

# Jerk Limited Input Shapers

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**Abstract**—The focus of this paper is on the design of jerk limited input shapers (time-delay filters). Closed form solutions for the jerk limited time-delay filter for undamped systems is derived followed by the formulation of the problem for damped systems. Since the jerk limited filter involves concatenating an integrator to a time-delay filter, a general filter design technique is proposed where smoothing of the shaped input can be achieved by concatenating transfer functions of first order, harmonic systems, etc.

## I. INTRODUCTION

Prefiltering of command inputs to systems with underdamped modes has been addressed by various researchers [1], [2], [3], [9]. Smith's *Posicast Controller* [1] was motivated by a simple wave cancellation concept for the elimination of the oscillatory motion of underdamped systems. This technique required exact knowledge of the damping and natural frequency of the plant to be able to eliminate residual vibrations. Singer and Seering [2] addressed this problem by proposing a technique to design a series of impulses whose amplitudes and application time were determined so as to force the residual energy and the sensitivity of the residual energy with respect to natural frequency or damping to zero. The filtered input was then generated by the convolution of the command input with the impulse sequence. Singh and Vadali [8] proposed a technique to design time-delay filters whose performance was identical to the *Input Shaping* technique proposed by Singer and Seering [2]. Over the past decade numerous papers have been published which deal with the design of discrete time and continuous time prefilterers for the robust vibrations control of maneuvering structures. These include the digital shaping filter by Murphy and Watanabe [11], multi-hump input shapers by Singhose et al., minimax filters by Singh [12], user specified time-delay filters by Singh and Vadali [8] besides others. The technique to desensitize the input profile to modeling errors have been used to address a slew of classic optimal control problems such as time-optimal [6], [7], [4], fuel-time optimal [5], and minimum power/jerk controllers [10].

The input shaping/time-delay filtering technique include information of specific modes in the design process. If it is necessary to roll off the energy over the high frequency spectrum, additional constraints need to be included in the design process. Jerk limits in the design process can result in control profiles which can be tracked by actuators and which can be used to minimize the excitation of the unmodeled high frequency modes of structures. Muenchhof

and Singh [14] present a detailed development of the design technique for the minimum-time jerk limited control profiles for maneuvering underdamped flexible structures. Lim et al. [13], propose a technique for the design of multi-input shapers which permits inclusion of constraints on the jerk. This paper addresses the problem of jerk limited input shapers for prefiltering command inputs to vibratory systems without rigid body modes. The paper by Muenchhof and Singh [14] addressed the problem of design of control profiles for systems with rigid body modes. The paper will start by addressing the design of a time-delay filters where the delay time and the gains of the delayed signals are all unknown. This will be followed by the presentation of a general concept to design input shapers by including additional dynamics to the time-delay filter such as harmonic oscillators and first order dynamics to permit smooth ramping up and ramping down of control profiles. The paper will conclude with some remarks.

## II. JERK LIMITED INPUT SHAPERS

### A. Undamped Systems

This section deals with the design of Jerk Limited Time-Delay filter (Input Shaper) which is schematically represented in Figure 1. The development which follows is for a single mode system, but can be easily extended for multiple mode systems.

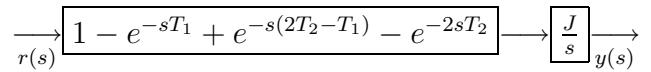


Fig. 1. Single Time-Delay Controlled System

The transfer function of the filter shown in Figure 1 without the integrator element is

$$G(s) = (1 - \exp(-sT_1) + \exp(-s(2T_2 - T_1)) - \exp(-2sT_2)), \quad (1)$$

The output of the transfer function  $G(s)$  subject to a unit step input is shown in Figure 2 and its time integral is represented as

$$y(t) = J(t - (t - T_1)\mathcal{H}(t - T_1) + (t - (2T_2 - T_1))\mathcal{H}(t - (2T_2 - T_1)) - (t - 2T_2)\mathcal{H}(t - 2T_2)), \quad (2)$$

where  $J$  is the permissible jerk and  $\mathcal{H}()$  is the Heaviside Step function.  $y(t)$  should equal 1 at steady state for a DC gain of unity which results in the constraint equation



Fig. 2. Parameterized Control Profile

$$y(2T_2) = J(2T_2 - (2T_2 - T_1) + (2T_2 - (2T_2 - T_1))) = 1, \quad (3)$$

or

$$T_1 = \frac{1}{2J}. \quad (4)$$

which implies that the first switch  $T_1$  is only a function of the permitted jerk. To cancel the undamped poles of the system, we require a pair of zeros of the time-delay filter to cancel the poles of the system. This results in the constraint equations

$$1 - \cos(\omega T_1) + \cos(\omega(2T_2 - T_1)) - \cos(2\omega T_2) = 0 \quad (5)$$

and

$$-\sin(\omega T_1) + \sin(\omega(2T_2 - T_1)) - \sin(2\omega T_2) = 0 \quad (6)$$

These two constraint equations are satisfied if

$$\sin(\omega T_2) = \sin(\omega(T_2 - T_1)). \quad (7)$$

Substituting Equation 4 into Equation 7, and simplifying we have

$$\tan(\omega T_2) = -\cot\left(\frac{\omega}{4J}\right) \quad (8)$$

which results in the closed form solutions

$$T_2 = \frac{(2n+1)\pi}{2\omega} + \frac{1}{4J}. \quad (9)$$

For specific values of  $\omega$  and  $J$ ,  $T_1$  can equal  $T_2$ , which corresponds to the first and the second switch collapsing. From Equations 4 and 7, this corresponds to

$$\sin(\omega T_2) = 0, \Rightarrow \cos\left(\frac{\omega}{4J}\right) = 0 \quad (10)$$

or

$$\frac{\omega}{4J} = (2m+1)\frac{\pi}{2}, \quad m=1,2,3\dots \quad (11)$$

So, for a given  $J$  or  $\omega$ , we can solve for  $\omega$  or  $J$  respectively for which  $T_1$  and  $T_2$  are equal, which corresponds to a simple ramp input to the system.

Figures 3 and 4 illustrate the variation of the switch times and the final time of the time-delay filter as a function of varying frequency and Jerk respectively. It is clear from Figure 4, that the first and second switch coincide which corresponds to the solid line intersecting the dashed line. Figure 4 is generated for  $J = 3$ , for which we have from Equation 11,  $\omega = 6\pi, 18\pi, 30\pi$ , for which the switches collapse, which corroborates the results in Figure 4

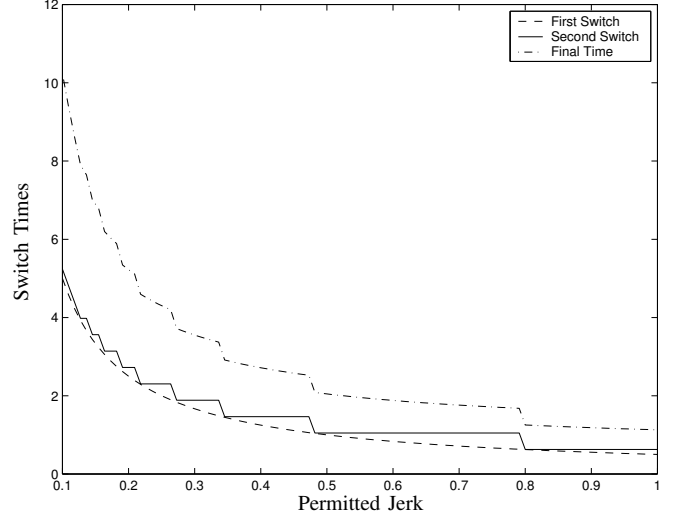


Fig. 3. Switch Time Variation vs Jerk for  $\omega = 15$

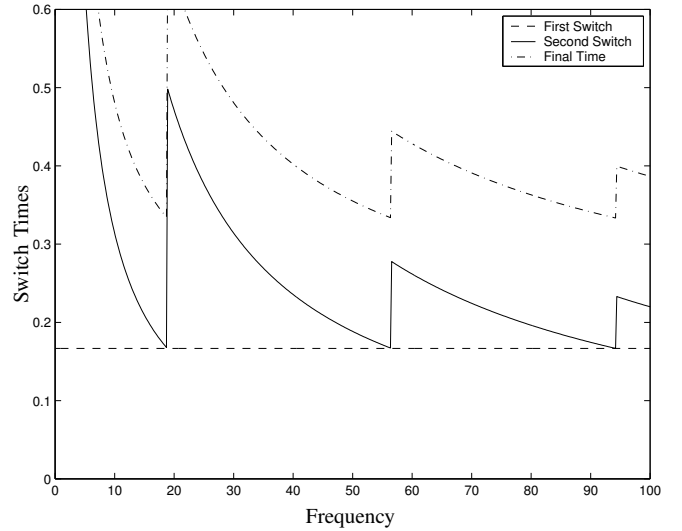


Fig. 4. Switch Time Variation vs Frequency

## B. Damped Systems

The jerk limited time delay filter for damped systems cannot be solved in closed form. The problem can be solved numerically by an optimization problem.

The jerk limited time-delay filter is parameterized as

$$G(s) = \frac{J}{s}(1 - \exp(-sT_1) + \exp(-sT_2) - \exp(-sT_3)). \quad (12)$$

To satisfy the requirement that the final value of the jerk limited time-delay filter be unity when it is driven by an unit step input results in the constraint equation

$$y(T_3) = J(T_3 + T_1 - T_2) = 1. \quad (13)$$

To cancel the damped poles of the system at  $s = \sigma \pm j\omega$ , we require a pair of zeros of the time-delay filter to cancel the damped poles of the system. This results in the constraint

equations

$$1 - e^{-\sigma T_1} \cos(\omega T_1) + e^{-\sigma T_2} \cos(\omega T_2) - e^{-\sigma T_3} \cos(\omega T_3) = 0 \quad (14)$$

and

$$-e^{-\sigma T_1} \sin(\omega T_1) + e^{-\sigma T_2} \sin(\omega T_2) - e^{-\sigma T_3} \sin(\omega T_3) = 0. \quad (15)$$

The optimization problem can be stated as minimization of  $T_3$  subject to the three equality constraints given by Equations 13, 14 and 15.

### III. ROBUST JERK LIMITED TIME-DELAY FILTER

Most systems have errors in estimated damping and natural frequencies which can result in significant residual errors when a rest-to-rest maneuver is performed. It is therefore imperative to design filters which can handle uncertainties in estimated model parameters. There are multiple approaches to achieve robustness. The simplest includes reducing the sensitivity of the residual energy of the modes, at the nominal values of estimated system parameters. If bounds and distributions of the uncertain parameters are available to the designer, the minimax approach proposed by Singh [4] can be used to arrive at filters which minimize the maximum magnitude of the residual energy in the domain of interest. In this work, robustness is achieved by placing multiple zeros of the time-delay filter at the location of the uncertain poles of the plant.

The added requirement of robustness results in a filter with increased number of parameters to be determined. The approach for the design of robust jerk limited time-delay filters is developed for damped systems with the knowledge that the undamped systems are a sub-set of the damped system. The robust jerk limited time-delay filter is parameterized as

$$G(s) = \frac{J}{s} (1 - \exp(-sT_1) + \exp(-sT_2) - \exp(-sT_3) + \exp(-sT_4) - \exp(-sT_5)). \quad (16)$$

To satisfy the requirement that the final value of the jerk limited time-delay filter be unity when it is driven by an unit step input results in the constraint equation

$$y(T_5) = J(T_5 - T_4 + T_3 - T_2 + T_1) = 1. \quad (17)$$

To cancel the damped poles of the system at  $s = \sigma \pm j\omega$ , we require a pair of zeros of the time-delay filter to cancel the damped poles of the system. This results in the constraint equations

$$1 - e^{-\sigma T_1} \cos(\omega T_1) + e^{-\sigma T_2} \cos(\omega T_2) - e^{-\sigma T_3} \cos(\omega T_3) + e^{-\sigma T_4} \cos(\omega T_4) - e^{-\sigma T_5} \cos(\omega T_5) = 0 \quad (18)$$

and

$$-e^{-\sigma T_1} \sin(\omega T_1) + e^{-\sigma T_2} \sin(\omega T_2) - e^{-\sigma T_3} \sin(\omega T_3) + e^{-\sigma T_4} \sin(\omega T_4) - e^{-\sigma T_5} \sin(\omega T_5) = 0. \quad (19)$$

The robustness is achieved by placing a second pair of zeros of the time-delay filter at the estimated location of the oscillatory poles of the system, which results in the equations

$$-T_1 e^{-\sigma T_1} \sin(\omega T_1) + T_2 e^{-\sigma T_2} \sin(\omega T_2) - T_3 e^{-\sigma T_3} \sin(\omega T_3) + T_4 e^{-\sigma T_4} \sin(\omega T_4) - T_5 e^{-\sigma T_5} \sin(\omega T_5) = 0 \quad (20)$$

and

$$-T_1 e^{-\sigma T_1} \cos(\omega T_1) + T_2 e^{-\sigma T_2} \cos(\omega T_2) - T_3 e^{-\sigma T_3} \cos(\omega T_3) + T_4 e^{-\sigma T_4} \cos(\omega T_4) - T_5 e^{-\sigma T_5} \cos(\omega T_5) = 0. \quad (21)$$

The optimization problem can now be stated as the minimization of  $T_5$  subject to the constraint given by Equations 17-21.

To illustrate the reduced sensitivity of the residual energy to variations in the frequency, the response of the system was studied for various values of model frequencies with a filter designed for a frequency of 15 rad/sec and a permitted jerk of 4. Figure 5 illustrates the improved performance of the robust jerk limited time-delay filter.

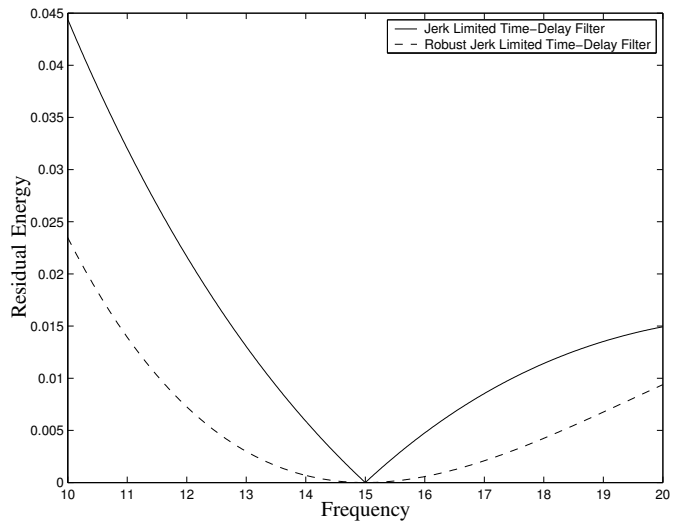


Fig. 5. Sensitivity Curve

### IV. JERK LIMITED TIME-DELAY FILTERS FOR MULTI-MODE SYSTEMS

The proposed approach can be used for the control of systems with multiple under-damped modes. A generic formulation is developed below. The number of parameters to be optimized for can be reduced for undamped systems by exploiting the symmetric characteristics of the time-delay filter. The transfer function of the time-delay filter is

$$G(s) = \frac{J}{s} \sum_{i=0}^N (-1)^i \exp(-sT_i) \quad (22)$$

where  $T_0 = 0$  and  $N$  is an odd number. The unknowns ( $T_i$ ) have to satisfy the constraint equation

$$\sum_{i=1}^N (-1)^{i+1} T_i = \frac{1}{J} \quad (23)$$

which satisfies the requirement that the final value of the output of the filter when it is subject to an unit step input is unity. To cancel the undamped or under-damped poles at

$$s_k = \sigma_k \pm j\omega_k \text{ for } k = 1, 2, 3, \dots \quad (24)$$

the following constraints have to be satisfied

$$\sum_{i=1}^N (-1)^i \exp(-\sigma_k T_i) \cos(\omega_k T_i) = 0 \text{ for } k = 1, 2, 3, \dots \quad (25)$$

and

$$\sum_{i=1}^N (-1)^i \exp(-\sigma_k T_i) \sin(\omega_k T_i) = 0 \text{ for } k = 1, 2, 3, \dots \quad (26)$$

The optimal solution is one which satisfies all the constraints and minimizes  $T_N$ .

To desensitize the filter to errors in estimated damping or frequency, the following constraint equations are added to the optimization problem

$$\sum_{i=1}^N (-1)^i T_i \exp(-\sigma_k T_i) \sin(\omega_k T_i) = 0 \text{ for } k = 1, 2, 3, \dots \quad (27)$$

and

$$\sum_{i=1}^N (-1)^i T_i \exp(-\sigma_k T_i) \cos(\omega_k T_i) = 0 \text{ for } k = 1, 2, 3, \dots \quad (28)$$

It can be seen that desensitizing the filter with respect to damping simultaneously desensitizes the filter to the frequency as well.

The design of jerk limited time-delay filters for user specified time-delays follows the process proposed by Singh and Vadali [8]. It is clear that additional number of delays are required since the delay times are no longer variables in the optimization process.

To illustrate the design of multi-mode jerk limited input shapers, consider the system

$$\frac{y(s)}{u(s)} = \frac{225}{s^4 + 34s^2 + 225} \quad (29)$$

which is characterized by two modes with frequencies 3 and 5. For a jerk constraint of 3, the jerk limited input shaper is designed. The dashed line and the solid line in Figure 6 illustrates the response of the system to a step input and the shaped input respectively. It is clear that the residual

vibration of the two modes is eliminated after shaping the input.

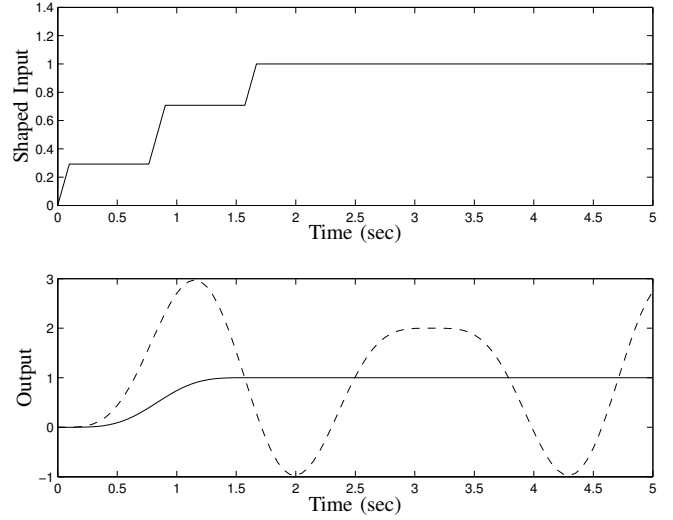


Fig. 6. Shaped Input and Comparison of system response

## V. FILTERED INPUT SHAPERS

The technique presented in this work where an integrator is concatenated to a time-delay filter to satisfy the constraint of jerk limited filter design can be extended by cascading other transfer functions such as that of first order systems, harmonic systems etc.

### A. First Order Filtered Input Shaper

Instead of using an integrator in conjunction with a time-delay filter to account for the limit on the permitted jerk, one can concatenate a first order filter to a time-delay filter to generate a smooth input which can then be used to drive a time-delay filter designed to cancel the underdamped poles of the system of interest. Figure 7 illustrates the proposed filter structure where  $T$  is a user selected time-delay which in the case of a discrete time implementation, can be an integral multiple of the sampling interval.

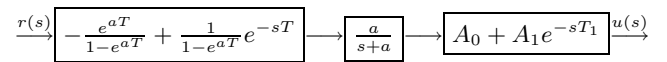


Fig. 7. First Order Filtered Time-Delay Filter

### B. Sinusoid Filtered Input Shaper

Filtering with a transfer function of a scaled sinusoid results in an input which emulates a step input but with zero initial and final slopes. The scaling of the sinusoid transfer function is to satisfy the requirement that the DC gain of the transfer function is unity. The sinusoid filtered time-delay filter is illustrated in Figure 8 which can be rewritten as shown in Figure 9.

Here the first time-delay filter cancels the oscillatory response of the scaled harmonic oscillator. This truncated

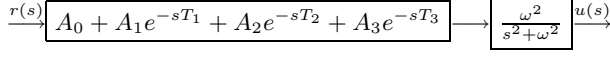


Fig. 8. Sinusoid Filtered Time-Delay Filter

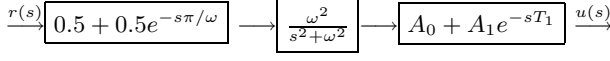


Fig. 9. Sinusoid Filtered Time-Delay Filter

harmonic response is then input to the second time-delay filter which is designed to cancel the oscillatory mode of the system. Figure 10 illustrates the control profile. The benefit of this approach can be gauged from the frequency response plots of the sinusoid filtered time-delay filter. Figure 11 illustrates the frequency response plots of the time-delay filter, jerk limited time-delay filter and a sinusoid filtered time-delay filter. The sinusoid filtered time delay filter has been designed such that the maximum jerk of the control profile is equal to the maximum permitted jerk. It can easily be seen that the magnitude plots of the sinusoid filtered time-delay filters rolls off much more rapidly compared to the time-delay filter and the jerk limited time-delay filter. Thus, this input will not significantly excite the unmodeled dynamics.

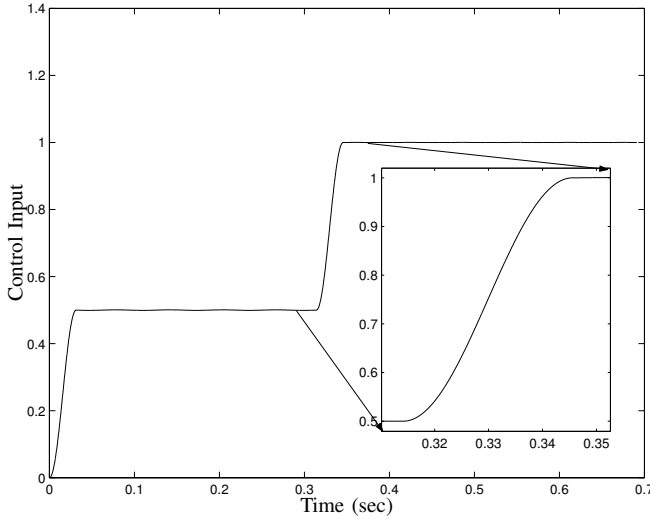


Fig. 10. Sinusoid Filtered, Input Shaped Control Input

### C. Jerk Limits

Consider a part of the sinusoid filtered time-delay filter illustrated in Figure 12.

The output  $p$  of the time-delay filter subject to a unit step input is

$$p(t) = \sin^2(\omega/2t) + \sin^2(\omega/2(t - \pi/\omega))\mathcal{H}(t - \frac{\pi}{\omega}) \quad (30)$$

and the rate of change of  $p$  which is the jerk is

$$\dot{p}(t) = \frac{\omega}{2} \sin(\omega t) + \frac{\omega}{2} \sin(\omega(t - \pi/\omega))\mathcal{H}(t - \frac{\pi}{\omega}) \quad (31)$$

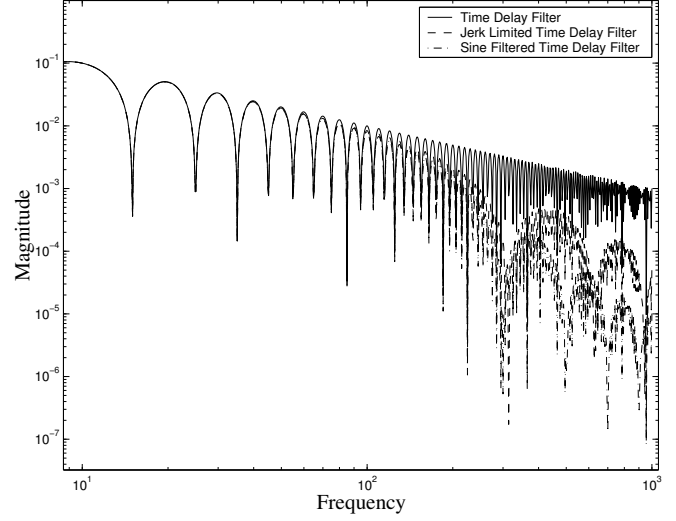


Fig. 11. Frequency Response

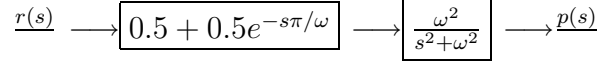


Fig. 12. Time-Delay Filter

which implies that the maximum magnitude of the jerk is  $\frac{\omega}{2}$  and occurs at time  $t = \frac{\pi}{2\omega}$ . This is the upper bound for the jerk. It can be seen that the jerk is zero at the start and the end of the maneuver which results in a very practical control profile. When the signal  $p$  is passed through the second time-delay filter, based on the damping present in the oscillatory pole to be cancelled, the jerk can lie in the limit

$$\frac{\omega}{4} \leq \text{Maximum Jerk} \leq \frac{\omega}{2}. \quad (32)$$

If the pole to be cancelled is undamped the maximum jerk is  $\frac{\omega}{4}$  since  $A_0$  and  $A_1$  are equal to 0.5. When the poles to be cancelled contain damping,  $A_0$  is greater than 0.5 and  $A_1$  is less than 0.5, resulting in the maximum jerk lying in the range specified by Equation 32.

This constraint is valid when the time-delay filter is designed to cancel the unwanted under-damped pole. However, if the under-damped pole has to be controlled using a robust time-delay filter, the limits on the jerk changes, since the robust time-delay filter uses smaller gains.

## VI. CONCLUSIONS

A simple technique to design filtered Input Shapers is proposed in this paper. The paper first addresses the problem of design of jerk limited time-delay filters which results in a ramping of the control input. This motivates the design of filtered Input shapers by concatenating transfer functions of a scaled harmonic oscillator in addition to others, to result in smooth control profiles. The roll off of the frequency response plots for the filtered Input Shapers is used to illustrate their benefits.

## VII. ACKNOWLEDGMENT

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