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Practical Stabilizability of a Class of Switched Systems

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***Abstract*— In this paper, we extend our previous practical stabilizability results to a class of switched systems with autonomous subsystems. We prove a sufficient condition for the local practical stabilizability of such systems. A switching law which leads to ϵ-practical stability is constructed in the proof. Finally, we illustrate the effectiveness of the switching law by a tracking problem example.**

I. INTRODUCTION

Recently, it has been observed that, under appropriate switching laws, switched systems whose subsystems have no common equilibrium may still exhibit interesting behaviors similar to those of a conventional stable system near an equilibrium. In [5], we formally define such behaviors as practical stability. Such practical stability notions are extensions of the traditional concepts on practical stability in [1] and [2], which are concerned with bringing the system trajectories to be within given bounds. Similar boundedness behaviors have also been observed by other researchers. Lin and Antsaklis in [3] study the ultimate boundedness problem for switched linear systems with uncertainties. Zhai and Michel in [6], [7] introduce the notion of practical stability for a class of switched systems. The notion in [6], [7] concerns the boundedness property of the system trajectory with respect to a given bound.

In [5], we have proposed a necessary and sufficient condition for the global practical asymptotic stabilizability of integrator switched systems (a more complete version of [5] can be found in [4]). In the present paper, we extend the result in [5] to a class of switched systems with autonomous subsystems which have no common equilibrium. The main result of the paper is a sufficient condition for the local practical stabilizability of such systems. Such a condition is an extension of the condition in [5] which concerns the convex cone of the subsystem vector fields. In the proof of the condition, we explicitly construct a valid switching law under which the system is ϵ-practically stable. The switching law is then applied to a tracking problem to show its effectiveness.

It should be noted that our results in this paper have the following distinct features as opposed to the available literature results (such as those in [3] and [6]). First, the notions and results we propose in this paper concern conditions for boundedness property with respect to any bound (not a given fixed bound) for switched systems with autonomous subsystems. Second, the condition we propose is easier to verify than conditions based on Lyapunov-like functions such as those in [6], [7]. Third, a switching law which achieves ϵ-practical stability is explicitly constructed in this paper.

II. PRELIMINARIES

In this paper, we consider switched systems consisting of autonomous subsystems

$$\dot{x} = f_i(x), \quad i \in I \stackrel{\triangle}{=} \{1, 2, \cdots, M\} \tag{1}$$

and a switching law orchestrating the active subsystem at each time instant. In (1), we assume that every $f_i(x)$ is locally Lipschitz continuous. The state trajectory of system (1) is determined by the initial state and the *switching sequence* defined as follows.

Definition 1 (Switching Sequence): A switching sequence σ in $[t_0, t_f]$ is defined as

$$\sigma = \big((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K)\big) \tag{2}$$

where $0 \le K < \infty$, $t_0 \le t_1 \le \cdots \le t_K \le t_f$, $i_k \in I$ for $k = 0, 1, \cdots, K$.

We also define $\Sigma_{[t_0,t_f]} = \{$switching sequence σ's in $[t_0, t_f]\}$ and $\Sigma_{[t_0,\infty)} = \{\ \sigma$ defined on $[t_0, \infty)$ satisfying $\sigma_{[t_0,t_f]} \in \Sigma_{[t_0,t_f]}$, $\forall t_f > t_0$, where $\sigma_{[t_0,t_f]}$ is the truncated version of σ in $[t_0, t_f]\}$ ■

σ indicates that subsystem i_k is active in $[t_k, t_{k+1})$. It can also be interpreted as a timed sequence of active subsystems indices. For a switched system to be well-behaved, we only consider **nonZeno** sequences which switch at most a finite number of times in any finite time interval $[t_0, t_f]$. Switching sequences are usually generated by switching laws defined below.

Definition 2 (Switching Law): For system (1), a switching law $\mathcal{S}$ is defined to be a mapping $\mathcal{S} : \mathbb{R}^n \times \mathbb{R} \to \bigcup_{t_0} \Sigma_{[t_0,\infty)}$ which specifies a switching sequence $\sigma = \sigma(x_0, t_0) \in \Sigma_{[t_0,\infty)}$ for any initial x_0 and t_0. ■

Remark 1: $\mathcal{S}$ is often determined by some rules or algorithms, which describe how to generate a switching sequence given (x_0, t_0), rather than mathematical formulae. In this paper, we specify switching laws using such descriptions. ■

Now we review some notions and results reported in [5] (slightly modified to be more suitable for the present paper). In the following, the vector (and matrix) norm $\|\cdot\|$ denotes the 2-norm; and $B[x, r]$ denotes the closed ball $\{y \in \mathbb{R}^n | \|y - x\| \le r\}$.

First we introduce the following practical stability notions for system (1). Without loss of generality, we only discuss the case of the origin and let the initial time be $t_0 = 0$.

Definition 3 (ϵ-Practical Stability): Assume a switching law $\mathcal{S}$ is given for system (1). Given an $\epsilon > 0$, system (1) is said to be *ϵ-practically stable around the origin under $\mathcal{S}$* if there exists a $\delta = \delta(\epsilon) > 0$ such that $x(t) \in B[0, \epsilon]$ for any $t \geq 0$ whenever $x(0) = x_0 \in B[0, \delta]$. ■

Definition 4 (Practical Stabilizability): System (1) is said to be *(locally) practically stabilizable around the origin* if for any $\epsilon > 0$, there exists a switching law $\mathcal{S} = \mathcal{S}(\epsilon)$ such that the system is ϵ-practically stable around the origin under $\mathcal{S}$. ■

In [5], we consider integrator switched systems whose subsystems are $f_i(x) = a_i \in \mathbb{R}^n$, $a_i \neq 0$, $i \in I$. For such systems, besides the above notions, we also introduce the notion of global ϵ-attractivity. The origin is said to be globally ϵ-attractive if for any $x_0 \in \mathbb{R}^n$, $\exists T = T(x_0) \geq 0$ such that $x(t) \in B[0, \epsilon]$, $\forall t \geq T$. Moreover, an integrator switched system is said to be globally practically asymptotically stablilizable around the origin if for any $\epsilon > 0$, $\exists \mathcal{S} = \mathcal{S}(\epsilon)$ such that the system is ϵ-practically stable and the origin is globally ϵ-attractive. The following theorem provides a necessary and sufficient condition for the global practical asymptotic stabilizability of such systems.

Theorem 1 ([5]): System (1) with $f_i(x) = a_i \in \mathbb{R}^n$, $a_i \neq 0$, $i \in I$, is globally practically asymptotically stabilizable around the origin if and only if $C = \mathbb{R}^n$, where C is the convex cone $C = \{\sum_{i=1}^{M} \lambda_i a_i | \lambda_1 \geq 0, \cdots, \lambda_M \geq 0\}$. ■

Remark 2: In this paper, we only need the notions in Definitions 3 and 4 due to the following reason. In general, unlike integrator switched systems, global ϵ-attractivity property is very difficult to be established for system (1). Therefore, we are more interested in local properties for such systems. Since local ϵ-attractivity can be implied by ϵ-practical stability (we can choose the attractive region to be $B[0, \delta]$), we hence do not explicitly study local attractivity in the sequel. ■

III. A SUFFICIENT CONDITION FOR PRACTICAL STABILIZABILITY

In this section, we present the main results of the paper. In the following, Theorem 2 extends the result of [5] and provides a sufficient condition for the local practical stabilizability of system (1). The proof of Theorem 2 is constructive, i.e., for any given $\epsilon > 0$, we actually construct a switching law which achieves the ϵ-practical stability of system (1) around the origin. However, unlike the case of integrator switched systems in [5], the condition in Theorem 2 is not necessary, as will be illustrated by an example.

In Theorem 2 and its proof, we assume that $f_i(0) \neq 0$ for any $i \in I$.

Theorem 2: System (1) in $\mathbb{R}^n$ is practically stabilizable around the origin if $C = \mathbb{R}^n$, where C is the convex cone $C = \{\sum_{i=1}^{M} \lambda_i f_i(0) | \lambda_1 \geq 0, \cdots, \lambda_M \geq 0\}$.

Before proving Theorem 2, let us first introduce the idea of the proof and several preliminary results that will be used in the proof. First of all, we note that given a time interval $[T_k, T_{k+1}]$, a trajectory $x(t)$ starting from $x(T_k)$ can be decomposed into two parts $x_a(t)$ and $x_b(t)$, i.e., $x(t) = x_a(t) + x_b(t)$ for $t \in [T_k, T_{k+1}]$. Here $x_a(t)$ corresponds to the trajectory generated by an integrator switched system consisting of subsystems

$$\dot{x}_a(t) = f_i(0), \ i \in I \tag{3}$$

for $t \in [T_k, T_{k+1}]$, under the same switching sequence as that corresponds to $x(t)$, and with the initial condition $x_a(T_k) = x(T_k)$. And $x_b(t)$ corresponds to the trajectory generated by a switched system consisting of subsystems

$$\dot{x}_b(t) = f_i(x) - f_i(0) \stackrel{\triangle}{=} \Delta f_i(x), \ i \in I \tag{4}$$

for $t \in [T_k, T_{k+1}]$, under the same switching sequence as that corresponds to $x(t)$, and with the initial condition $x_b(T_k) = 0$.

The idea of the proof is as follows. We will partition the whole time range $[0, \infty)$ into time interval $[T_k, T_{k+1}]$'s, $k = 0, 1, 2, \cdots$, and on each interval decompose the state trajectory $x(t)$ into $x_a(t)$, $x_b(t)$ as mentioned above. We will construct a switching law $\mathcal{S}(\epsilon)$ for any given $\epsilon > 0$, such that $\exists 0 < \delta < \frac{\epsilon}{2}$ so that $x(0) \in B[0, \delta]$ implies that $\|x_a(t)\| \leq \frac{\epsilon}{2}$ and $\|x_b(t)\| \leq \delta < \frac{\epsilon}{2}$, $\forall t \geq 0$. In this way, we can conclude that $\|x(t)\| \leq \|x_a(t)\| + \|x_b(t)\| \leq \frac{\epsilon}{2} + \delta < \epsilon$, $\forall t \geq 0$. Moreover, the time intervals are actually generated by $\mathcal{S}(\epsilon)$.

Let us £rst consider the switched system whose dynamics is (3). Note that this system corresponds to an integrator switched system during each time interval $[T_k, T_{k+1}]$. From Theorem 1 we can see that the condition $C = \mathbb{R}^n$ in Theorem 2 is actually a necessary and sufficient condition for the global practical asymptotic stabilizability of the integrator switched system. Moreover, from the insight of the proof of Theorem 1 (see [4]), we can prove the following lemma.

Lemma 1: For system (3) satisfying the condition $C = \mathbb{R}^n$, there exists a constant $G_1 > 0$ such that every $x \in B[0, 1]$ can be expressed as

$$x = \sum_{i=1}^{M} \gamma_i f_i(0) \tag{5}$$

where the γ_i's satisfy $\gamma_i \leq 0$ for $1 \leq i \leq M$ and

$$\sum_{i=1}^{M} |\gamma_i| \leq G_1. \tag{6}$$

Proof of Lemma 1: We prove this lemma by constructing the expression (5) for any $x \in B[0, 1]$. First let us consider the unit vectors $e_1, \cdots, e_n$ in $\mathbb{R}^n$ and their negatives $-e_1, \cdots, -e_n$. We denote them as $\hat{e}_1 = e_1$, $\cdots$, $\hat{e}_n = e_n$, $\hat{e}_{n+1} = -e_1$, $\cdots$, $\hat{e}_{2n} = -e_n$. Since $C = \mathbb{R}^n$, they have the representations

$$\hat{e}_1 = \sum_{i=1}^{M} \lambda_{1,i} f_i(0), \cdots, \hat{e}_{2n} = \sum_{i=1}^{M} \lambda_{2n,i} f_i(0) \tag{7}$$

with $\lambda_{k,i} \geq 0$. Furthermore, note that every vector $x \in B[0,1]$ can be represented as $x = \sum_{k=1}^{n} \alpha_k e_k$ where $\alpha_k \in \mathbb{R}$, $\sum_{k=1}^{n} \alpha_k^2 \leq 1$. By using the $\hat{e}_k$'s, x can be represented as

$$x = \sum_{k=1}^{2n} \beta_k \hat{e}_k, \tag{8}$$

where $\beta_k = \begin{cases} \alpha_k, & \text{if } \alpha_k \leq 0 \\ 0, & \text{if } \alpha_k > 0 \end{cases}$ for $1 \leq k \leq n$ and $\beta_k = \begin{cases} -\alpha_{k-n}, & \text{if } \alpha_{k-n} > 0 \\ 0, & \text{if } \alpha_{k-n} \leq 0 \end{cases}$ for $n+1 \leq k \leq 2n$. Note that every $\beta_k \leq 0$ and $\sum_{k=1}^{2n} \beta_k^2 = \sum_{k=1}^{n} \alpha_k^2 \leq 1$.

Substituting (7) into (8), we can write x as

$$\begin{aligned} x &= \sum_{k=1}^{2n} \beta_k \hat{e}_k = \sum_{k=1}^{2n} \beta_k \big(\sum_{i=1}^{M} \lambda_{k,i} f_i(0) \big) \\ &= \sum_{i=1}^{M} \big(\sum_{k=1}^{2n} \beta_k \lambda_{k,i} \big) f_i(0) = \sum_{i=1}^{M} \gamma_i f_i(0) \end{aligned} \tag{9}$$

where $\gamma_i = \sum_{k=1}^{2n} \beta_k \lambda_{k,i} \leq 0$ for any $1 \leq i \leq M$ and

$$\begin{aligned} \sum_{i=1}^{M} |\gamma_i| &\leq \sum_{i=1}^{M} \Big(\big(\sum_{k=1}^{2n} \beta_k^2 \big)^{\frac{1}{2}} \big(\sum_{k=1}^{2n} \lambda_{k,i}^2 \big)^{\frac{1}{2}} \Big) \\ &\leq \sum_{i=1}^{M} \big(\sum_{k=1}^{2n} \lambda_{k,i}^2 \big)^{\frac{1}{2}} \triangleq G_1. \end{aligned} \tag{10}$$

In (10), since not all $\lambda_{k,i}$'s are equal to 0, we have $G_1 > 0$. ∎

Remark 3: In fact, in (5), if $x \neq 0$, then not all γ_i's are equal to 0. ∎

The following switching law can be applied to the switched system (3) to generate a nonZeno switching sequence for initial $x(0) \in B[0,1]$.

<u>**Switching Law A**</u> (for integrator switched system (3) with $x_a(0) = x(0) \in B[0,1]$):

(1). Assume that the system trajectory starts from $x_a(0) \in B[0,1]$ at time 0. Set $k = 0$, $T_k = 0$ and the current state $x_a(T_k) = x_a(0)$.

(2). Obtain the expression of the current state $x_a(T_k) = \sum_{i=1}^{M} \gamma_i f_i(0)$ as in (5). First switch to subsystem 1 and stay for time $|\gamma_1|$, then switch to subsystem 2 and stay for time $|\gamma_2|$ and so on. In other words, we obtain a switching sequence $\big((T_k, 1), (T_k + |\gamma_1|, 2), (T_k + |\gamma_1| + |\gamma_2|, 3), \cdots, (T_k + |\gamma_1| + \cdots + |\gamma_{M-1}|, M)\big)$ from time T_k to $\tilde{T}_k \triangleq T_k + \sum_{i=1}^{M} |\gamma_i|$.

(3). From time $\tilde{T}_k$ on, we let subsystem M be active until the state trajectory intersects the unit sphere.

(4). When the state intersects the unit sphere, set $k = k+1$ and denote T_k to be the time instant of intersection (note that $x_a(T_k)$ is the intersecting point). Go back to step (2). ∎

Under Switching Law A, note that $x_a(\tilde{T}_k) = 0$, $\forall k \geq 0$. Since for any $x_a(T_k)$ on the unit sphere, to drive the state to the origin, no switching law will take less than the time duration $\frac{1}{\max_{1 \leq i \leq M} \|f_i(0)\|}$, we conclude that it takes no less than time $\frac{1}{\max_{1 \leq i \leq M} \|f_i(0)\|}$ and no more than M switchings to complete one iteration of steps (2), (3), and (4) in Switching Law A (for any $x(T_k)$ on the unit sphere). Therefore, using Switching Law A, we obtain a nonZeno switching sequence for initial state $x_a(0) \in B[0,1]$. Moreover, for any $x_a(0) \in B[0,1]$, we can show that under Switching Law A, there exists a $G > 1$ such that the trajectory $x_a(t) \in B[0,G]$ for any $t \geq 0 = T_0$. The G can be chosen as follows. Consider any $T_0 \leq t \leq \tilde{T}_0$, we have

$$\begin{aligned} \|x_a(t)\| &\leq \|x_0\| + \sum_{i=1}^{M} |\gamma_i| \cdot \max_{1 \leq i \leq M} \|f_i(0)\| \\ &\leq 1 + G_1 \cdot \max_{1 \leq i \leq M} \|f_i(0)\| \end{aligned} \tag{11}$$

Defining $G \triangleq 1 + G_1 \cdot \max_{1 \leq i \leq M} \|f_i(0)\|$, we then have $G > 1$ and $\|x_a(t)\| \leq G$ for $t \in [T_0, \tilde{T}_0]$. For $t \in [\tilde{T}_0, T_1]$ where $T_1 = \tilde{T}_0 + \frac{1}{\|f_M(0)\|}$, by Switching Law A, the trajectory $x_a(t) \in B[0,1]$ and hence is in $B[0,G]$. Similarly, such arguments can be applied to any time interval $[T_k, T_{k+1}]$ to establish the validity of $x_a(t) \in B[0,G]$, $\forall t \geq 0$.

Corollary 1: Given any $\epsilon_1 > 0$, any $x \in B[0, \frac{\epsilon_1}{2G}]$ can be expressed as

$$x = \sum_{i=1}^{M} \hat{\gamma}_i f_i(0) \tag{12}$$

where $\hat{\gamma}_i \leq 0$ for $1 \leq i \leq M$ and

$$\sum_{i=1}^{M} |\hat{\gamma}_i| \leq \frac{\epsilon_1 G_1}{2G}. \tag{13}$$

Proof of Corollary 1: For every $x \in B[0, \frac{\epsilon_1}{2G}]$, we have $\frac{2G}{\epsilon_1} x \in B[0,1]$ and can be expressed as

$$\frac{2G}{\epsilon_1} x = \sum_{i=1}^{M} \gamma_i f_i(0). \tag{14}$$

The conclusion of the corollary then follows by defining $\hat{\gamma}_i \triangleq \frac{\epsilon_1}{2G} \gamma_i$.

Remark 4: Substituting γ_i's by $\hat{\gamma}_i$'s, and the unit sphere by the $\frac{\epsilon_1}{2G}$-sphere in Switching Law A, a similar law can be developed for system (3) with $x_a(0) \in B[0, \frac{\epsilon_1}{2G}]$ such that $x_a(\tilde{T}_k) = 0$, $\forall k \geq 0$, and $\|x_a(t)\| \leq \frac{\epsilon_1}{2}$, $\forall t \geq 0$. ∎

In the following, we define $\delta \triangleq \frac{\epsilon_1}{2G}$. The following switching law is a modification of Switching Law A and is suitable for switched system (1) with $x(0) \in B[0, \delta]$.

<u>**Switching Law B**</u> (for switched system (1) with $x(0) \in B[0, \delta]$):

(1). Assume that the system trajectory starts from $x(0) \in B[0, \delta]$ at time 0. Set $k = 0$, $T_k = 0$ and the current state $x(T_k) = x(0)$.

(2). Obtain the expression of the current state $x(T_k) = \sum_{i=1}^{M} \hat{\gamma}_i f_i(0)$. First switch to subsystem 1 and stay for time $|\hat{\gamma}_1|$, then switch to subsystem 2 and stay for time $|\hat{\gamma}_2|$ and so on. In other words, we obtain a switching sequence $\big((T_k, 1), (T_k + |\hat{\gamma}_1|, 2), (T_k + |\hat{\gamma}_1| + |\hat{\gamma}_2|, 3), \cdots, (T_k + |\hat{\gamma}_1| + \cdots + |\hat{\gamma}_{M-1}|, M)\big)$ from time T_k to $\tilde{T}_k \triangleq T_k + \sum_{i=1}^{M} |\hat{\gamma}_i|$.

(3). From time $\tilde{T}_k$ on, we let subsystem M be active until the state trajectory intersects the δ-sphere.

(4). When the state intersects the δ-sphere, set $k = k + 1$ and denote T_k to be the time instant of intersection (note that $x(T_k)$ is the intersecting point). Go back to step (2). ■

Equipped with Corollary 1 and Switching Law B, we are now ready to present the proof of Theorem 2. The proof is based on the application of Switching Law B to switched system (1). Before proceeding, a remark is in order.

Remark 5: Note that at each T_k, $x(t)$ can be decomposed into $x_a(T_k) = x(T_k)$ and $x_b(T_k) = 0$. Thus when Switching Law B is applied to switched system (1), we have $x_a(\tilde{T}_k) = 0$ and $\|x_a(t)\| \leq \frac{\epsilon_1}{2G} \cdot G = \frac{\epsilon_1}{2}$ (see Remark 4). However, $x(\tilde{T}_k)$ may not be equal to 0, and $x(T_{k+1})$ may not even exist (since it is possible that after certain $\tilde{T}_k$, the trajectory generated by step (3) may never intersect the δ-sphere). Of course, if T_{k+1} does exist, we must have $\|x(T_{k+1})\| = \delta$. Also, in order to carry out step (3), we need to justify that $x(\tilde{T}_k) \in B[0, \delta]$. This will be shown to be true in our proof of Theorem 2 in the following. ■

Proof of Theorem 2: Consider $x_b(t)$ for the switched system (4) under the switching sequence generated by Switching Law B. First note that during $[T_0, \tilde{T}_0]$, $x_b(t)$ can be expressed as

$$x_b(t) = \int_{T_0}^{t} \Delta f_{i(\tau)}\big(x(\tau)\big)\, d\tau \tag{15}$$

where $i(\tau)$ indicates the active subsystem at time instant τ (also note that $x_b(T_0) = 0$). Assume that $x(t) \in B[0, \epsilon_1]$ for any $t \in [T_0, \tilde{T}_0]$. Under this assumption, we have $\|x_b(t)\| \leq \max_{i \in I, x \in B[0,\epsilon_1]} \|\Delta f_i(x)\| \cdot (t - T_0)$ for any $t \in [T_0, \tilde{T}_0]$.

Define

$$K(\epsilon_1) \triangleq \max_{i \in I, x \in B[0,\epsilon_1]} \|\Delta f_i(x)\|. \tag{16}$$

Since every $f_i(x)$ is locally Lipschitz continuous, we have that every $\Delta f_i(x)$ is locally Lipschitz continuous and hence $K(\epsilon_1) \to 0$ as $\epsilon_1 \to 0$. Now consider any $t \in [T_0, \tilde{T}_0]$. Since

$$\begin{aligned}
\|x_b(t)\| &\leq \max_{i \in I, x \in B[0,\epsilon_1]} \|\Delta f_i(x)\| \cdot (t - T_0) \\
&\leq \max_{i \in I, x \in B[0,\epsilon_1]} \|\Delta f_i(x)\| \cdot (\tilde{T}_0 - T_0) \\
&= K(\epsilon_1) \cdot \sum_{i=1}^{M} |\hat{\gamma}_i| \qquad (17) \\
&\leq K(\epsilon_1) \cdot \frac{\epsilon_1 G_1}{2G}. \qquad (18)
\end{aligned}$$

From the continuity of $K(\epsilon_1)$, there exists an ϵ_0 such that for any ϵ_1 satisfying $0 < \epsilon_1 \leq \epsilon_0$, we have $K(\epsilon_1) \leq \frac{1}{G_1}$. For such an ϵ_1, we then have

$$\|x_b(t)\| \leq K(\epsilon_1) \cdot \frac{\epsilon_1 G_1}{2G} \leq \frac{\epsilon_1}{2G} = \delta < \frac{\epsilon_1}{2}. \tag{19}$$

To completely justify the validity of (19), we need to verify that our assumption $x(t) \in B[0, \epsilon_1]$ is true for any $t \in [T_0, \tilde{T}_0]$. We prove its validity by contradiction. Assume that there exists a $t_1 \in [T_0, \tilde{T}_0]$ at which the state trajectory intersects the ϵ_1-sphere for the first time. For such a t_1, we must have

$$\begin{aligned}
\|x(t_1)\| &= \|x_a(t_1) + x_b(t_1)\| \\
&\leq \|x_a(t_1)\| + \|x_b(t_1)\| \\
&\leq \frac{\epsilon_1}{2} + \int_0^{t_1} \|\Delta f_{i(\tau)}\big(x(\tau)\big)\|\, d\tau \\
&\leq \frac{\epsilon_1}{2} + K(\epsilon_1) \cdot (t_1 - T_0) \\
&\leq \frac{\epsilon_1}{2} + K(\epsilon_1) \cdot \sum_{i=1}^{M} |\hat{\gamma}_i| \\
&\leq \frac{\epsilon_1}{2} + K(\epsilon_1) \cdot \frac{\epsilon_1 G_1}{2G} \\
&\leq \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2G} < \frac{\epsilon_1}{2} + \frac{\epsilon_1}{2} = \epsilon_1 \qquad (20)
\end{aligned}$$

which is a contradiction to $\|x(t_1)\| = \epsilon_1$. Hence we conclude that if $x(0) \in B[0, \delta]$, then $x(t) \in B[0, \epsilon_1]$ for any $t \in [T_0, \tilde{T}_0]$. In particular, at $t = \tilde{T}_0$, we have $x_a(\tilde{T}_0) = 0$ and $x_b(\tilde{T}_0) \in B[0, \delta]$; hence $x(\tilde{T}_0) \in B[0, \delta]$. For any $t \in [\tilde{T}_0, T_1]$, due to step (3) in Switching Law B, we have $x(t) \in B[0, \delta] \subset B[0, \epsilon_1]$. Using a similar argument as the above, we can then prove that $x(t) \in B[0, \epsilon_1]$ for $t \in [T_1, T_2]$ and $t \in [T_2, T_3]$ and so on, which consequently establishes that $x(t) \in B[0, \epsilon_1]$ for any $t \geq 0$.

Finally, for any given $\epsilon > 0$, we can choose an ϵ_1 satisfying $0 < \epsilon_1 \leq \min\{\epsilon, \epsilon_0\}$, and then choose $\delta = \frac{\epsilon_1}{2G}$. In this way, we have whenever $x(0) \in B[0, \delta]$, $x(t) \in B[0, \epsilon_1]$ and hence $x(t) \in B[0, \epsilon]$. ■

Remark 6: From the proof of Theorem 2, we note that even if $\epsilon > 0$ is given, we usually need to choose $\epsilon_1 \leq \min\{\epsilon, \epsilon_0\}$ in order to determine δ. Here ϵ_0 can be chosen to be the maximum value of ϵ_0 such that $K(\epsilon_0) \leq \frac{1}{G_1}$, and is independent of ϵ. If the given $\epsilon > \epsilon_0$, we can see that the trajectory $x(t)$ will actually be in $B[0, \epsilon_0]$; hence ϵ will be an overestimate of the trajectory bound. However, it may not be possible in this case to relax δ by a small amount so as to make ϵ a tight bound. This is because if $x(0)$ deviates from 0 too much, it may then be impossible to keep the trajectory in $B[0, \epsilon]$. This explains why the result is local in nature. ■

In order to apply Theorem 2, we need to verify the validity of the condition $C = \mathbb{R}^n$. The following lemma in [5] provides us with an easier way to verify it (the proof of it can be found in [4]).

Lemma 2 ([5]): $C = \mathbb{R}^n$ if and only if there exists a subset $\{f_{i_1}(0), \cdots, f_{i_l}(0)\}$ of $\{f_1(0), \cdots, f_M(0)\}$ which satisfies the following conditions:

(a). span$\{f_{i_1}(0), \cdots, f_{i_l}(0)\} = \mathbb{R}^n$ and

(b). there exist $\lambda_j > 0$, $j = 1, \cdots, l$, such that $\sum_{j=1}^{l} \lambda_j f_{i_j}(0) = 0$. ■

Finally in this section, we point out that, in general, the condition in Theorem 2 is not necessary for the practical stabilizability of system (1). The following example illustrates this fact.

Example 1: Consider a switched system in $\mathbb{R}^2$ which consists of

$$\text{subsystem 1: } \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -x_2 \end{bmatrix} = f_1(x) \quad (21)$$

$$\text{subsystem 2: } \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -x_2 \end{bmatrix} = f_2(x) \quad (22)$$

Since $f_1(x)$ and $f_2(x)$ are continuously differentiable, they are locally Lipschitz continuous. For this system, $C = \{\lambda_1 f_1(0) + \lambda_2 f_2(0) | \lambda_1 \geq 0, \lambda_2 \geq 0\} = \{[a,\ 0]^T | a \in \mathbb{R}\} \neq \mathbb{R}^2$. However, the system is locally practically stabilizable around the origin. Since given any $\epsilon > 0$, we can choose $\delta = \frac{\epsilon}{4}$ and construct a switching law which alternates the active subsystem once for every time duration $\frac{\epsilon}{4}$. Under this switching law, for any $x(0) \in B[0, \delta]$, we can show $x(t) \in B[0, \epsilon]$ for any $t \geq 0$ as follows. In this case it is easy to see that $x_1(t)$ will be decreasing (under subsystem 1) and increasing (under subsystem 2), yet all within the range $[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$. And since for both subsystems $x_2(t)$ satisfies $\dot{x}_2 = -x_2$. We have $x_2(t) = e^{-t}x_2(0)$ where $x_2(0)$ is the initial condition of x_2. Hence $|x_2(t)| \leq |x_2(0)| \leq \frac{\epsilon}{4}$ for any $t \geq 0$. Consequently we have $\|x(t)\| = \sqrt{x_1^2(t) + x_2^2(t)} \leq \sqrt{\frac{\epsilon^2}{4} + x_2^2(0)} \leq \sqrt{\frac{\epsilon^2}{4} + \frac{\epsilon^2}{16}} < \epsilon$. ■

IV. A TRACKING PROBLEM EXAMPLE

In this section, we apply Switching Law B proposed in the proof of Theorem 2 to a tracking problem example.

Example 2: Consider a switched system in $\mathbb{R}^2$ which consists of

$$\text{subsystem 1: } \dot{y} = Ay + b_1 \quad (23)$$

$$\text{subsystem 2: } \dot{y} = Ay + b_2 \quad (24)$$

$$\text{subsystem 3: } \dot{y} = Ay + b_3 \quad (25)$$

where $A = \begin{bmatrix} -0.1 & 0.2 \\ -0.2 & -0.1 \end{bmatrix}$, $b_1 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}$, and $b_3 = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}$. We want to develop a switching law such that the switched system state $y(t)$ would approximately track the state trajectory of the dynamical system

$$\dot{z} = Az \quad (26)$$

in the sense that $\|y(t) - z(t)\| \leq \epsilon$ (ϵ is a prespecified tolerance level) for $t \geq 0$. Here we assume that $y(0) = z(0)$.

We can transform the above tracking problem into a practical stabilization problem by defining $x(t) \stackrel{\triangle}{=} y(t) - z(t)$ and considering the following switched system

$$\text{subsystem 1: } \dot{x} = f_1(x) = Ax + b_1 \quad (27)$$

$$\text{subsystem 2: } \dot{x} = f_2(x) = Ax + b_2 \quad (28)$$

$$\text{subsystem 3: } \dot{x} = f_3(x) = Ax + b_3 \quad (29)$$

with $x(0) = 0$ (such a system comes from the subtraction of (26) from (23)-(25)). To solve the tracking problem, we only need to design a switching law under which the switched system with subsystems (27)-(29) becomes ϵ-practical stable around the origin. In can readily be seen that the convex cone of $f_i(0) = b_i$, $i = 1, 2, 3$ is $\mathbb{R}^2$ by using Lemma 2. Hence the condition in Theorem 2 is satisfied. Therefore we can apply Switching Law B to the system (27)-(29) to render it ϵ-practically stable around the origin.

The parameters used in Switching Law B can be obtained as follows. First note that the $\lambda_{k,i}$'s in (7) can be obtained by expressing $\hat{e}_k = \sum_{i=1}^{3} \lambda_{k,i} f_i(0) = \sum_{i=1}^{3} \lambda_{k,i} b_i$. A choice of these parameters can be obtained by noting that $\hat{e}_1 = 2b_1$, $\hat{e}_2 = 2b_1 + 2b_2$, $\hat{e}_3 = b_2 + b_3$, $\hat{e}_4 = 2b_1 + 2b_3$. With this choice, we then have $G_1 = \sum_{i=1}^{3} (\sum_{k=1}^{4} \lambda_{k,i}^2)^{\frac{1}{2}} = 7.9362$ and $G = 1 + G_1 \cdot \max_{1 \leq i \leq 3} \|b_i\| = 6.6118$. Also note that $K(\epsilon_1) \leq \|A\| \cdot \epsilon_1$. So if we choose $\epsilon_0 = \frac{1}{\|A\| G_1} = 0.5635$, for any $0 < \epsilon_1 \leq \epsilon_0$, we will have $x(t) \in B[0, \epsilon_1]$ for $t \geq 0$ if $x(0) \in B[0, \delta]$.

Assume we are given $\epsilon = 0.1$, we can simply choose $\epsilon_1 = \min\{\epsilon, \epsilon_0\} = \epsilon = 0.1$ and consequently $\delta = \frac{\epsilon_1}{2G} = 0.0076$. Once we have the parameters $\lambda_{k,i}$'s, G_1, G, ϵ_1, and δ, Switching Law B can readily be applied. The $x(t)$ thus obtained can then be reinterpreted into $y(t)$. Figure 1 shows the desired trajectory $z(t)$ to be tracked and Figure 2 shows the trajectory $y(t)$ for a finite time duration. It can be seen that $y(t)$ approximately tracks $z(t)$ and satisfies $\|y(t) - z(t)\| \leq 0.1$. A closer look into a portion of the trajectories $y(t)$ and $z(t)$ is shown in Figure 3. ■

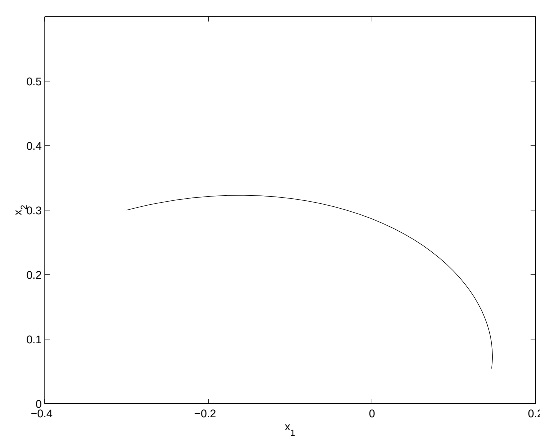

Fig. 1. The trajectory $z(t)$ with $z(0) = [-0.3,\ 0.3]^T$ for $t \in [0, 10]$.

V. CONCLUSION

This paper reports some results on practical stabilization problems of a class of switched systems with autonomous

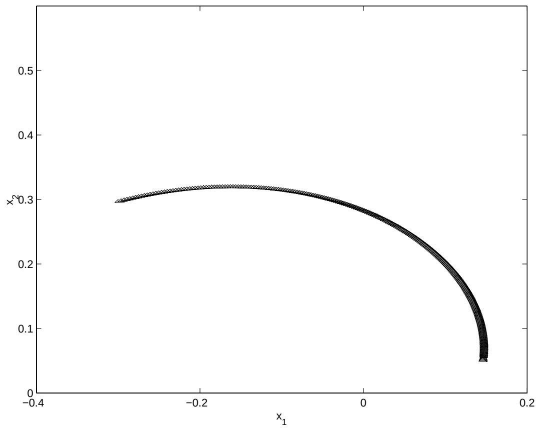

Fig. 2. The trajectory $y(t)$ (under Switching Law B) with $y(0) = [-0.3,\ 0.3]^T$ for $t \in [0, 10]$.

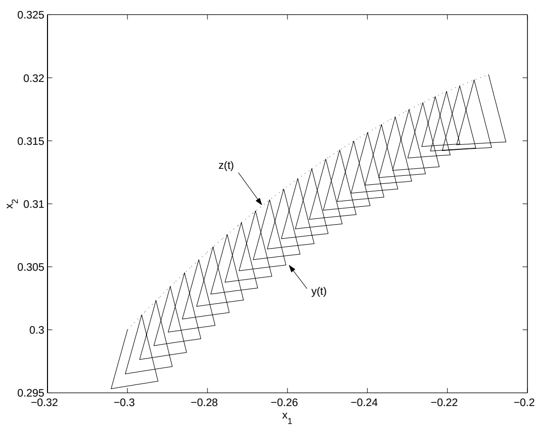

Fig. 3. The trajectory $y(t)$ (solid curve, under Switching Law B) and $z(t)$ (dotted curve) with $y(0) = z(0) = [-0.3,\ 0.3]^T$ for $t \in [0, 1]$.

subsystems. A sufficient condition for the practical stabilizability of such systems was proved. Moreover, a switching law for ϵ-practical stability was constructed. The switching law can easily be implemented. The research in this paper is a continuation of our previous studies in [5] and is a further step towards the studies of more general stabilization and tracking problem of switched systems. Future research includes the extensions of the results to the studies of local behaviors of switched systems with time-varying subsystems and tracking of more general trajectories of nonlinear systems.

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