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Necessary and Sufficient Conditions for Stability of a Class of Second Order Switched Systems

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Abstract— **For a special class of systems, it is shown that the existence of a Common Quadratic Lyapunov Function (CQLF) is necessary and sufficient for the stability of an associated switched system under arbitrary switching. Furthermore, it is shown that the existence of a CQLF for N $(N > 2)$ subsystems is equivalent to the existence of a CQLF for every pair of subsystems. An algorithm is proposed to compute a CQLF for the subsystems, when it exists, using the left and right eigenvectors of a critical matrix obtained from a matrix pencil.**

Index Terms— **switched systems, stability, common quadratic Lyapunov function, M–matrix**

I. Problem Statement

Consider the switched system

$$\Sigma_s : \dot{x}(t) = A(t)x(t), A(t) \in \mathcal{A} = \{A_1, A_2, \ldots, A_N\} \quad (1)$$

where $x(t) \in \Re^2$ is the state, and $A_i \in \Re^{2\times 2}$, $i = 1, 2, \ldots, N$ are the system matrices for the subsystems

$$\Sigma_i : \ \dot{x}(t) = A_i x(t) \ , \ i = 1, 2, \ldots, N. \qquad (2)$$

Throughout the paper, the negative of each matrix A_i (i.e., $-A_i$) is assumed be an M–matrix[1]. Therefore, each matrix $A = [a_{ij}]$ in the set $\mathcal{A}$ satisfies $a_{ii} < 0, \ i = 1, 2$ and $a_{ij} \geq 0$, and is Hurwitz. The objective of this paper is to derive necessary and sufficient conditions for the stability of the switched system (1) under arbitrary switching between the system matrices $A_i, i = 1, 2, \ldots, N$.

Clearly, if a common quadratic Lyapunov function (CQLF) exists for the subsystems Σ_i, $i = 1, 2, \ldots, N$, then the switched system (1) is stable under arbitrary switching. The converse of this statement is not true in general [3], [4]. However, in this paper, we prove that the converse is indeed true for a specific class of systems.

The following notation will be used in sequel. Let $T = [t_{ij}] \in \Re^{n\times m}$. T is said to be a non–negative matrix, and denoted as $T \succeq 0$ if $t_{ij} \geq 0$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Similarly, for two matrices $T_1, T_2 \in \Re^{n\times m}$, we write $T_1 \succeq T_2$ if $T_1 - T_2 \succeq 0$. For $Q \in \Re^{n\times n}$, $Q > 0$ denotes that Q is positive definite.

II. Main Results

We first consider the two subsystem case.[2]

Proposition 1: For the systems in (1) and (2) with $N = 2$, the following statements are equivalent.

(i) The switched system Σ_s is stable under arbitrary switching.
(ii) The matrix pencil $\alpha A_1 + (1 - \alpha)A_2$ is Hurwitz for all $\alpha \in [0, 1]$.
(iii) A CQLF exists for the subsystems Σ_1 and Σ_2.

Remark 1: (i) In general, the existence of a CQLF is not necessary for the stability of a switched system under arbitrary switching [3], [4]. However, for the special class of systems under consideration, Proposition 1 states that stability of the switched system in (1) is equivalent to the existence of a CQLF for the subsystems in (2).

(ii) The stability properties of the switched system in (1) can be determined by checking the stability of the matrix pencil, $\alpha A_1 + (1 - \alpha)A_2$, $\alpha \in [0, 1]$, which can further be reduced to checking whether A_1 is Hurwitz and whether the matrix product $A_1^{-1}A_2$ has any negative eigenvalues [5]. Furthermore, the critical value of α for which $\alpha A_1 + (1 - \alpha)A_2$ has the largest real eigenvalue can be found easily [4].

Lemma 1: If the diagonal entries of the matrices for the systems in (1) and (2) with $N = 2$ are equal to -1, then the following statements are equivalent.

(i) The switched system Σ_s is stable under arbitrary switching.
(ii) The matrix pencil $\alpha A_1 + (1 - \alpha)A_2$ is Hurwitz for all $\alpha \in [0, 1]$.
(iii) A *diagonal* CQLF exists for subsystems Σ_1 and Σ_2.

Remark 2: This type of system, whose system matrices have -1 on the diagonals, are widely encountered in problems of power control for wireless networks [6], [7].

The results of Lemma 1 can be extended as follows.

Theorem 1: If the diagonal entries of the matrices for the systems in (1) and (2) with $N > 2$ are equal to -1, then the following statements are equivalent.

(i) The switched system Σ_s is stable under arbitrary switching.
(ii) All matrices in the convex hull $\sum_{i=1}^{N} \alpha_i A_i$ are Hurwitz for $\alpha_i \geq 0$, $i = 1, 2, \ldots, N$, and $\sum_{j=1}^{N} \alpha_j = 1$.
(iii) The matrix pencils $\alpha A_i + (1 - \alpha)A_j$ are Hurwitz for all $\alpha \in [0, 1]$, and all $i, j = 1, 2, \ldots, N$, $i \neq j$.
(iv) A *diagonal* CQLF exists for every pair of subsystems Σ_i and Σ_j, $i, j = 1, 2, \ldots, N$, $i \neq j$.
(v) A *diagonal* CQLF exists for the subsystems Σ_i, $i = 1, 2, \ldots, N$ and it can be computed using the following algorithm.

Algorithm 1 **:** (a) Among all pairs of the matrices A_i and A_j, $i, j = 1, 2, \ldots, N$, $i < j$, determine (using Lemma 1) the matrix $A = \alpha A_i + (1-\alpha)A_j$ which has the largest real eigenvalue for some $\alpha \in [0, 1]$.

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[1]See [1], [2] for the definition and properties of M–matrices.

[2]The proofs of the results in this section are relegated to the Appendix.

(b) Solve for $v = [v_1, v_2]^T$ from $Av = \lambda v$.
(c) Compute the diagonal common Lyapunov matrix D as $D = \text{diag}[v_2/v_1, v_1/v_2]$. □

Remark 3: In general, for $N > 2$, the existence of a CQLF for every pair of subsystems is necessary but not sufficient for the existence of a CQLF for all of the subsystems [4], [8]. However, the above theorem states that this condition is also sufficient for the special class of systems under consideration.

APPENDIX

A. Proof of Proposition 1

(i) ⇒ (ii): Suppose that the matrix pencil $\alpha A_1 + (1-\alpha)A_2$ is unstable for some $\alpha = \alpha_c \in [0,1]$. Then $A_{eq} = \alpha_c A_1 + (1-\alpha_c)A_2$ is unstable which means that alternating switching between systems Σ_1 and Σ_2 utilizing Σ_1 for $\alpha_c T$ units of time and Σ_2 for $(1-\alpha_c)T$ units of time, with sufficiently small time interval T leads to an unstable switched system.

(ii)⇒(iii): Assume that the matrix pencil $\alpha A_1 + (1-\alpha)A_2$ is Hurwitz for all $\alpha \in [0,1]$. It can be shown that $\alpha A_1 + (1-\alpha)A_2^{-1}$ is also Hurwitz for all $\alpha \in [0,1]$. Hence, by the result in [8], a CQLF exists for Σ_1 and Σ_2.

(iii) ⇒ (i): Trivial. □

B. Proof of Lemma 1

The assertions (i) ⇒ (ii) and (iii) ⇒ (i) follow from Proposition 1. In order to prove (ii) ⇒ (iii), suppose that the matrix pencil achieves its largest real eigenvalue $\lambda < 0$ for $\alpha = \alpha_c \in [0,1]$ and $A = \alpha_c A_1 + (1-\alpha_c)A_2$. Let $Av = \lambda v$ and $A^T w = \lambda w$. Since $-A$ is an M–matrix, it can be shown that $v, w \succ 0$ and $w = [v_2, v_1]^T$ where $v = [v_1, v_2]^T$ [1]. Define a diagonal matrix $D = \text{diag}[w_1/v_1, w_2/v_2]$. From $A^T w = \lambda w \prec 0$ and $Av = \lambda v \prec 0$, it follows that $(A^T D + DA)v \prec 0$. Hence, $-(A^T D + DA)$ is an M–matrix [1] and since it is also symmetric, $(A^T D + DA) < 0$, and D is a Lyapunov solution for A. We now proceed to show that it is also a Lyapunov solution for A_1 and A_2. We consider two cases: (i) $\alpha_c \in \{0,1\}$ and (ii) $\alpha_c \in (0,1)$.

Case (i): Without loss of generality, assume $\alpha_c = 1$. Denote the matrices as $A_i = \begin{bmatrix} -1 & a_i \\ b_i & -1 \end{bmatrix}$, $i = 1, 2$. The largest eigenvalue of A_i is $\lambda_i = -1 + \sqrt{a_i b_i}$. As $\alpha_c = 1$, $\lambda_1 \geq \lambda_2$, hence $a_1 b_1 \geq a_2 b_2$. This may happen when (a) $A_1 \succeq A_2$ (i.e. $a_1 \geq a_2,\ b_1 \geq b_2$), (b) Otherwise, i.e. when either $(a_1 < a_2, b_1 > b_2)$ or $(a_1 > a_2, b_1 < b_2)$.

The first subcase (a), ($A_1 \succeq A_2$) is trivial; as any diagonal Lyapunov matrix for A_1 would be a Lyapunov matrix for A_2. Consider the 2nd subcase (b). Here, $(a_1 - a_2)(b_1 - b_2) < 0$. The largest eigenvalue of the matrix pencil, when given as a function of α, achieves its maximum for $\alpha_{max} = p/q$ where $p = a_1 b_2 + a_2 b_1 - 2a_2 b_2$ and $q = -2(a_1 - a_2)(b_1 - b_2)$. As $\alpha \in [0,1]$, so $\alpha_c = 1$ when $\alpha_{max} \geq 1$. Hence, $\alpha_c = 1$, if $p \geq q > 0$, i.e., we have

$$a_1 b_2 + a_2 b_1 \leq 2a_1 b_1 \ \text{ and } \ 0 \leq a_2 b_2 \leq a_1 b_1 < 1. \quad (3)$$

The eigenvectors of A_1 and A_1^T corresponding to λ_1 can be computed as $v = [\frac{1}{\sqrt{b_1}}, \frac{1}{\sqrt{a_1}}]^T$ and $w = [\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{b_1}}]^T$. Define $D = \text{diag}[w_1/v_1, w_2/v_2]$. As before, D is a Lyapunov matrix for A_1. Using (3), $(A_2^T D + DA_2)v \prec 0$ and hence, D is a Lyapunov matrix for A_2 as well.

Case (ii): $\alpha_c \in (0,1)$. Let $A = \alpha_c A_1 + (1-\alpha_c)A_2$, and consider the matrix pencil $B = \beta A + (1-\beta)A_1,\ \beta \in [0,1]$ which achieves its largest real eigenvalue for $\beta = 1$. Hence, this case reduces to case (i) discussed above, and a diagonal common Lyapunov matrix, D, can be calculated for A and A_1 using the eigenvectors of A. Similar arguments hold for the matrices, A and A_2. Thus, D is a Lyapunov matrix for both A_1 and A_2. □

C. Proof of Theorem 1

The assertions (i) ⇒ (ii), (ii) ⇒ (iii) and (v) ⇒ (i) are straightforward. The claim (iii) ⇒ (iv) follows from the equivalence of the items (ii) and (iii) in Lemma 1. In order to prove (iv) ⇒ (v), we proceed as follows. Let every pair of subsystems Σ_i and Σ_j, $i, j = 1, 2, \ldots, N$, $i \neq j$, have a diagonal CQLF. The subsystem Σ_i has a set of normalized diagonal Lyapunov matrices $D_i =$diag$[1, d_i]$, where d_i can be seen to lie in a convex set, $d_i \in (d_i^-, d_i^+)$, $d_i^- \geq 0$, $d_i^+ > 0$ [8], [4]. Similarly for Σ_j, we have $D_j =$diag$[1, d_j]$, $d_j \in (d_j^-, d_j^+)$, $d_j^- \geq 0$, $d_j^+ > 0$. As Σ_i and Σ_j have a diagonal CQLF, the set $(d_i^-, d_i^+) \cap (d_j^-, d_j^+)$ is non–empty. Since this is true for every pair of subsystems, we have $\bigcap_{i=1}^{N} (d_i^-, d_i^+) \neq \emptyset$ which establishes that there exists a diagonal CQLF, $x^T D x$, for all of the subsystems. Note that the set of common Lyapunov functions can be computed by intersecting the intervals (d_i^-, d_i^+) whose limits can be determined by solving a second order algebraic equation for each matrix A_i. An alternative solution is Algorithm 1 which follows from the proof of Lemma 1. (QED) □

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