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Identificability of Multi-leaks in a Pipeline

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Abstract— This contribution discusses the sensibility of the transient response of a fluid in a pipeline with respect to the existence of two leaks which occur simultaneosly assuming that only pressure and flow rate are measured at the extremes of the line. It is shown that the four unknown parameters of the fluid model produced by two simultaneous leaks are identificable considering the transient response of the fluid. The analysis is made generating a family of nonlinear lumped parameters models of the fluid with only two unknown parameters and introducing the parameter z_{eq} which parametrizes the subset of two indistinguishable leaks with steady state data. The evaluation of the quadratic integral error of the output vector by simulation for any two members of the family with parameter z_{eq} shows the possibility to identify the unknown parameters of two leaks considering the transient behavior at the extremes of the pipeline. Thus, the identification of the parameters can be achieved off-line using the family of models with data of the transient response; diverse optimization algorithms can be implemented for this purpose. To estimate the parameters of a water pilot pipeline with leaks, here an Extended Kalman Filter of reduced order has been implemented in a recursive structure. The advantages of the novel family of models and its specifications for the leak location problem are shown with an example.

I. INTRODUCTION

The automatic detection and location of multi-leaks in pipelines is a challenge for the process control and supervision engineering. Several schemes of leak location based in the mathematical model of the fluid have been developed. Shields [1] and Kowalczuk [2] designed residual generators using a finite dimension model and assuming fix space discretization in the set of PDE which describes the fluid behavior. However, these methods are not robust with respect to uncertainty in the position and can be only applied to detect and locate the leaks in limited cases, because first the leaks points are in advance fixed, and second the isolation algorithm of the leaks is established in steady state condition of the residual. The first assumption provokes false locations and the second one generates a non unique solution when at least two leaks appear simultaneously. In spite of the strong detectability property of this particular problem [3], the two mentioned drawbacks make less useful these methods. Since in a real application the isolation task has a high priority, Verde et al [4] formulated the leak location problem with two coupled residuals, in which the leak position is adapted minimizing the sensitivity of one residual and maximizing the sensitivity of other one. This procedure is robust with respect to the location of sequential leaks. However, if leaks appears at the same time, false positions are estimated.

On the other hand, it has been recently shown that the isolation problem for two leaks is not feasible only with steady state data of the fluid in a pipeline [5]. This fact motivates the following study about the identificability problem of leaks in a pipeline when only pressures and flow rates are measured at the extremes of the line. The analysis is made evaluating the sensitivity of the transient response of the fluid with respect to positions and outflows produced by the leaks. One concludes that the transient of the fluid is rich enough to isolate the characteristics of the two leaks. To estimate the parameters an Extended Kalman Filter is used and the results with simulation data show the feasibility of the procedure.

This paper is organized as follows. In section II the nonlinear dynamic model assuming two leaks non uniform distributed in the pipeline is given together with the statics relations which satisfy the physical variables of the system. In particular, a key parameter which characterizes a subset of leaks called z_{eq} is defined. Section III discusses the limitation about the leak localization when only steady state properties are used and two leaks appear simultaneously. In section IV an Extended Kalman Filter to identify the unknown parameters is presented and finally the conclusions are given in the last section.

II. PIPELINE MODEL

Consider the one-dimensional nonlinear model of a fluid in a pipeline with distributed parameters given by

$$\frac{\partial Q(z,t)}{\partial t} + gA\frac{\partial H(z,t)}{\partial z} + \mu Q(z,t)\,|Q(z,t)| = 0 \qquad (1)$$

$$b^2\frac{\partial Q(z,t)}{\partial z} + gA\frac{\partial H(z,t)}{\partial t} = 0 \qquad (2)$$

which is obtained using momentum and energy equations and assuming incompressible fluid [6], with $H(z,t)$ the pressure head (m), $Q(z,t)$ the flow rate (m^3/s), z the length coordinate (m), t the time coordinate (s), g the gravity (m/s^2), A the section cross-area (m^2), D

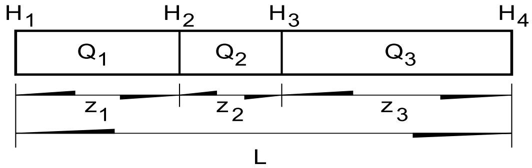

Fig. 1. Variables definition with non uniform sections

the pipeline diameter (m), b the speed of sound (m/s) and $\mu = \frac{f}{2DA}$ where f is the friction coefficient.

Considering the existence of two leaks arbitrarily located at point $p_1 = z_1$ and $p_2 = z_1 + z_2$ of a pipeline with length L, the outflow associated to each leak

$$Q_f(p_i, t) = \lambda_i \sqrt{H(p_i, t)} \quad \text{for } i = 1, 2 \qquad (3)$$

with $\lambda_i > 0$ produces a discontinuity in the set of eqs. (1) and (2). If the up and down stream flows and the pressures at the extremes of the line are measured, diverse combination of input and output variables can be defined to describe the system (1,2). Choosing the output and input vectors given respectively by

$$y = \begin{bmatrix} Q(t,0) & Q(t,L) \end{bmatrix}^T \quad u = \begin{bmatrix} H(t,0) & H(t,L) \end{bmatrix}^T$$

Verde [7] discretized the differentials with respect to z (1,2) by a first order approximation to get a lumped parameters model, obtaining

$$\begin{bmatrix} \dot{Q}_1 \\ \dot{H}_2 \\ \dot{Q}_2 \\ \dot{H}_3 \\ \dot{Q}_3 \end{bmatrix} = \begin{bmatrix} -\mu Q_1^2 - \frac{a_1}{z_1}(H_2 - u_1) \\ \frac{a_2}{z_2}(Q_1 - Q_2 - \sqrt{H_2}\lambda_1) \\ -\mu Q_2^2 + \frac{a_1}{z_2}(H_2 - H_3) \\ \frac{a_2}{z_2}(Q_2 - Q_3 - \sqrt{H_3}\lambda_2) \\ -\mu Q_3^2 + \frac{a1}{z_3}(H_3 - u_2) \end{bmatrix} \qquad (4)$$

where the flow Q_i and pressure H_i are the discretized variables given in Fig. 1 and the sections sizes of the pipeline satisfy

$$z_1 + z_2 + z_3 = L, \qquad (5)$$

with a_1 and a_2 constants.

One can see from Fig. 1, that the discretization depends on the leaks separation and so long there is not a leak the section definition is arbitrary. Moreover, defining

$$\begin{bmatrix} y_{1s} & y_{2s} \end{bmatrix} := \lim_{t\to\infty} \begin{bmatrix} y_1 & y_2 \end{bmatrix}$$

$$\begin{bmatrix} u_{1s} & u_{2s} \end{bmatrix} = \lim_{t\to\infty} \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

one obtains the following steady state condition for one leak

$$\text{If} \quad \lambda_1 \neq 0, \quad \frac{a_1}{\mu}(u_{1s} - u_{2s}) = z_1 y_{1s}^2 + (z_2 + z_3) y_{2s}^2 \qquad (6)$$

$$\text{If} \quad \lambda_2 \neq 0, \quad \frac{a_1}{\mu}(u_{1s} - u_{2s}) = (z_1 + z_2) y_{1s}^2 + z_3 y_{2s}^2 \qquad (7)$$

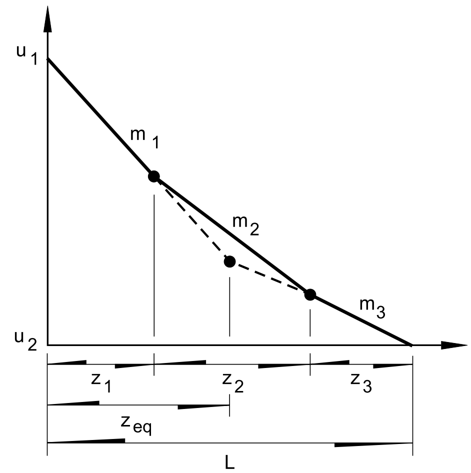

Fig. 2. Pressure drops with 2 leaks at points z_1 and $z_1 + z_2$

and for the two-leak case

$$y_{1s} - y_{2s} = \lambda_1 \sqrt{u_{1s} - \frac{\mu z_1}{a_1} y_{1s}^2} + \lambda_2 \sqrt{u_{2s} + \frac{\mu z_3}{a_1} y_{2s}^2} \qquad (8)$$

To characterize the set for two leaks, Verde et al [5] defined the parameter z_{eq}

$$z_{eq} := \frac{a_1(u_{1s} - u_{2s})}{\mu(y_{1s}^2 - y_{2s}^2)} - \frac{L y_{2s}^2}{(y_{1s}^2 - y_{2s}^2)} \qquad (9)$$

for $y_{1s} \neq y_{2s}$, and it can be estimated straightforward from the data of the pipeline with

$$z_1 < z_{eq}, \qquad z_2 > z_{eq} - z_1 \qquad (10)$$

Fig. 2 shows the pressure behavior in steady state under these conditions. One can see that if leaks are located at (z_1, z_2), the profile of the pressure along the first and last section is equivalent to the profile of one leak located at point z_{eq}. In the case of one leak, $z_{eq} = z_1$, and z_2 could take any value satisfying the second inequality of (10). Moreover from the (9) and Fig. 2 one obtains additionally

$$(y_{1s}^2 - y_{2s}^2)(z_{eq} - z_1) = z_2(Q_{2s}^2 - y_{2s}^2) \qquad (11)$$

with $Q_{2s} = \lim_{t\to\infty} Q_2(t)$.

Since, there are three eqs. (8), (9) and (11) and four unknown parameters, there are an infinite number of quadruplets $\wp_j(z_{eq}) = (z_{1_j}, \lambda_{1_j}, z_{2_j}, \lambda_{2_j})$ for a given z_{eq} satisfying

$$\lim_{t\to\infty} (y\wp_j(z_{eq}) - y\wp_k(z_{eq})) = 0 \quad j \neq k \qquad (12)$$

with $y\wp_{\tilde{n}}(z_{eq})$ the output vector for the set $\wp_{\tilde{n}}(z_{eq})$. It means, model (4) in steady state is not feasible to identify the parameter of the two leaks. Thus, z_{eq} characterizes the subset of unidentificable pair of leaks which cannot be isolated in term of the measurable variables in steady state (inlet-flow, outlet-flow and total pressure drop).

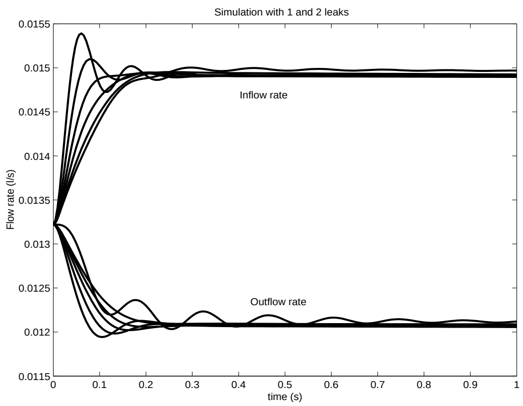

Fig. 3. Transient response simulation with 2 leaks for $z_{eq} = 50m$

It is to remark, that the input-output model (4) is insensitive to the unknown parameters (z_1, z_2) before the occurrence of the leaks.

Fig. 3 shows a set of responses of inlet-flow and outlet-flow for $z_{eq} = 50m$ simulated with the six couples of leaks given in Table I. It can be seen from the responses that the flow rates are deviated one from the other only during the transients; after the transitions, the responses achieve the same output values.

TABLE I

Set of indistinguishable leaks

	$Pair_1$	$Pair_2$	$Pair_3$	$Pair_4$	$Pair_5$	$Pair_6$
z_1	50.	10.	17.	25.	33.5	42.
z_2	0.	100.	83.	62.	41.6	20.
λ_1	0.001	0.0005	0.0005	0.0005	0.0005	0.0005
λ_2	0.	0.0005	0.0005	0.0005	0.0005	0.0005

As consequence, the only possibility to isolate the leaks is to tackle the multi-leaks problem considering the transient response of the fluid for a given z_{eq} and using the parameters identification framework or a pattern recognition procedure in spite of the required persistent excitation condition. It is important to remark that this loss of distinguishability for two leaks cannot be recovered by any residual evaluation method, even fuzzy logic, based on steady state conditions of the residual.

III. PARAMETERS IDENTIFICABILITY

In order to characterize the family of dynamics models for a given z_{eq} after the occurrence of the leaks and proving that (6) and (7) are not satisfied, one obtains, from eqs. (11), (9) and (8) the static relationships for the parameters

$$z_1 = z_{eq} - c_1 z_2 \quad (13)$$

$$\lambda_1 = \frac{y_{1s} - Q_{2s}}{\sqrt{u_{1s} - \frac{\mu z_1}{a_1} y_{1s}^2}} \quad (14)$$

with

$$c_1 = \frac{Q_{2s}^2 - y_{2s}^2}{y_{1s}^2 - y_{2s}^2}.$$

Assuming that the flow in steady state Q_{2s} is known when both leaks are present and substituting the above steady state relationships into model (4) one gets the family of models

$$\begin{aligned}
\dot{\tilde{Q}}_1 &= -\mu \tilde{Q}_1^2 + \frac{a_1}{z_{eq} - c_1 z_2}(u_1 - \tilde{H}_2) \\
\dot{\tilde{H}}_2 &= \frac{a_2}{z_2}\left(\tilde{Q}_1 - \tilde{Q}_2 - \frac{c_2}{\sqrt{\sigma(z_2)}}\sqrt{\tilde{H}_2}\right) \\
\dot{\tilde{Q}}_2 &= -\mu \tilde{Q}_2^2 + \frac{a_1}{z_2}(\tilde{H}_2 - \tilde{H}_3) \\
\dot{\tilde{H}}_3 &= \frac{a_2}{z_2}\left(\tilde{Q}_2 - \tilde{Q}_3 - \lambda_2\sqrt{\tilde{H}_3}\right) \\
\dot{\tilde{Q}}_3 &= -\mu \tilde{Q}_3^2 + \frac{a_1}{L - z_{eq} + c_1 z_2 - z_2}(\tilde{H}_3 - u_2)
\end{aligned} \quad (15)$$

with

$$\sigma(z_2) = H_{1e} - \frac{\mu}{a_1}(z_{eq} - c_1 z_2) y_{1s}^2$$

and $c_2 = y_{1s} - Q_{2s}$.

Note that this model depends only on two unknown parameters (z_2, λ_2) and its state variables describe the transient behavior of the fluid in the pipeline with two simultaneous leaks for a given z_{eq}.

Using (10) and (8) one can obtain the intervals for the unknown parameters (z_2, λ_2) of the family z_{eq}

$$z_2 \in \left[0, \frac{z_{eq} - Um}{c_1}\right] \quad (16)$$

$$\lambda_2 \in [\lambda_{2_{min}}, \lambda_{2_{max}}] \quad (17)$$

with a threshold at the extremes Um and

$$\lambda_{2min} = \frac{(y_{2s} - Q_{2s})}{\sqrt{u_{2s} - \frac{\mu Um}{a_1} y_{2s}^2}} \quad (18)$$

$$\lambda_{2max} = \frac{(y_{2s} - Q_{2s})}{\sqrt{u_{2s} - \frac{\mu(L - z_{eq})}{a_1} y_{2s}^2}} \quad (19)$$

Using both models (4) and (15), one proposes to tackle the leaks diagnosis problem in two steps. First a residual generator is applied considering model (4), which detects the abnormal condition $y_{1s} - y_{2s} \neq 0$ and evaluates the parameter z_{eq}, and as second step an identification task for unknown parameters (z_2, λ_2) for a given Q_{2s}, using (15), must be solved. It means, after the detection of a leak, the algorithm must be switched from model (4) with four unknown parameters to the model (15) with only two unknown and the unknown parameters are identified using the latest model and re-processing the data off-line.

This novel idea to switch from one to other model is the key to solve the isolation problem with two leaks. It is to note, that this model is totally general for the effects of the leaks and diverse procedures can be used to estimate the unknown parameters.

To identify the positions and leaks flows, one can re-process the data of the transient response, since this process is achieved off-line. Other possibility could be the addition of an excitation in the physical system to exhibit the mode of the fluid in leak conditions during the identification procedure.

To study the sensitivity of the transient response of a family z_{eq} with respect to the pair (z_2, λ_2), the following quadratic integral criteria of the output $y\wp_i(z_{eq})$ is considered

$$J_{j,k}(z_{eq}) = \int_0^N (y\wp_{jk}(z_{eq}))^T W (y\wp_{jk}(z_{eq}))dt \qquad (20)$$

with $y\wp_{jk}(z_{eq}) = y\wp_j(z_{eq}) - y\wp_k(z_{eq})$ for a given z_{eq} and $j \neq k$, where $N > max\,(ts_k, ts_{\tilde{n}})$ with ts_k and $ts_{\tilde{n}}$ the settling time of the output for two members j and k of the family, respectively and W a symmetric weighting matrix.

The behavior of the error (20) for a fix given $\wp_k(z_{eq})$ and moving the set $\wp_k(z_{eq})$ can be done by simulation.

Taking the parameters of the water pilot plant of the National University [8] for $z_{eq} = 120m$ and leaks parameters $\wp_k(120) = (105.3, 1.38 \times 10^{-4}, 20, 4.35 \times 10^{-4})$, the integral (20) with $W = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ and a set of $\wp_j(z_{eq})$ with $j \neq k$ and constrains $z_2 \in [15, 26]$ and $\lambda_2 \in \left[4.28 \times 10^{-4}, 4.45 \times 10^{-4}\right]$ has been evaluated. Figs. 4 and 5 show its form and its level surfaces respectively and it can be seen that the form of the error is convex. Figures 6 and 7 show the error for other two families ($z_{eq} = 61m$ and $z_{eq} = 92m$).

The plots of the error assures the identificability of the parameters with various optimization algorithms. In [8] the error (20) for diverse families z_{eq} has been studied in detail and the study concluded in general that:

- the weighting matrix W affects strongly the form of the level surfaces; the downstream flow is less sensitive to parameter variations than the upstream flow.
- two very closed leaks ($z_2 \approx 0$) behave as one leak.
- the function generated by (20) with the constraints (16-19) is convex.

IV. IDENTIFICATION PROCESS

To estimate the parameters (λ_2, z_2), diverse pattern recognition processes in a recursive framework can be

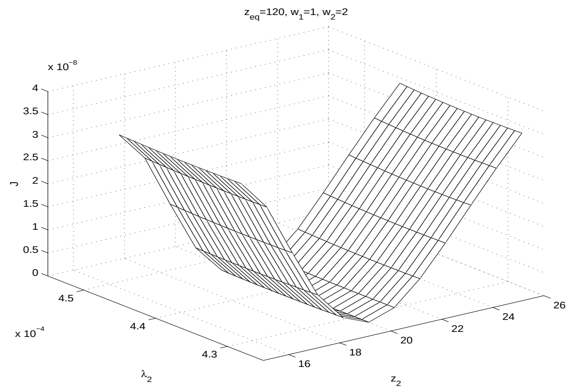

Fig. 4. Criterion $J_{i,k}(z_{eq})$ for $z_{eq} = 120m$ considering $z_2 = [15, 26]$ and $\lambda_2 = \left[4.28 \times 10^{-4}, 4.45 \times 10^{-4}\right]$

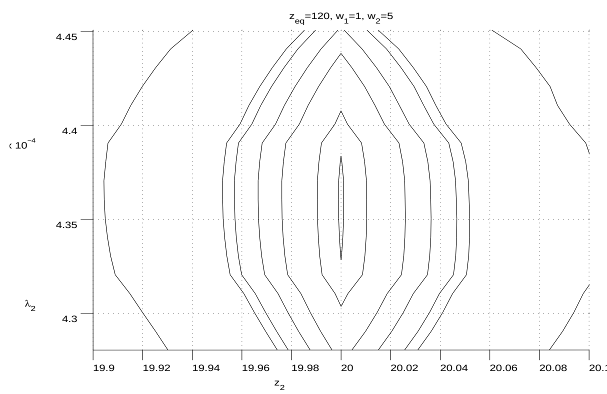

Fig. 5. Level Surfaces for $z_{eq} = 120m$

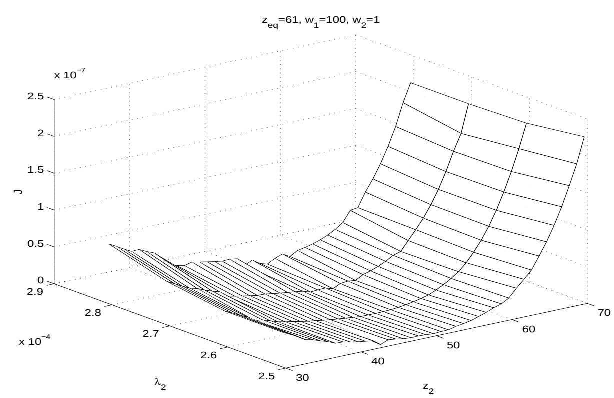

Fig. 6. Criterion $J_{i,k}(z_{eq})$ for $z_{eq} = 61m$ considering $z_2 = [30, 70]$ and $\lambda_2 = \left[2.50 \times 10^{-4}, 2.81 \times 10^{-4}\right]$

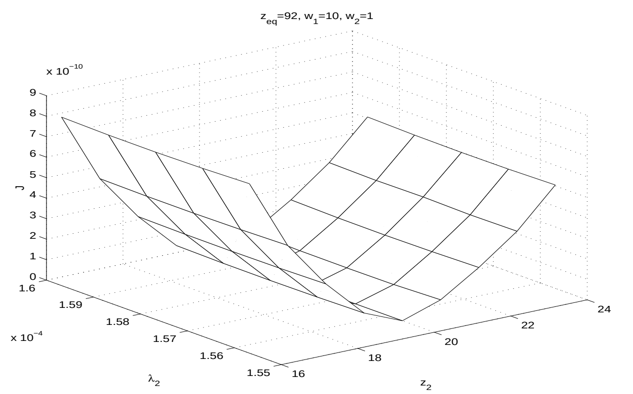

Fig. 7. Criterion $J_{i,k}(z_{eq})$ for $z_{eq} = 92m$ considering $z_2 = [16, 24]$ and $\lambda_2 = \left[1.55 \times 10^{-4}, 1.59 \times 10^{-4}\right]$

applied. Since the parameter can be identified only using data during the transient response of the output, one must re-use the data set if the settling time of the system response is shorter than the converge time of the estimator.

In particular here an Extended Kalman Filter of reduced order, including the parameters as states [9] is used. To design the Kalman filter a model of reduced order is obtained through a linear transformation and the unknown parameters z_2 and λ_2 are added as new state variables. From (15), one gets

$$\begin{aligned}
\dot{w}_1 &= \mu y_1^2 \left(c_1 z_2 - z_{eq}\right) - \frac{\mu}{4 z_2}\rho^2 + a_1 \left(u_1 - w_2\right) \\
\dot{w}_2 &= \frac{a_2}{z_2^2}\left(w_1 - z_{eq} y_1\right) + \frac{a_2}{z_2}\left(c_1 z_2 y_1 - w_3 - \lambda_2 \sqrt{w_2}\right) \\
\dot{w}_3 &= -\mu w_3^2 + \frac{a_1}{L - z_{eq} + c_1 z_2 - z_2}\left(w_2 - u_2\right) \\
\dot{z}_2 &= 0 \\
\dot{\lambda}_2 &= 0
\end{aligned}$$

with output $y_2 = w_3$ and $\rho = 2w_1 - 2z_{eq} y_1^2 + 2c_1 z_2 y_1$.

This model can be rewritten as

$$\begin{aligned}
\dot{w}_a &= f(w_a, u_a) \\
y_2 &= [0 \quad 0 \quad 1 \quad 0 \quad 0] w_a = C w_a
\end{aligned}$$

where $w_a = [w_1 \quad w_2 \quad w_3 \quad z_2 \quad \lambda_2]^T$ is the new state vector and $u_a = [u_1 \quad u_2 \quad y_1]^T$ the new input vector.

Following [9], the filter is given by

$$\begin{aligned}
\dot{\hat{w}}_a &= f(\hat{w}_a, u_a) + K(t) e(t) \\
\hat{y}_2 &= C \hat{w}_a
\end{aligned}$$

with the error signal $e(t) = y_2 - \hat{y}_2$ and $K(t)$ a time-varying matrix.

To calculate the matrix gain the Riccati differential matrix equation must be solved

$$\dot{P} = (A + \alpha I) P + P\left(A^T + \alpha I\right) - P C^T R^{-1} C P - Q \tag{21}$$

with a positive real number $\alpha > 0$ and positive definite matrix Q and $R > 0$ and

$$A(t) = \frac{\partial f}{\partial w_a}\left(w_a, u_a\right)|_{\hat{w}_a} \tag{22}$$

and the filter gain is given by

$$K(t) = P(t) C^T R^{-1} \tag{23}$$

For the water pilot plant given in [8] assuming two leaks with set

$$\begin{aligned}
\lambda_1 &= 6.12 \times 10^{-5} m^3/s \qquad & z_1 &= 39.98m \\
\lambda_2 &= 1.05 \times 10^{-4} m^3/s \qquad & z_2 &= 44.00m
\end{aligned}$$

one obtains a family of model with $z_{eq} = 66m$. Taking in (21) the values $\alpha = 0.001$, $R = 1$ and

$$Q = \begin{bmatrix} 0.01 & 0 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0 & 1 \times 10^7 & 0 \\ 0 & 0 & 0 & 0 & 0.01 \end{bmatrix}$$

with initial conditions $\hat{\lambda}_2(0) = 3.0534 \times 10^{-4} (m^3/s)$ and $\hat{z}_2(0) = 45(m)$ the EKF estimates the values $\hat{\lambda}_2 = 1.0773 \times 10^{-4}$ and $\hat{z}_2 = 44.14$.

The evolutions of the estimated variables during the identification process are given in Fig. 8. These plots are obtained by simulation. From the plots of the parameters estimations, one can see that the EKF generates a bias in the position of the leak; however considering the size of the pipeline ($132m$), an error of $0.15m$ can be neglected. From the output error plot one can see that the error becomes approximately zero before the parameters converge their values. This fact justifies the parameter estimation bias in z_2 and demands a faster EKF to eliminate the bias.

V. CONCLUSIONS

Taking into account the loss of identificability of the four unknown parameters which characterize the existence of two simultaneous leaks in a pipeline when it is assumed known only pressures and flows at the ends of the line in steady state, a family of models in term of a parameter z_{eq} is derived. This novel family for a given z_{eq} captures the dynamics effects of the indistinguishable set with only two known parameters and it has not been published before. It is shown that the unknown parameters of the family are identificable during the transient response of the upstream and downstream flow. The key to achieve a identificable parameters set is the parameter z_{eq}. The success of the model is shown locating two leaks of a water pilot plane using off-line an Extended Kalman Filter of reduced order with noisy data.

Thus, the open problem of detection and isolation in a pipeline for two leaks is solved. However, since the

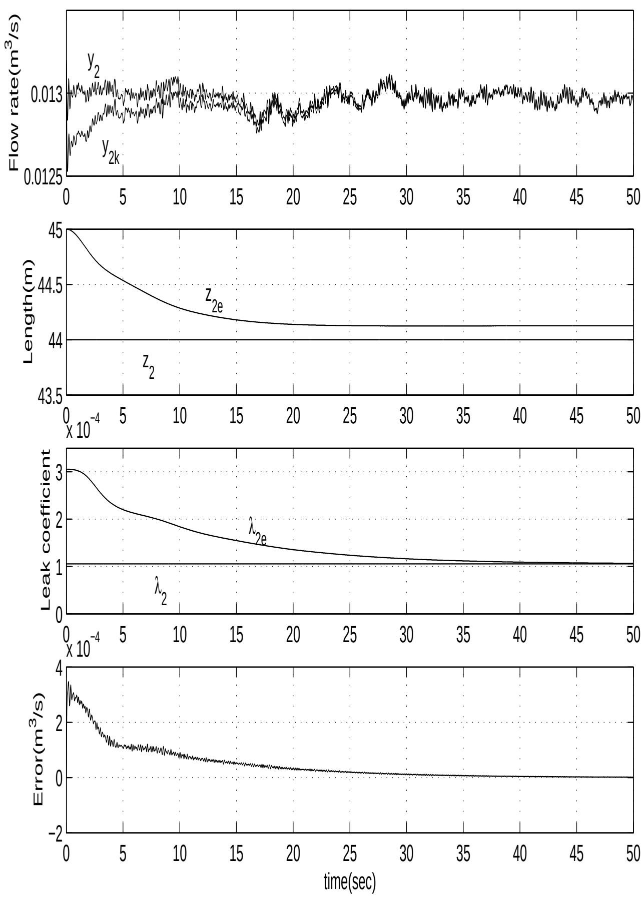

Fig. 8. Evolution of estimated parameters for $z_{eq} = 66m$

settling time of the response of the flows is limited, one must propose an efficient optimization algorithm to reduce the convergence time in general. Moreover, the extension of this procedure to a general case of more than two leaks demands extra measurements of the internal variables of the fluid.

VI. ACKNOWLEDGMENTS

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