Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Robust Control Synthesis with General Frequency Domain Specifications: Static Gain Feedback Case

Tetsuya Iwasaki
Mechanical and Aerospace Engineering
University of Virginia, 122 Engineer's Way
Charlottesville, VA 22904-4746
iwasaki@virginia.edu

Shinji Hara
Information Physics and Computing
The University of Tokyo, 7-3-1 Hongo
Bunkyo, Tokyo 113-8656, Japan
Shinji_Hara@ipc.i.u-tokyo.ac.jp

***Abstract*— This paper considers a robust control synthesis problem for uncertain linear systems to meet design specifications given in terms of multiple frequency domain inequalities in (semi)finite ranges. In this paper, we restrict our attention to static gain feedback controllers. We will develop a new multiplier method that allows for reduction of synthesis conditions to linear matrix inequality problems. We study conditions under which the reduction is exact (nonconservative) in the single-objective nominal setting. Although the multiplier method is conservative in the general setting of multi-objective robust control, numerical design examples demonstrate the utility of the method for the state feedback case.**

I. Introduction

Design specifications for practical control synthesis are often given in terms of frequency domain inequalities (FDIs). Most state space approaches to such design problems rely on the Kalman-Yakubovich-Popov (KYP) lemma [1], [2] that converts an FDI to a linear matrix inequality (LMI) which is numerically tractable. While the standard KYP lemma characterizes FDIs in the entire frequency range, practical requirements are usually described by multiple FDIs in (semi)finite ranges; e.g., small sensitivity in a low frequency range and control roll-off in a high frequency range. Hence some sort of "adaptors," such as the weighting functions, have been used to fit the requirements into the KYP framework. However, the design iterations to search for the right weighting functions can be tedious and time consuming, and the controller complexity (order) tends to increase with the complexity of the weighting functions.

The objective of this paper is to develop a state space design theory that is capable of directly treating multiple FDI specifications in various frequency ranges without introducing weighting functions. To our knowledge, this problem has not been addressed in the literature. Our approach is based on the generalized Kalman-Yakubovich-Popov (GKYP) lemma [3], [4], recently developed by the authors, that provides an LMI characterization of FDIs in (semi)finite frequency ranges. We will first give a dual version of the GKYP lemma which is more suitable than the primal for feedback synthesis. A multiplier method is then developed to render the synthesis conditions convex through a simple change of variable, in the static gain feedback setting. We discuss cases where the multiplier method is nonconservative for the single-objective nominal design. The method is extended, with some conservatism, for the multi-objective robust control synthesis for systems with polytopic uncertainties. Design examples will demonstrate applicability of our results.

We use the following notation. For a matrix M, its transpose, complex conjugate transpose, the Moore-Penrose inverse, and the null space are denoted by M^{T}, M^*, $M^\dagger$, and $\mathcal{N}(M)$, respectively. The Hermitian part of a square matrix M is denoted by $\mathrm{He}(M) := M + M^*$. The symbol $\mathbf{H}_n$ stands for the set of $n \times n$ Hermitian matrices. The subscript n will be omitted if $n = 2$. For matrices Φ and P, $\Phi \otimes P$ means their Kronecker product. For $G \in \mathbb{C}^{n\times m}$ and $\Pi \in \mathbf{H}_{n+m}$, a function $\sigma: \mathbb{C}^{n\times m} \times \mathbf{H}_{n+m} \to \mathbf{H}_m$ is defined by

$$\sigma(G,\Pi) := \begin{bmatrix} G \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I_m \end{bmatrix}.$$

Given a positive integer q, let $\mathbf{Z}_q$ be the set of positive integers up to q, i.e., $\mathbf{Z}_q := \{\ 1,\ 2,\ \ldots,\ q\ \}$.

II. Problem statement and formulation

A. Problem statement

Consider the plant $G(\lambda)$ described by

$$\begin{bmatrix} \lambda x \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{1}$$

with a static gain feedback control $u = Ky$ where λ is the frequency variable (s for continuous-time and z for discrete-time cases), and $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n_w}$, $u(t) \in \mathbb{R}^{n_u}$, $z(t) \in \mathbb{R}^{n_z}$, and $y(t) \in \mathbb{R}^{n_y}$. Denote by $G(\lambda) \star K$ the closed-loop transfer function from w to z. The control synthesis problem of our interest is, given $\Pi \in \mathbf{H}_{n_w+n_z}$ and $\Phi, \Psi \in \mathbf{H}$, find a stabilizing feedback gain K such that

$$\sigma((G(\lambda)\star K)^*, \Pi) < 0 \quad \forall\ \lambda \in \bar{\mathbf{\Lambda}}(\Phi^{\mathsf{T}}, \Psi^{\mathsf{T}}). \tag{2}$$

where

$$\mathbf{\Lambda}(\Phi,\Psi) := \{\ \lambda \in \mathbb{C} \mid \sigma(\lambda,\Phi) = 0,\ \sigma(\lambda,\Psi) \geq 0\ \} \tag{3}$$

and $\bar{\mathbf{\Lambda}} := \mathbf{\Lambda}$ if $\mathbf{\Lambda}$ is bounded and $\bar{\mathbf{\Lambda}} := \mathbf{\Lambda} \cup \{\infty\}$ if unbounded.

For clarity of exposition, we shall restrict our attention to this single-objective nominal control problem in the main

This research is supported in part by NSF/ECS/CAREER 0237708 and by The Ministry of Education, Science, Sport and Culture, Japan, under Grant 14550439, and by the 21st Century COE Program on Information Science and Technology Strategic Core.

body of our theoretical developments. However, we will later discuss extensions to a more general problem where there are multiple FDI constraints of the above form as well as some uncertainty in the plant model.

B. Problem formulation via a dual GKYP lemma

Consider a transfer function

$$G(\lambda) = C(\lambda I - A)^{-1}B + D, \tag{4}$$

where $A \in \mathbb{C}^{n\times n}$, $D \in \mathbb{C}^{n_z\times n_w}$. The GKYP lemma in [4] provides a characterization of the FDI: $\sigma(G(\lambda), \Pi) < 0$ for all $\lambda \in \bar{\mathbf{\Lambda}}(\Phi, \Psi)$. The following result provides a dual version of the GKYP lemma.

Theorem 1: *Let $\Phi, \Psi \in \mathbf{H}$, $\Pi \in \mathbf{H}_{n_w+n_z}$, and $G(\lambda)$ in (4) be given and consider $\mathbf{\Lambda}(\Phi^{\mathsf{T}}, \Psi^{\mathsf{T}})$ defined by (3). Suppose $\mathbf{\Lambda}$ represents curves on the complex plane and A has no eigenvalues in $\mathbf{\Lambda}$. The following statements are equivalent.*

(i) *$\sigma(G(\lambda)^*, \Pi) < 0$ holds for all $\lambda \in \bar{\mathbf{\Lambda}}(\Phi^{\mathsf{T}}, \Psi^{\mathsf{T}})$.*

(ii) *There exist $P = P^*$ and $Q = Q^* > 0$ such that*

$$N \begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 \\ 0 & \Pi \end{bmatrix} N^* < 0, \tag{5}$$

$$N := \begin{bmatrix} \mathcal{M} & I \end{bmatrix} T, \quad \mathcal{M} := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where T is the permutation matrix such that

$$[\ M_1\ M_2\ M_3\ M_4\ ]T = [\ M_1\ M_3\ M_2\ M_4\ ] \tag{6}$$

for arbitrary matrices M_1, M_2, M_3, and M_4 with column dimensions n, n_w, n, and n_z, respectively.

With the result of Theorem 1, and ignoring the stability requirement for the moment, a synthesis problem may be formulated as the search for the parameters $Q > 0$, P, and K satisfying (5) with $\mathcal{M}$ defined to be the state space matrices of $G(\lambda) \star K$ as follows:

$$\begin{aligned} \mathcal{M} \ &:= \ \mathcal{A} + \mathcal{B}K\mathcal{C} \\ &= \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} K \begin{bmatrix} C_2 & D_{21} \end{bmatrix}. \end{aligned} \tag{7}$$

The resulting condition is not convex due to the product terms between the parameters P, Q, and K. In the next section, we shall develop a multiplier method to re-parametrize the condition so that the problem becomes convex. Throughout the paper, we will assume that $\mathcal{C}$ has full row rank without loss of generality.

III. Synthesis with nullspace filling multiplier

A. Basic idea

By the Finsler's theorem [5], condition (5) is equivalent to the existence of a multiplier $\mathcal{W}$ such that

$$T \begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 \\ 0 & \Pi \end{bmatrix} T^* < \mathrm{He} \begin{bmatrix} -I \\ \mathcal{M} \end{bmatrix} \mathcal{W}. \tag{8}$$

Note that (5) holds if and only if the left hand side (LHS) of (8) is negative definite on the range space of $[\ \mathcal{M} \ \ I\ ]^*$. The role of the multiplier $\mathcal{W}$ is to fill the orthogonal subspace, the nullspace of $[\ \mathcal{M} \ \ I\ ]$, so that the LHS with this modification becomes negative definite as in (8). The synthesis problem now is to compute $Q > 0$, P, $\mathcal{W}$ and K satisfying condition (8). This is still a nonconvex problem due to the product term between K and $\mathcal{W}$.

To make the problem tractable, we shall restrict the class of multipliers $\mathcal{W}$ to be

$$\begin{aligned} \mathbf{W}(\mathcal{C}, R) := \{\ & \mathcal{C}^\dagger W R + (I - \mathcal{C}^\dagger\mathcal{C})V \mid W \in \mathbb{C}^{n_y\times n_y}, \\ & \det(W) \neq 0,\ V \in \mathbb{C}^{(n+n_w)\times(2n+n_w+n_z)}\ \} \end{aligned} \tag{9}$$

where $R \in \mathbb{C}^{n_y\times(2n+n_w+n_z)}$ is a matrix to be specified later. In this case, the product term can be made linear in terms of the new variable $\mathcal{K} := KW$ as follows:

$$\mathcal{M}\mathcal{W} = (\mathcal{A} + \mathcal{B}K\mathcal{C})(\mathcal{C}^\dagger W R + (I - \mathcal{C}^\dagger\mathcal{C})V) = \mathcal{A}\mathcal{W} + \mathcal{B}\mathcal{K}R.$$

Thus, the synthesis equation (8) becomes an LMI in terms of the parameters Q, P, $\mathcal{W}$, and $\mathcal{K}$. Moreover, the above change of variable is invertible and the feedback gain can be found by $K = \mathcal{K}W^{-1}$.

We now turn our attention to the choice of R. We would like to specify R so that the restriction $\mathcal{W} \in \mathbf{W}(\mathcal{C}, R)$ can be made without introducing conservatism. The following theorem gives a characterization of such R and summarizes the synthesis LMI.

Theorem 2: *Let $R \in \mathbb{C}^{n_y\times(2n+n_w+n_z)}$, $\Phi, \Psi \in \mathbf{H}$, $\Pi \in \mathbf{H}_{n_w+n_z}$, $P, Q \in \mathbf{H}_n$, and the system in (1) be given. The following statements are equivalent.*

(i) *There exist a feedback gain K and a real scalar $\mu > 0$ such that conditions (5) and*

$$S(TXT^* - \mu R^* R)S^* < 0 \tag{10}$$

$$S := \begin{bmatrix} \mathcal{M} & I \\ \mathcal{C} & 0 \end{bmatrix}, \quad X := \begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 \\ 0 & \Pi \end{bmatrix}$$

are satisfied where $\mathcal{M}$ is defined in (7).

(ii) *There exist matrices $\mathcal{W} \in \mathbf{W}(\mathcal{C}, R)$, and $\mathcal{K}$ such that*

$$T \begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 \\ 0 & \Pi \end{bmatrix} T^* < \mathrm{He} \begin{bmatrix} -\mathcal{W} \\ \mathcal{A}\mathcal{W} + \mathcal{B}\mathcal{K}R \end{bmatrix}.$$

If (ii) holds, a gain in (i) can be given by $K := \mathcal{K}W^{-1}$.

The nullspace filling multiplier introduced above may be considered as a generalization of the multipliers developed by de Oliveira, Bernussou, Geromel, and others, for robust stability analysis of systems with polytopic uncertainties [6]–[10]. The main advantage of this type of formulation is that there is no product term between the system parameters $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and the "Lyapunov" parameters (P, Q). An implication is that one can easily obtain vertex-type results for robust control analysis as well as synthesis. Moreover, the formulation is also useful for multi-objective control as we discuss later.

The condition in statement (ii) of Theorem 2 is given in terms of LMIs and hence can be numerically solved by semidefinite programming. Here, we note that the nonsingularity constraint on $W \in \mathbf{W}(\mathcal{C}, R)$ can be ignored when solving the LMIs because a perturbation argument applies due to the strictness of the LMI.

We see that statement (ii) gives a *sufficient* condition for existence of K satisfying (5), regardless of the choice of R. Moreover, the condition is also *necessary* if R is chosen to satisfy (10). It can be verified that a particular choice:

$$R = \mathcal{C} \begin{bmatrix} I & -\mathcal{M}^* \end{bmatrix},$$

satisfies (10) for a sufficiently large $\mu > 0$, provided P, Q, and K solve (5). This means that an appropriate R exists whenever the original synthesis problem is feasible. However, the above choice of R is not practical because it depends on the unknown controller parameter. The next section will discuss some heuristic and exact choices of R that are directly useful for synthesis.

B. Specific choices of R

First, we shall specialize Theorem 2 for some specific cases of $\mathcal{C}$ and give particular choices of R that lead to LMI synthesis conditions which are nonconservative. Later, we will discuss some heuristic choices of R leading to sufficient conditions for synthesis. To this end, let us introduce the following:

Assumption 1:

(a) $\mathbf{\Lambda}$ *represents curves on the complex plane, and* Ψ *is active in* $\mathbf{\Lambda}$ *in the sense that* $\mathbf{\Lambda}(\Phi^\mathsf{T}, \Psi^\mathsf{T}) \neq \mathbf{\Lambda}(\Phi^\mathsf{T}, 0)$.

(b) *At least* n_z *eigenvalues of* Π *are negative.*

The first part of Item (a) is a natural condition within the framework of our control specifications expressed in terms of restricted frequency inequalities. The second part means that the frequency range $\mathbf{\Lambda}$ is not the entire $j\omega$ axis nor the unit circle, but a partial segment (or segments) of it. (See [4] for details.) Item (b) is a necessary condition for feasibility of the control synthesis problem and hence can be imposed without loss of generality.

Corollary 1 (Full Information): *Let* $\Phi, \Psi \in \mathbf{H}$, $\Pi \in \mathbf{H}_{n_w+n_z}$, $P, Q \in \mathbf{H}_n$, *and the system in (1) be given. Suppose* $Q > 0$ *and Assumption 1 hold, and consider the full information case*

$$\mathcal{C} = \begin{bmatrix} C_2 & D_{21} \end{bmatrix} = I.$$

Then there exists a feedback gain K *satisfying (5) if and only if statement (ii) in Theorem 2 holds, provided* R *is chosen as follows. Let* $N \in \mathbb{C}^{(2n+n_w+n_z)\times(n+n_z)}$ *be a full column rank matrix such that* $N^*XN < 0$ *for all* P *and* $Q > 0$. *Then define* R *to be a full row rank matrix with* n_y *rows such that* $RTN = 0$. *In particular, one such* N *is given by*

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}, \quad N_1 = \begin{bmatrix} pI_n \\ qI_n \end{bmatrix}, \quad N_2^* \Pi N_2 < 0$$

where $r := \begin{bmatrix} p & q \end{bmatrix}^* \in \mathbb{C}^2$ *is such that* $r^*\Phi r = 0$ *and* $r^*\Psi r < 0$, *and the column dimension of* N_2 *is* n_z. *Existence of such* r *and* N_2 *is guaranteed by Assumption 1.*

Corollary 1 gives an exact solution to the full information synthesis problem with an FDI in a bounded frequency range. If the FDI specification is given for the entire frequency range (i.e., the second condition in Assumption 1(a) is violated), Corollary 1 can be modified as follows. First note that the parameter Q can be set to zero without loss of generality, and the parameter P may be required to be positive definite to enforce a stability constraint. We can then specify an appropriate R by choosing r so that $r^*\Phi r$ is negative and modifying N accordingly.

Corollary 2 (State Feedback): *Let* $\Phi, \Psi \in \mathbf{H}$, $\Pi \in \mathbf{H}_{n_w+n_z}$, $P, Q \in \mathbf{H}_n$, *and the system in (1) be given. Suppose* $Q > 0$ *and Assumption 1 hold, and consider the state feedback case*

$$\begin{bmatrix} C_2 & D_{21} \end{bmatrix} = \begin{bmatrix} I_n & 0 \end{bmatrix}. \qquad (11)$$

Suppose further that:

$$\Phi_{11} = 0, \quad \Psi_{11} < 0, \quad \sigma(D_{11}^*, \Pi) < 0. \qquad (12)$$

Then there exists a feedback gain K *satisfying (5) if and only if statement (ii) in Theorem 2 holds, provided we choose*

$$R := \begin{bmatrix} 0_n & 0 & I_n & 0 \end{bmatrix}$$

where R *is partitioned so that the numbers of columns are* n, n_w, n, *and* n_z *from left to right.*

The conditions in (12) are satisfied when the control specifications are given in terms of a continuous-time ($\Phi_{11} = 0$), bounded frequency ($\Psi_{11} < 0$) inequality condition that holds at infinite frequency ($\sigma(\boldsymbol{D}^*, \Pi) < 0$). The last condition is met if, for instance, $D_{11} = 0$ and the origin is included in the feasible domain defined by the set of $\boldsymbol{G}$ such that $\sigma(\boldsymbol{G}^*, \Pi) < 0$.

Next, we will consider the state feedback case and present some potentially conservative but reasonable choices for R. Consider the case of continuous-time, small gain condition in the low frequency range. In this case, R can be chosen as

$$R = \begin{bmatrix} 0 & 0 & I_n & (D_{11}B_1^\dagger)^* \end{bmatrix}. \qquad (13)$$

For the continuous-time, small gain condition in the high frequency range, we have

$$R = \begin{bmatrix} I_n & 0 & 0 & 0 \end{bmatrix}. \qquad (14)$$

We claim that these choices are reasonable because the the upper left $n \times n$ and lower right $n_z \times n_z$ block matrices of $\mathcal{N}(RS^*)(ST)X(ST)^*\mathcal{N}(RS^*)$ are both negative definite, which are necessary conditions for existence of μ satisfying (10). Although these are not sufficient in general, this approach allows us to choose R that is independent of P and Q. We will later illustrate applicability of the heuristic choices of R presented here through numerical design examples.

IV. Extensions

A. Multi-objective, robust control

We consider the following problem: Find K such that

$$\sigma((G_k(\lambda) \star K)^*, \Pi_k) < 0 \quad \forall\, \lambda \in \bar{\mathbf{\Lambda}}(\Phi_k^{\mathsf{T}}, \Psi_k^{\mathsf{T}}) \tag{15}$$

holds for all $k \in \mathbf{Z}_q$ where each $G_k(\lambda)$ is a given plant, and (Φ_k, Ψ_k, Π_k) defines a frequency domain specification to be achieved for the closed-loop system $G_k(\lambda) \star K$. The plant $G_k(\lambda)$ may represent a vertex of a set of uncertain systems for robust control, or a plant with a selected disturbance-performance (i.e., w-z) channel for multi-objective control. The following result can be obtained from Theorem 2 in a straightforward manner, and hence its proof is omitted.

Corollary 3: *Let $R_k \in \mathbb{C}^{n_y \times (2n+n_w+n_z)}$, $\Phi_k, \Psi_k \in \mathbf{H}$, $\Pi_k \in \mathbf{H}_{n_w+n_z}$, and systems $G_k(\lambda)$ as in (1) be given where $k \in \mathbf{Z}_q$. There exists a static feedback gain K such that the frequency domain specifications (15) are satisfied for all $k \in \mathbf{Z}_q$ if there exist scalars $\alpha_k > 0$ and matrices $P_k = P_k^*$, $Q_k = Q_k^* > 0$, $\mathcal{W}_k \in \mathbf{W}(\mathcal{C}_k, R_k)$, and $\mathcal{K}$ such that all $\mathcal{W}_k$ have a common W and*

$$T_k \begin{bmatrix} \Phi_k \otimes P_k + \Psi_k \otimes Q_k & 0 \\ 0 & \alpha_k \Pi_k \end{bmatrix} T_k^* < \mathrm{He} \begin{bmatrix} -\mathcal{W}_k \\ \mathcal{A}_k \mathcal{W}_k + \mathcal{B}_k \mathcal{K} R_k \end{bmatrix} \tag{16}$$

holds for all $k \in \mathbf{Z}_q$, where T_k and $(\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k)$ are defined as in (6) and (7) using the input/state/output dimensions and the state space matrices of $G_k(\lambda)$. In this case, one such gain is given by $K := \mathcal{K} W^{-1}$.

Corollary 3 gives a sufficient condition for the existence of static feedback gain that achieves the multiple FDI specifications in (15). The condition is given in terms of LMIs and can be solved numerically. The associated degree of conservatism is dependent upon the choices of R_k. In the full information or state feedback case, some reasonable choices have been proposed in the previous section. It should be noted that this formulation does not assume common "Lyapunov matrices" (P, Q) as in the quadratic stability literature [11] or in the more recent multi-objective control [12], [13], but rather, (P, Q) can be interpreted as "parameter-dependent" as discussed in [6], [7], [14]. Thus we expect reduced conservatism when compared with these existing techniques for multi-objective robust control. It should be emphasized, however, that the main contribution of this paper is not the conservatism reduction but the synthesis method to meet FDI specifications in (semi)finite frequency ranges, which have not been addressed in the literature.

The above formulation naturally captures the multi-objective control in the sense that the control gain K is designed so that each specification defined by (Φ_k, Ψ_k, Π_k) is met for the corresponding plant $G_k(\lambda)$. However, the formulation is not suitable for robust control synthesis in its present form. To elaborate on this point, let $\mathcal{I}$ be a subset of indices $\mathbf{Z}_q$ corresponding to a robust performance specification (Φ_o, Ψ_o, Π_o) to be satisfied by a family of plants defined by the convex hull of the state space matrices of $G_k(\lambda)$ with $k \in \mathcal{I}$. In this case, we have $(\Phi_k, \Psi_k, \Pi_k) = (\Phi_o, \Psi_o, \Pi_o)$, $T_k = T_o$, and $R_k = R_o$ for all $k \in \mathcal{I}$. We shall assume that the family of plants share a common measured output signal, i.e., $\mathcal{C}_k = \mathcal{C}_o$ for all $k \in \mathcal{I}$. The sufficient condition in Corollary 3 guarantees the performance (Φ_o, Ψ_o, Π_o) for each vertex plant $G_k(\lambda)$, but not for every plants in the convex hull. This deficiency can be overcome, with some additional conservatism, by using a common $\mathcal{W}_o = \mathcal{W}_k$ for all $k \in \mathcal{I}$ (see the example in Section V for a relaxed version of this idea). In this case, every coefficients of $\mathcal{A}_k$ and $\mathcal{B}_k$ for $k \in \mathcal{I}$ become independent of k, allowing for an arbitrary convex combination of (16) to be taken to conclude the robust performance.

B. Regional pole constraints

The design specifications in (15) encompass frequency domain shaping of closed-loop transfer functions. However, the closed-loop stability has not been captured, and hence one may wish to include a stability constraint, or more generally, regional pole constraints, as an additional design specification. The following lemma readily follows from [15], [16] and gives a basic result for an eigenvalue characterization.

Lemma 1: *Let $A \in \mathbb{C}^{n \times n}$ and $\Phi \in \mathbf{H}$ be given. Suppose $\det(\Phi) < 0$. Then the following are equivalent.*

(i) *Each eigenvalue λ of A satisfies $\sigma(\lambda, \Phi^{\mathsf{T}}) < 0$.*
(ii) *There exists $P = P^* > 0$ such that $\sigma(A^*, \Phi \otimes P) < 0$.*
(iii) *There exist W and $P = P^* > 0$ such that*

$$\Phi \otimes P < \mathrm{He} \begin{bmatrix} -I \\ A \end{bmatrix} W \begin{bmatrix} -qI & pI \end{bmatrix}$$

where $r := [\; p \quad q \;]^* \in \mathbb{C}^2$ is an arbitrary fixed vector satisfying $r^* \Phi r < 0$.

The set of eigenvalues characterized in (i) captures, through certain choices of Φ, the half plane and the inside or outside of a circle on the complex plane. See [4] for details.

The condition in (iii) can be used to give additional constraints in the design equations discussed in the previous sections. In particular, we replace A with the closed-loop matrix $A + B_2 K$ in the state feedback case, and introduce the change of variable $\mathcal{K} := KW$. As a result, we add the following constraint to the design:

$$\Phi \otimes P < \mathrm{He} \begin{bmatrix} -W \\ AW + B_2 \mathcal{K} \end{bmatrix} \begin{bmatrix} -qI & pI \end{bmatrix}$$

Clearly, multiple inequalities of the same form can be added to enforce (robust) regional pole constraints expressed as the intersection of half planes and circles. In this case, as in Corollary 3, different Φ, A, B_2, and P may be used for each inequality but W and $\mathcal{K}$ have to be common for all inequality constraints.

V. Design example

We consider the classical ACC benchmark problem of cart-spring system. The plant is described by

$$\dot{x} = Ax + B_1 w + B_2 u, \quad z = Cx,$$

$$A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & 0 & 0 \\ k & -k & 0 & 0 \end{bmatrix}, \quad B_1 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 := \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C := \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

where $k = 1$ is the spring constant and the unit mass is assumed for each cart. Our objective is to design a stabilizing state feedback controller $u = Kx$ such that

$$|T_{zw}(j\omega)| < \gamma, \quad \forall\, |\omega| \le \varpi_\ell$$

$$|T_{uw}(j\omega)| < \rho, \quad \forall\, |\omega| \ge \varpi_h$$

hold, where T_{zw} and T_{uw} are the closed-loop transfer functions from w to z and u, respectively, γ and ρ are the performance bounds, and ϖ_ℓ and ϖ_h are the cut-off frequencies in the low/high ranges. From Lemma 1 and Theorem 2, the synthesis conditions are given by

$$\begin{bmatrix} 0 & P_s \\ P_s & 0 \end{bmatrix} < \mathrm{He} \begin{bmatrix} -W \\ AW + B\mathcal{K} \end{bmatrix} \begin{bmatrix} I & I \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} -Q_\ell & 0 & P_\ell & 0 \\ 0 & \alpha & 0 & 0 \\ P_\ell & 0 & \varpi_\ell^2 Q_\ell & 0 \\ 0 & 0 & 0 & -\alpha\gamma^2 \end{bmatrix} < \mathrm{He} \begin{bmatrix} -I & 0 & 0 \\ 0 & -1 & 0 \\ A & B_1 & B_2 \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} WR_\ell \\ V_\ell \\ \mathcal{K}R_\ell \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} Q_h & 0 & P_h & 0 \\ 0 & \beta & 0 & 0 \\ P_\ell & 0 & -\varpi_h^2 Q_h & 0 \\ 0 & 0 & 0 & -\beta\rho^2 \end{bmatrix} < \mathrm{He} \begin{bmatrix} -I & 0 & 0 \\ 0 & -1 & 0 \\ A & B_1 & B_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} WR_h \\ V_h \\ \mathcal{K}R_h \end{bmatrix} \tag{19}$$

where W, $\mathcal{K}$, $P_s = P_s^* > 0$, $P_\ell = P_\ell^*$, $Q_\ell = Q_\ell^* > 0$, V_ℓ, $P_h = P_h^*$, $Q_h = Q_h^* > 0$, V_h, and $\alpha, \beta > 0$ are the (real) variables and R_ℓ and R_h are given by R in (13) and (14), respectively. If these equations admit a solution, a feasible state feedback gain is given by $K = \mathcal{K}W^{-1}$.

We fixed the values ϖ_ℓ, ϖ_h, and γ as

$$\varpi_\ell = 2, \quad \varpi_h = 3, \quad \gamma = 2,$$

and then minimized ρ. The optimal value of ρ and the corresponding feedback gain K are found to be

$$\rho_{\min} = 0.52,$$
$$K = \begin{bmatrix} -1.4414 & 0.0802 & -1.7213 & -0.8622 \end{bmatrix}.$$

The resulting closed-loop transfer functions are shown in Fig 1 where the shaded regions indicate the bounds on the gain of the transfer functions. We see that the upper bounds are relatively tight, showing that the associated conservatism is not significant.

For the sake of illustration, we have changed the frequency interval of the constraint $|T_{zw}| < \gamma$ from $\varpi_\ell = 2$ to $\varpi_\ell = 1$. By this change, the natural frequency of the cart-spring system ($\sqrt{2}$ [rad/s]) is now outside of the frequency range. All the other parameters are fixed as before and the feasibility problem is solved. The resulting design is shown in Fig. 2. We see that the large peak in $|T_{zw}|$ is now allowed and the time response is lightly damped. If we minimize ρ, then it can get as low as 0.27 at the expense of a larger peak value $\|T_{zw}\|_\infty = 6.8$.

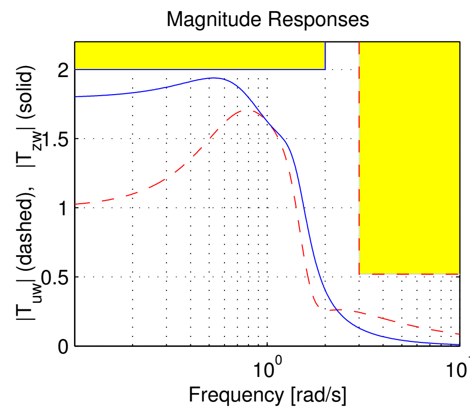

Fig. 1. Transfer functions (Nominal design)

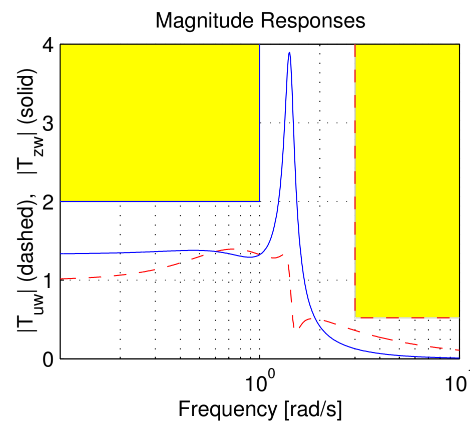

Fig. 2. Transfer functions (Nominal bad design)

The result shown in Fig. 1 is compared with a mixed H_2/H_∞ multi-objective design [12]. In particular, the conditions

$$\|T_{zw}\|_\infty < \gamma, \quad \|T_{uw}\|_2 < \rho$$

can be conservatively reduced to the search for matrices $X = X^\mathsf{T}$ and $\mathcal{K}$ satisfying

$$\begin{bmatrix} \mathrm{He}(AX + B_2\mathcal{K}) & XC^* & B_1 \\ CX & -\gamma I & 0 \\ B_1^* & 0 & -\gamma I \end{bmatrix} < 0,$$

$$\mathrm{He}(AX + B_2\mathcal{K}) + B_1 B_1^* < 0, \quad \begin{bmatrix} \rho^2 & \mathcal{K} \\ \mathcal{K}^* & X \end{bmatrix} > 0.$$

Once these LMIs are solved for X and $\mathcal{K}$, a state feedback gain is found by $K = \mathcal{K}X^{-1}$.

We have fixed the value of γ to be the same as that in our design, i.e., $\gamma = 2$, and then minimized ρ. This problem is meant to find the minimum energy control that achieves the same regulation performance as before. As a result, we obtained $\rho_{\min} = 2.00$ and

$$K = \left[\begin{array}{cccc} -2.8835 & 0.3727 & -2.4005 & -2.7767 \end{array} \right].$$

The corresponding closed-loop frequency response is shown in Fig. 3. We remark two things. First, the bound on the H_∞ norm of the transfer function T_{zw} is not very tight and the gap between the H_∞ norm bound and the actual norm shows the degree of conservatism for this design. Second, the transfer function T_{uw} does not roll off as much as in the previous design so that the peak value of $|u(t)|$ is three times larger, showing a limitation of the H_2 norm as a measure for the "control effort."

Next we consider the case where the spring constant k is uncertain but is known to lie in the interval $[1, 2]$. In this case, we will have the synthesis equations (17)-(19) for $k = 1$, and in addition, copies of these equations for $k = 2$ where the variables W and $\mathcal{K}$ are common but the others are not (e.g., we have two different P_h's for $k = 1$ and $k = 2$). Minimizing γ subject to these 6 LMIs and computing the gain by $K = \mathcal{K}W^{-1}$, we have

$$\varpi_\ell = 2, \quad \varpi_h = 4, \quad \rho = 0.5, \quad \gamma = 2.66,$$

$$K = \left[\begin{array}{cccc} -2.6736 & 0.8938 & -2.2190 & -1.1612 \end{array} \right].$$

The resulting closed-loop response shown in Fig. 4, where the two solid curves show responses for cases $k = 1$ and 2, and similarly for the dashed curves. We see that conservatism is still moderate even for this robust design, suggesting a potential for practical applications.

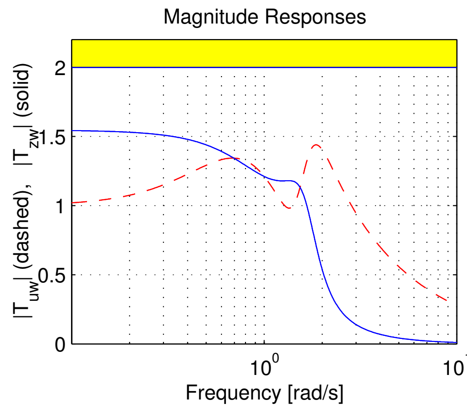

Fig. 3. Transfer functions (H_2/H_∞ design)

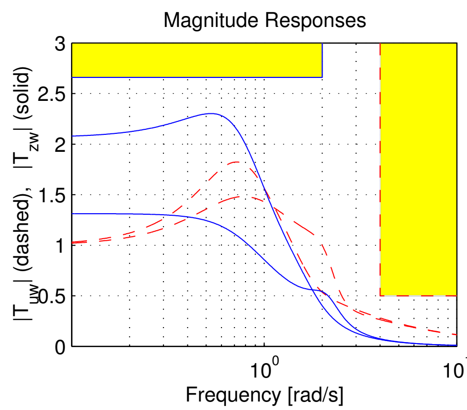

Fig. 4. Transfer functions (Robust design)

VI. Conclusion

We have developed methods for synthesizing static feedback controllers to achieve, for a family of plants, multiple FDI specifications in (semi)finite frequency ranges. Sufficient conditions for existence of feasible controllers are given in terms of LMIs, and some special cases, where the conditions become also necessary, are discussed. Utility of our result for the general state feedback design is demonstrated through numerical examples.

VII. References

[1] B. D. O. Anderson. *SIAM J. Contr.*, 5:171–182, 1967.

[2] A. Rantzer. *Sys. Contr. Lett.*, 28(1):7–10, 1996.

[3] T. Iwasaki and S. Hara. *Proc. American Contr. Conf.*, pages 3828–3833, 2003.

[4] T. Iwasaki and S. Hara. Generalized KYP lemma: Unified characterization of frequency domain inequalities with applications to system design. *Technical Report of The Univ. Tokyo*, METR2003-27, August 2003. http:\\www.keisu.t.u-tokyo.ac.jp/Research/techrep.0.html.

[5] R. E. Skelton, T. Iwasaki, and K. M. Grigoriadis. *A Unified Algebraic Approach to Linear Control Design*. Taylor & Francis, 1997.

[6] M.C. de Oliveira, J. Bernussou, and J.C. Geromel. *Sys. Contr. Lett.*, 37:261–265, 1999.

[7] J.C. Geromel, M.C. de Oliveira, and J. Bernussou. *SIAM J. Contr. Optim.*, 41(3):700–711, 2002.

[8] D. Henrion, D. Arzelier, and D. Peaucelle. *Automatica*, 39:1479–1485, 2003.

[9] Y. Ebihara and T. Hagiwara. *Proc. IEEE Conf. Decision Contr.*, pages 4179–4184, 2002.

[10] T. Shimomura, M. Takahashi, and T. Fujii. *IEEE Conf. Dec. Contr.*, pages 2157–2162, 2001.

[11] B. R. Barmish. *J. Optimiz. Theory Appl.*, 46(4), 1985.

[12] C. Scherer, P. Gahinet, and M. Chilali. *IEEE Trans. Auto. Contr.*, 42(7):896–911, 1997.

[13] I. Masubuchi, A. Ohara, and N. Suda. *Int. J. Robust and Nonlinear Contr.*, 8:669–686, 1998.

[14] P. Apkarian, H.D. Tuan, and J. Bernussou. *IEEE Trans. Auto. Contr.*, 46(12):1941–1946, 2001.

[15] D. Henrion and G. Meinsma. *IEEE Trans. Auto. Contr.*, 46(8):1285–1288, 2001.

[16] D. Peaucelle and D. Arzelier. *IEEE Trans. Auto. Contr.*, 46(4):624–630, 2001.