\mathbb{R})$ represent the set of all $n \times n$ real matrices. Given a matrix $A = [a_{ij}] \in M_n(\mathbb{R})$, the directed graph of A, denoted by $\Gamma(A)$, is the directed graph on n vertices V_i, $i \in \mathcal{I}$, such that there is a directed edge in $\Gamma(A)$ from V_j to V_i if and only if $a_{ij} \neq 0$ (c.f. [11]).

Let $\xi_i \in \mathbb{R}$, $i \in \mathcal{I}$, represent the ith information state associated with the ith agent. The set of agents $\mathcal{A}$ is said to achieve consensus asymptotically if for any $\xi_i(0)$, $i \in \mathcal{I}$, $\|\xi_i(t) - \xi_j(t)\| \to 0$ as $t \to \infty$ for each $(i,j) \in \mathcal{I}$.

Given T as the sampling period, we propose the following discrete-time consensus scheme:

$$\xi_i[k+1] = \frac{1}{\sum_{j=1}^n \alpha_{ij}[k] G_{ij}[k]} \sum_{j=1}^n \alpha_{ij}[k] G_{ij}[k] \xi_j[k], \quad (1)$$

where $k \in \{0, 1, 2, \cdots\}$ is the discrete time index, $(i,j) \in \mathcal{I}$, $\alpha_{ij}[k] > 0$ is a weighting factor, $G_{ii}[k] \triangleq 1$, and $G_{ij}[k]$ equals one if information flows from A_j to A_i at time $t = kT$ and zero otherwise, $\forall j \neq i$. Eq. (1) can be written in matrix form as

$$\xi[k+1] = D[k]\xi[k], \quad (2)$$

where $\xi = [\xi_1, \cdots, \xi_n]^T$, $D = [d_{ij}]$, $(i,j) \in \mathcal{I}$, with $d_{ij} = \frac{\alpha_{ij}[k] G_{ij}[k]}{\sum_{j=1}^n \alpha_{ij}[k] G_{ij}[k]}$.

In addition, we propose the following continuous-time consensus scheme:

$$\dot{\xi}_i(t) = -\sum_{j=1}^n \sigma_{ij}(t) G_{ij}(t)(\xi_i(t) - \xi_j(t)), \quad (3)$$

where $(i,j) \in \mathcal{I}$, $\sigma_{ij}(t) > 0$ is the weighting factor, $G_{ii}(t) \triangleq 1$, and $G_{ij}(t)$ equals one if information flows from A_j to A_i at time t and zero otherwise, $\forall j \neq i$. Eq. (3) can be written in matrix form as

$$\dot{\xi}(t) = C(t)\xi(t), \quad (4)$$

where $C = [c_{ij}]$, $(i,j) \in \mathcal{I}$, with $c_{ii} = -\sum_{j \neq i}(\sigma_{ij}(t) G_{ij}(t))$ and $c_{ij} = \sigma_{ij}(t) G_{ij}(t)$, $j \neq i$.

Note that the interaction topology $\mathcal{G}$ may be dynamically changing due to unreliable transmission or limited communication/sensing range. This implies that $G_{ij}[k]$ in Eq. (1) and $G_{ij}(t)$ in Eq. (3) may be time-varying. We use $\mathcal{G}[k]$ and $\mathcal{G}(t)$ to denote the dynamically changing interaction topologies corresponding to Eq. (1) and Eq. (3) respectively. We also allow the weighting factors $\alpha_{ij}[k]$ in Eq. (1) and $\sigma_{ij}(t)$ in Eq. (3) to be dynamically changing to represent possibly time-varying relative confidence of each agent's information state or relative reliabilities of different information exchange links between agents. As a result, both matrix $D[k]$ in Eq. (1) and matrix $C(t)$ in Eq. (3) are dynamically changing over time.

Compared to the models in [3], we do not constrain the weighting factors $\alpha_{ij}[k]$ in Eq. (1) other than to require that they are positive. This provides needed flexibility for some applications. The Vicsek model and simplified Vicsek model used in [3] can be thought of as special cases of our discrete time consensus scheme. If we let $\alpha_{ij}[k] \triangleq 1$ in Eq. (1), we obtain the Vicsek model. Also the simplified Vicsek model can be obtained if we let $\alpha_{ij}[k] \triangleq \frac{1}{g}$, $\forall j \neq i$, and $\alpha_{ii}[k] \triangleq 1 - \sum_{j \neq i} \frac{1}{g} G_{ij}[k]$, where $g > n$ is a constant. Compared to [5], where the interaction graph is assumed to be time-invariant and weighting factors σ_{ij} are specified a priori to be constant and equal to each other, we study continuous time consensus scheme with dynamically changing interaction topologies and weighting factors. The continuous update rule in [3] can also be regarded as a special case of our continuous update scheme by letting $\sigma_{ij} \triangleq \frac{1}{n}$.

III. Consensus of Information Under Dynamically Changing Interaction Topologies

Let $\mathbf{1}$ denote an $n \times 1$ column vector with all the entries equal to 1. Also let I_n denote the $n \times n$ identity matrix. A matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ is nonnegative, denoted as $A \geq 0$, if all its entries are nonnegative. Furthermore, if all its row sums are $+1$, A is said to be a (row) stochastic matrix [11]. A stochastic matrix P is called indecomposable and aperiodic (SIA) if $\lim_{n \to \infty} P^n = \mathbf{1}y^T$, where y is some column vector [13]. For nonnegative matrices, $A \geq B$ implies that $A - B$ is a nonnegative matrix. It is easy to verify that if $A \geq \rho B$, $\forall \rho > 0$, and the directed graph of B has a spanning tree, then the directed graph of A has a spanning tree.

We need the following two lemmas. The first lemma is from [3] and the second lemma is originally from [13] and restated in [3].

Lemma 3.1: [3] Let $m \geq 2$ be a positive integer and let $P_1, P_2, \cdots, P_m$ be nonnegative $n \times n$ matrices with positive diagonal elements, then

$$P_1 P_2 \cdots P_m \geq \gamma(P_1 + P_2 + \cdots + P_m),$$

where $\gamma > 0$ can be specified from matrices P_i, $i = 1, \cdots, m$.

Lemma 3.2: [13] Let $S_1, S_2, \cdots, S_k$ be a finite set of SIA matrices with the property that for each sequence $S_{i_1}, S_{i_2}, \cdots, S_{i_j}$ of positive length, the matrix product

$S_{i_j}S_{i_{j-1}}\cdots S_{i_1}$ is SIA. Then for each infinite sequence $S_{i_1}, S_{i_2}, \cdots$ there exists a column vector y such that

$$\lim_{j\to\infty} S_{i_j}S_{i_{j-1}}\cdots S_{i_1} = \mathbf{1}y^T.$$

We also need the following lemmas to derive our main results.

Lemma 3.3: Given a matrix $A = [a_{ij}] \in M_n(\mathbb{R})$, where $a_{ii} \leq 0$, $a_{ij} \geq 0$, $\forall i \neq j$, and $\sum_{j=1}^n a_{ij} = 0$ for each j, then A has at least one zero eigenvalue and all of the non-zero eigenvalues are in the open left half plane. Furthermore, A has exactly one zero eigenvalue if and only if the directed graph associated with A has a spanning tree.

Proof: See Corollary 1 in [10]. ■

Lemma 3.4: If a nonnegative matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ has the same positive constant row sums given by $\mu > 0$, then μ is an eigenvalue of A with an associated eigenvector $\mathbf{1}$ and $\rho(A) = \mu$, where $\rho(\cdot)$ denotes the spectral radius. In addition, the eigenvalue μ of A has algebraic multiplicity equal to one, if and only if the graph associated with A has a spanning tree. Furthermore, if the graph associated with A has a spanning tree and $a_{ii} > 0$, then μ is the unique eigenvalue of maximum modulus.

Proof: The first part of the lemma follows directly from the properties of nonnegative matrices (c.f. [11]).

For the second part of the lemma, we need to show both the necessary and sufficient conditions.

(Sufficiency.) If the graph associated with A has a spanning tree, then the graph associated with $B = A - \mu I_n$ also has a spanning tree. We know that $\lambda_i(A) = \lambda_i(B) + \mu$, where $i = 1, \cdots, n$, and $\lambda_i(\cdot)$ represents the ith eigenvalue. Noting that B satisfies the conditions in Lemma 3.3, we know that zero is an eigenvalue of B with algebraic multiplicity equal to one, which implies that μ is an eigenvalue of A with algebraic multiplicity equal to one.

(Necessity.) If the graph associated with A does not have a spanning tree, we know that $B = A - \mu I_n$ has more than one zero eigenvalue from Lemma 3.3, which in turn implies that A has more than one eigenvalue equal to μ.

For the third part of the lemma, the Gersgorin disc theorem [11] implies that all the eigenvalues of A are located in the union of the n discs given by

$$\bigcup_{i=1}^n \{z \in \mathbf{C} : |z - a_{ii}| \leq \sum_{j\neq i} |a_{ij}|\},$$

where $\mathbf{C}$ is the set of complex numbers. Noting that $a_{ii} > 0$, it is easy to see that this union is included in the circle given by $\{z \in \mathbf{C} : |z| \leq \mu\}$ with only one intersection at $z = \mu$. Thus we know that $|\lambda| < \mu$ for every eigenvalue of A satisfying $\lambda \neq \mu$. Combining the second part of the lemma, we know that μ is the unique eigenvalue of maximum modulus. ■

Corollary 3.1: A stochastic matrix has algebraic multiplicity equal to one for its eigenvalue $\lambda = 1$ if and only if the graph associated with the matrix has a spanning tree. Furthermore, a stochastic matrix with positive diagonal elements has the property that $|\lambda| < 1$ for every eigenvalue not equal to one.

Lemma 3.5: If $A \in M_n$ and $A \geq 0$, then $\rho(A)$ is an eigenvalue of A and there is a nonnegative vector $x \geq 0$, $x \neq 0$, such that $Ax = \rho(A)x$.

Proof: See Theorem 8.3.1 in [11]. ■

Lemma 3.6: Let $A = [a_{ij}] \in M_n(\mathbb{R})$ be a stochastic matrix. If A has an eigenvalue $\lambda = 1$ with algebraic multiplicity equal to one, and all the other eigenvalues satisfy $|\lambda| < 1$, then A is SIA, that is, $\lim_{m\to\infty} A^m \to \mathbf{1}\nu^T$, where ν satisfies $A^T\nu = \nu$ and $\mathbf{1}^T\nu = 1$. Furthermore, each element of ν is nonnegative.

Proof: The first part of the lemma follows Lemma 8.2.7 in [11]. For the second part, it is obvious that A^T is also nonnegative and has $\rho(A^T) = 1$ as an eigenvalue with algebraic multiplicity equal to one. Thus Lemma 3.5 implies that the eigenspace of A^T associated with eigenvalue $\lambda = 1$ is given by cx, where $c \in \mathbf{C}$, $c \neq 0$, and x is a nonnegative eigenvector. Since ν is also an eigenvector of A^T associated with eigenvalue $\lambda = 1$ and satisfies $\mathbf{1}^T\nu = 1$, it follows that each element of ν must be nonnegative. ■

A. Consensus Using Discrete Time Update Scheme

As a first step toward the general case, we first show necessary and sufficient condition for consensus of information using discrete time update scheme (1) with a time-invariant interaction topology and constant weighting factors, that is, a constant matrix D.

Theorem 3.2: With a time-invariant interaction topology and constant weighting factors, the discrete time update scheme (1) achieves consensus asymptotically for $\mathcal{A}$ if and only if the associated interaction graph $\mathcal{G}$ has a spanning tree.

Proof: (Sufficiency.) To show that ξ_i can achieve global consensus asymptotically, it is equivalent to show that $D^k \to \mathbf{1}c^T$, where c is some column vector, which implies that $\xi_i(k) \to c^T\xi(0)$, $\forall i \in \mathcal{I}$, as $k \to \infty$.

Obviously D is a stochastic matrix with positive diagonal entries. The fact that graph $\mathcal{G}$ has a spanning tree also implies that the directed graph of D has a spanning tree. Combining Corollary 3.1 and Lemma 3.6, we know that $\lim_{k\to\infty} D^k \to \mathbf{1}\nu^T$, where ν satisfies the properties defined in Lemma 3.6.

(Necessity.) If $\mathcal{G}$ does not have a spanning tree, neither does the directed graph of D, which implies, by Corollary 3.1, that the algebraic multiplicity of eigenvalue $\lambda = 1$ of D is $m > 1$. Therefore, the Jordan decomposition of D^k has the form $D^k = MJ^kM^{-1}$, where M is full rank and J^k is lower triangular with m diagonal elements equal to one. Therefore, the rank of $\lim_{k\to\infty} D^k$ is at least $m > 1$ which implies that $\mathcal{A}$ cannot reach consensus asymptotically. ■

The next lemma sets the stage for showing that under certain conditions, the existence of a spanning tree is sufficient for consensus under dynamically changing interaction topologies using the discrete update scheme (1).

Lemma 3.7: If the union of a set of directed graphs $\{\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, \cdots, \mathcal{G}_{i_m}\} \subset \bar{\mathcal{G}}$ has a spanning tree, then the matrix product $D_{i_m}\cdots D_{i_2}D_{i_1}$ is SIA, where D_{i_j} is a stochastic matrix corresponding to each directed graph $\mathcal{G}_{i_j}$ in Eq. (2).

Proof: From Lemma 3.1, we know that $D_{i_m}\cdots D_{i_2}D_{i_1} \geq \gamma\sum_{j=1}^m D_{i_j}$ for some $\gamma > 0$.

Since the union of $\{\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, \cdots, \mathcal{G}_{i_m}\}$ has a spanning tree, we know that the directed graph of matrix $\sum_{j=1}^m D_{i_j}$ has a spanning tree, which in turn implies that the directed graph of the matrix product $D_{i_m} \cdots D_{i_2} D_{i_1}$ has a spanning tree. Also the matrix product $D_{i_m} \cdots D_{i_2} D_{i_1}$ is a stochastic matrix with positive diagonal entries since stochastic matrices with positive diagonal entries are closed under matrix multiplication.

Combining Corollary 3.1 and Lemma 3.6, we know that the matrix product $D_{i_1} D_{i_2} \cdots D_{i_m}$ is SIA. ■

The following theorem extends the discrete time convergence result of [3].

Theorem 3.3: Let $\mathcal{G}[k] \in \bar{\mathcal{G}}$ be a switching interaction graph at time $t = kT$. Also let $\alpha_{ij}[k] \in \bar{\alpha}$, where $\bar{\alpha}$ is a finite set of arbitrary positive numbers. The discrete update scheme (1) achieves consensus asymptotically for $\mathcal{A}$ if there exists an infinite sequence of uniformly bounded, non-overlapping time intervals $[k_j T, (k_j + l_j)T)$, $j = 1, 2, \cdots$, starting at $k_1 = 0$, with the property that each interval $[(k_j + l_j)T, k_{j+1}T)$ is uniformly bounded and the union of the graphs across each such interval has a spanning tree. Furthermore, if the union of the graphs after some finite time does not have a spanning tree, then consensus cannot be achieved asymptotically for $\mathcal{A}$.

Proof: Let $\bar{D}$ denote the set of all possible matrices $D[k]$ under dynamically changing interaction topologies and weighting factors $\alpha_{ij}[k]$. We know that $\bar{D}$ is a finite set since both set $\bar{\mathcal{G}}$ and set $\bar{\alpha}$ are finite.

Consider the jth time interval $[k_j T, k_{j+1}T)$, which includes the time interval $[k_j T, (k_j + l_j)T)$ and must be uniformly bounded since both $[k_j T, (k_j + l_j)T)$ and $[(k_j + l_j)T, k_{j+1}T)$ are uniformly bounded. Also the sequence of time intervals $[k_j T, k_{j+1}T)$, $j = 1, 2, \cdots$, are contiguous.

The union of the graphs across $[k_j T, k_{j+1}T)$, denoted as $\bar{\mathcal{G}}[k_j]$, has a spanning tree since the union of the graphs across $[k_j T, (k_j + l_j)T)$ has a spanning tree. Let $\{D[k_j], D[k_j + 1], \cdots, D[k_{j+1} - 1]\}$ be the set of stochastic matrices corresponding to each graph in the union $\bar{\mathcal{G}}[k_j]$. Following Lemma 3.7, the matrix product $D[k_{j+1} - 1] \cdots D[k_j + 1]D[k_j]$, $j = 1, 2, \cdots$, is SIA. Then by applying Lemma 3.2 and mimicking a similar proof for Theorem 2 in [3], the first part can be proved.

If the union of the graphs after some finite time $\hat{t}$ does not have a spanning tree, then during the infinite time interval $[\hat{t}, \infty)$, there exist at least two agents such that there is no path in the union of the graphs that contains these two agents, which then implies that information of these two agents cannot reach consensus. ■

B. Consensus Using Continuous Time Update Scheme

The continuous-time analog of Theorem 3.2 has been shown in [10]. Therefore, we will focus on demonstrating that under certain conditions, the existence of a spanning tree is also sufficient for consensus under dynamically changing interaction topologies using the continuous time update scheme. To do so, we need the following lemma.

Lemma 3.8: If the union of the directed graphs $\{\mathcal{G}_{t_1}, \mathcal{G}_{t_2}, \cdots, \mathcal{G}_{t_m}\} \subset \bar{\mathcal{G}}$ has a spanning tree and C_{t_i} is the matrix corresponding to each directed graph $\mathcal{G}_{t_i}$ in Eq. (4), then the matrix product $e^{C_{t_m}\Delta t_m} \cdots e^{C_{t_2}\Delta t_2} e^{C_{t_1}\Delta t_1}$ is SIA, where $\Delta t_i > 0$ are bounded.

Proof: From Eq. (4), each matrix C_{t_i} satisfies the properties defined in Lemma 3.3. Thus each C_{t_i} can be written as the sum of a nonnegative matrix M_{t_i} and $-\eta_{t_i} I_n$, where η_{t_i} is the maximum absolute value of the diagonal entries of C_{t_i}, $i = 1, \cdots, m$.

From Lemma 1 in [10], we know that $e^{C_{t_i}\Delta t_i} = e^{-\eta_{t_i}\Delta t_i} e^{M_{t_i}\Delta t_i} \geq \rho_i M_{t_i}$ for some $\rho_i > 0$. Since the union of the directed graphs $\{\mathcal{G}_{t_1}, \mathcal{G}_{t_2}, \cdots, \mathcal{G}_{t_m}\}$ has a spanning tree, we know that the union of the directed graphs of M_{t_i} has a spanning tree, which in turn implies that the union of the directed graphs of $e^{C_{t_i}\Delta t_i}$ has a spanning tree. From Lemma 3.1, we know that $e^{C_{t_m}\Delta t_m} \cdots e^{C_{t_2}\Delta t_2} e^{C_{t_1}\Delta t_1} \geq \gamma \sum_{i=1}^{m} e^{C_{t_i}\Delta t_i}$ for some $\gamma > 0$, which implies that the above matrix product also has a spanning tree.

It can also be verified that each matrix $e^{C_{t_i}\Delta t_i}$ is a stochastic matrix with positive diagonal entries, which implies that the above matrix product is also stochastic with positive diagonal entries.

Combining Corollary 3.1 and Lemma 3.6, we know that the above matrix product is SIA. ■

In this paper, we also apply dwell time (c.f. [14], [3]) to the continuous time update scheme (4), which implies that the interaction graph and weighting factors are constrained to change only at discrete times, that is, the matrix $C(t)$ is piecewise constant.

Eq. (4) can be rewritten as

$$\dot{\xi}(t) = C(t_i)\xi(t), \quad t \in [t_i, t_i + \tau_i) \tag{5}$$

where t_0 is the initial time and $t_1, t_2, \cdots$ is an infinite time sequence at which the interaction graph or weighting factors change, resulting in a change in $C(t)$.

Let $\tau_i = t_{i+1} - t_i$ be the dwell time, $i = 0, 1, \cdots$. Note that the solution to Eq. (5) is given by $\xi(t) = e^{C(t_k)(t-t_k)} e^{C(t_{k-1})\tau_{k-1}} \cdots e^{C(t_1)\tau_1} e^{C(t_0)\tau_0} \xi(0)$, where k is the largest nonnegative integer satisfying $t_k \leq t$. Let $\bar{\tau}$ be a finite set of arbitrary positive numbers. Let Υ be an infinite set generated from set $\bar{\tau}$, which is closed under addition, and multiplications by positive integers. We assume that $\tau_i \in \Upsilon$, $i = 0, 1, \cdots$. By choosing the set $\bar{\tau}$ properly, dwell time can be chosen from an infinite set Υ, which somewhat simulates the case when the interaction graph $\mathcal{G}$ changes dynamically over time.

The following theorem extends the continuous time convergence result in [3].

Theorem 3.4: Let $t_1, t_2, \cdots$ be an infinite time sequence at which the interaction graph or weighting factors switch and $\tau_i = t_{i+1} - t_i \in \Upsilon$, $i = 0, 1, \cdots$. Let $\mathcal{G}(t_i) \in \bar{\mathcal{G}}$ be a switching interaction graph at time $t = t_i$ and $\sigma_{ij}(t_i) \in \bar{\sigma}$, where $\bar{\sigma}$ is a finite set of arbitrary positive numbers. The continuous time update scheme (3) achieves consensus asymptotically for $\mathcal{A}$ if there exists an infinite sequence of uniformly bounded, non-overlapping time intervals $[t_{i_j}, t_{i_j + l_j})$, $j = 1, 2, \cdots$, starting at $t_{i_1} = t_0$, with the property that each interval $[t_{i_j + l_j}, t_{i_{j+1}})$ is uniformly bounded and the union of the graphs across each such interval has a spanning tree. Furthermore, if the union of the graphs after some finite time does not have a spanning tree, then consensus cannot be achieved asymptotically for $\mathcal{A}$.

Proof: The set of all possible matrices $e^{C(t_i)\tau_i}$, where $\tau_i \in \Upsilon$, under dynamically changing interaction topologies and weighting factors can be chosen or constructed by matrix multiplications from the set $\bar{E} = \{e^{C(t_i)\tau_i}, \tau_i \in \bar{\tau}\}$. Clearly $\bar{E}$ is finite since $\bar{\mathcal{G}}$, $\bar{\sigma}$, and $\bar{\tau}$ are all finite.

Consider the jth time interval $[t_{i_j}, t_{i_{j+1}})$, which includes the time interval $[t_{i_j}, t_{i_j+l_j})$ and must be uniformly bounded since both $[t_{i_j}, t_{i_j+l_j})$ and $[t_{i_j+l_j}, t_{i_{j+1}})$ are uniformly bounded. Also the sequence of time intervals $[t_{i_j}, t_{i_{j+1}})$, $j = 1, 2, \cdots$, are contiguous.

The union of the graphs across $[t_{i_j}, t_{i_{j+1}})$, denoted as $\bar{\mathcal{G}}(t_{i_j})$, has a spanning tree since the union of graphs across $[t_{i_j}, t_{i_j+l_j})$ has a spanning tree. Let $\{C(t_{i_j}), C(t_{i_j+1}), \cdots, C(t_{i_{j+1}-1})\}$ be a set of matrices corresponding to each graph in the union $\bar{\mathcal{G}}(t_{i_j})$. Following Lemma 3.8, the matrix product $e^{C(t_{i_{j+1}-1})\tau_{i_{j+1}-1}} \cdots e^{C(t_{i_j+1})\tau_{i_j+1}} e^{C(t_{i_j})\tau_{i_j}}$, $j = 1, 2, \cdots$, is SIA. Then, the first part follows from Lemma 3.2 and an argument similar to the proof of Theorem 2 in [3].

The second part is similar to that in Theorem 3.3. ■

C. Discussion

The contribution of this paper is that the results in [3], which are limited to undirected graphs, are extended to directed graphs. Therefore, unidirectional information exchange is allowed instead of requiring bidirectional information exchange. This will be important in applications where bidirectional communication or sensing are not available.

Ref. [3] shows that consensus of information (the heading of each agent in their context) can be achieved if the union of a collection of graphs is connected frequently enough. This paper demonstrates that the same result can be achieved as long as the union of the graphs has a spanning tree, which is a milder requirement than being connected and implies that one half of the information exchange links required in [3] can be removed without adversely affecting the convergence result. In this sense, the results for convergence in [3] can be thought of as a special case of a more general result. Of course, the final achieved equilibrium points will depend on the property of the directed graphs. For example, compared to strongly connected graphs, graphs that are not strongly connected will reach different final equilibrium points (see [10] for an analysis of the final equilibrium points).

The leader following scenario described in [3] can also be thought of as a special case of our result. If there is one agent in the group which does not have any incoming link, but the union of the interaction graphs has a spanning tree frequently enough, then this agent must be the root of the spanning tree, i.e, the leader. Since consensus is guaranteed, the information state of the other agents asymptotically converges to the information state of the leader. Therefore, the scenario discussed in [3] of being linked to a leader frequently enough is a special case of having a spanning tree with the leader as the root, frequently enough.

For the continuous model used in [3], the switching times of the interaction graph is constrained to be separated by τ_D time units, where τ_D is a constant dwell time. Our continuous update scheme allows the switching times to be within an infinite set of positive numbers generated by any finite set of positive numbers, which is better suited to simulating the random switching of interaction graphs. Therefore, the continuous scheme in [3] can be thought of a special case of our result by letting $\bar{\tau} = \{\tau_d\}$ and $\Upsilon = \{k\tau_d | k = 1, 2, \cdots\}$.

Unlike the update schemes in [3], we do not constrain the weighting factors in our discrete and continuous update schemes, other than to require that they are positive. This provides flexibility to account for relative confidence and relative reliabilities of information from different agents.

An additional contribution of this paper is a new result for nonnegative matrices with the same positive row sums. The Perron-Frobenius Theorem states that if a nonnegative matrix A is irreducible, that is, the directed graph of A is strongly connected, then the spectral radius of A is a simple eigenvalue. We show that the irreducibility condition is too stringent for nonnegative matrices with the same positive row sums. Lemma 3.4 shows that for a nonnegative matrix A with identical positive row sums, the spectral radius of A (the row sum in this case) is a simple eigenvalue if and only if the directed graph of A has a spanning tree. In other words, A may be reducible but retains its spectral radius as a simple eigenvalue. Furthermore, if A has a spanning tree and positive diagonal entries, we know that the spectral radius of A is the unique eigenvalue of maximum modulus.

IV. Simulation Results

In this section, we simulate information consensus for five agents under dynamically changing interaction topologies using the discrete time update scheme (2) and the continuous time update scheme (5) respectively.

For simplicity, we constrain the possible interaction graphs for these five agents to be within the set $\mathcal{G}_s = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5\}$ as shown in Fig. 1, which is obviously a subset of $\bar{\mathcal{G}}$. For the discrete time update scheme, we assume that the interaction graph switches randomly in $\mathcal{G}_s$ at each time $t = kT$, where $k = 0, 1, 2, \cdots$ and T is 0.5 seconds. For the continuous update scheme, we assume that the interaction graph switches randomly in $\mathcal{G}_s$ at each random time $t = t_k$, $k = 0, 1, 2, \cdots$. The weighting factors in Eqs. (2) and (5) are chosen arbitrarily a priori for each directed graph in $\mathcal{G}_s$ to satisfy $\alpha_{ij}[k] > 0$ and $\sigma_{ij}(t_k) > 0$, $(i, j) \in \mathcal{I}$ and $k = 0, 1, 2, \cdots$.

Note that each directed graph in $\mathcal{G}_s$ does not have a spanning tree but that the union of these graphs do have a spanning tree is evident from Fig. 2. As the switching between graphs in $\mathcal{G}_s$ is random, the condition for consensus will be generically satisfied. Alternatively, it is obvious that the union of these graphs is not connected, which implies that the conditions in [3] are not satisfied. Simulation results show that asymptotic consensus is achieved using both the discrete time update scheme and the continuous time update scheme.

The initial information state was selected arbitrarily as $\xi_i = 0.2 * i$, $i = 1, \cdots, 5$. Fig. 3 shows the consensus results using both the discrete time update scheme and the continuous time update scheme. Note that $\xi_i(t)$, $i = 1, \cdots, 5$, reaches consensus for both cases.

Consider now a leader following scenario where the information graph switches in $\mathcal{G}_s' \stackrel{\triangle}{=} \mathcal{G}_s \setminus \mathcal{G}_1$. As a result, there is no information exchange link from A_3 to A_1. In this case, the union of the information graphs has a spanning tree, however, unlike the previous case there is

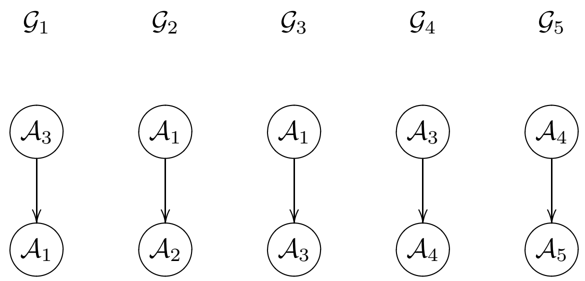

Fig. 1. Possible interaction topologies for $\mathcal{A} = \{A_i | i = 1, \cdots, 5\}$.

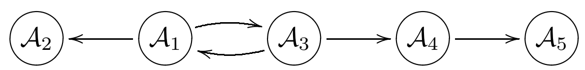

Fig. 2. The union of $\mathcal{G}_s$.

no incoming information link to A_1. Fig. 4 shows the consensus results using both the discrete time update scheme and the continuous time update scheme. Note that $\xi_i(t)$, $i = 2, \cdots, 5$, converges asymptotically to $\xi_1(0)$ as expected. This is similar to the leader following case in [3] except that we do not need the followers to be jointly linked to the leader, that is, the union of the directed graphs is not necessarily connected.

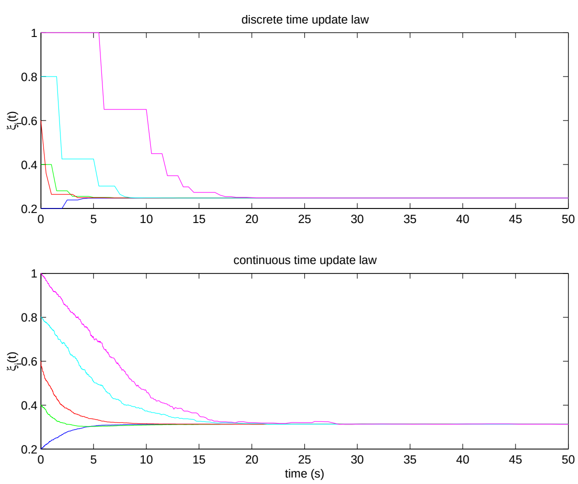

Fig. 3. Consensus with $\mathcal{G}[k]$ and $\mathcal{G}(t_k)$ randomly switching from $\mathcal{G}_s$.

V. Conclusion

This paper has considered the problem of information consensus under dynamically changing interaction topologies and weighting factors. We have applied directed graphs to represent information exchanges among multiple agents, taking into account the general case of unidirectional information exchange. We also proposed discrete and continuous update schemes for information consensus and gave conditions for asymptotic consensus under dynamically changing interaction topologies and weighting factors using these update schemes. Simulation examples were presented to illustrate the results.

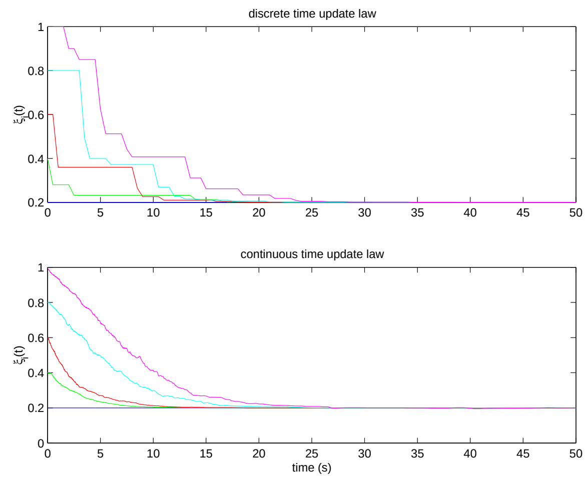

Fig. 4. Consensus with $\mathcal{G}[k]$ and $\mathcal{G}(t_k)$ randomly switching from $\mathcal{G}_s'$.

Acknowledgments

This work was funded by AFOSR grants F49620-01-1-0091 and F49620-02-C-0094, and by DARPA grant NBCH1020013.

References

[1] J. A. Fax and R. M. Murray, "Information fbw and cooperative control of vehicle formations," in *IFAC World Congress*, (Barcelona, Spain), 2002.

[2] J. A. Fax and R. M. Murray, "Graph Laplacians and stabilization of vehicle formations," in *IFAC World Congress*, (Barcelona, Spain), 2002.

[3] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, pp. 988–1001, June 2003.

[4] R. Olfati-Saber and R. M. Murray, "Agreement problems in networks with directed graphs and switching topology," in *Proceedings of the CDC*, (Maui, Hawaii), December 2003.

[5] R. W. Beard and V. Stepanyan, "Information consensus in distributed multiple vehicle coordinated control," in *Proceedings of the CDC*, (Maui, Hawaii), December 2003.

[6] T. Vicsek, A. Czirok, E. B. Jacob, I. Cohen, and O. Schochet, "Novel type of phase transitions in a system of self-driven particles," *Physical Review Letters*, vol. 75, pp. 1226–1229, 1995.

[7] J. S. Bellingham, M. Tillerson, M. Alighanbari, and J. P. How, "Cooperative path planning for multiple UAVs in dynamic and uncertain environments," in *Proceedings of the CDC*, (Las Vegas, NV), pp. 2816–2822, December 2002.

[8] W. Ren and R. W. Beard, "Decentralized scheme for spacecraft formation fying via the virtual structure approach," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 27, pp. 73–82, January-February 2004.

[9] T. W. McLain and R. W. Beard, "Coordination variables, coordination functions, and cooperative timing missions," in *Proceedings of the ACC*, (Denver, CO), pp. 296–301, June 2003.

[10] W. Ren, R. W. Beard, and T. W. McLain, *Coordination Variables and Consensus Building in Multiple Vehicle Systems*. Proceedings of the Block Island Workshop on Cooperative Control, Springer-Verlag Series: Lecture Notes in Control and Information Sciences, 2003. (to appear).

[11] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.

[12] C. Godsil and G. Royle, *Algebraic Graph Theory*. New York: Springer Graduate Texts in Mathematics #207, 2001.

[13] J. Wolfowitz, "Products of indecomposable, aperiodic, stochastic matrices," *Proceedings of the American Mathematical Society*, vol. 15, pp. 733–736, 1963.

[14] A. S. Morse, "Supervisory control of families of linear set-point controllers-part 1: Exact matching," *IEEE Transactions on Automatic Control*, vol. 41, no. 10, pp. 1413–1431, 1996.