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New Sufficient Conditions for Stability Analysis of Time Delay Systems using Dissipativity Theory

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***Abstract*— In this paper, we extend the concepts of dissipativity and exponential dissipativity to provide new sufficient conditions for guaranteeing asymptotic stability of a time delay dynamical system. Specifically, representing a time delay dynamical system as a negative feedback interconnection of a finite-dimensional linear dynamical system and an infinite-dimensional time delay operator, we show that the time delay operator is dissipative. As a special case of this result we show that the storage functional of the dissipative delay operator involves an integral term identical to the integral term appearing in standard Lyapunov-Krasovskii functionals. Finally, using stability of feedback interconnection results for dissipative systems, we develop new sufficient conditions for asymptotic stability of time delay dynamical systems. The overall approach provides an explicit framework for constructing Lyapunov-Krasovskii functionals as well as deriving new sufficient conditions for stability analysis of asymptotically stable time delay dynamical systems based on the dissipativity properties of the time delay operator.**

I. Introduction

Modern complex engineering systems involve a multitude of information and communication networks. A key physical limitation of such systems is that power transfers between interconnecting system components are not instantaneous and realistic models for capturing the dynamics of such systems should account for information in transit [1]. To accurately describe the evolution of these complex systems, it is necessary to include in any mathematical model of the system dynamics some information of the past system states. This leads to (infinite-dimensional) delay dynamical systems. Time-delay dynamical systems have been extensively studied in the literature (see [1–10] and the numerous references therein). Since time delay can severely degrade system performance and in many cases drive the system to instability, stability analysis of time delay dynamical systems remains a very important area of research [8–11]. A key method for analyzing stability of time delay dynamical systems is Lyapunov's second method as applied to functional differential equations. Specifically, stability analysis of a given linear time delay dynamical system is typically shown using a Lyapunov-Krasovskii functional [4], [11]. Standard Lyapunov-Krasovskii functionals involve a fixed quadratic function and an integral functional explicitly dependent on the system time delay. As in classical absolute stability theory [12], the fixed quadratic part of the Lyapunov-Krasovskii functional is associated with the stability of the forward delay-independent part of the retarded dynamical system. However, the system-theoretic foundation of the integral part of the Lyapunov-Krasovskii functional is less understood.

In this paper, we extend the notions of dissipativity [13] and exponential dissipativity [14] theory to derive new sufficient conditions for guaranteeing asymptotic stability of time delay dynamical systems. Specifically, we introduce the notion of *dynamic dissipativity*; namely, $(\Sigma, \hat{Q})$-dissipativity, where Σ is a dynamical system and $\hat{Q}$ is a symmetric matrix. By choosing a certain dynamical system Σ and a symmetric matrix $\hat{Q}$ it can be shown that a system $\mathcal{G}$ is $(\Sigma, \hat{Q})$-dissipative if and only if $\mathcal{G}$ is dissipative with respect to a quadratic supply rate. Thus, $(\Sigma, \hat{Q})$-disspaitivity provides a nontrivial extension of dissipativity theory with respect to a quadratic supply rate. Based on $(\Sigma, \hat{Q})$-dissipativity theory, we then provide a result on stability of negative feedback interconnection of $(\Sigma, \hat{Q})$-dissipative systems. Next, representing a time delay dynamical system as a negative feedback interconnection of a finite-dimensional linear dynamical system and an infinite-dimensional time delay operator, we show that the time delay operator is $(\Sigma_{\rm d}, \hat{Q}_{\rm d})$-dissipative. Furthermore, for a special choice of $\Sigma_{\rm d}$ and $\hat{Q}_{\rm d}$, we show that the storage functional of the time-delay operator involves an integral term which is identical to the integral term appearing in the Lyapunov-Krasovskii functional. Thus the overall approach provides an explicit framework for constructing Lyapunov-Krasovskii functionals as well as deriving new sufficient conditions for stability analysis of asymptotically stable time delay dynamical systems based on the dissipativity properties of the time delay operator.

II. Mathematical Preliminaries

In this section we introduce notation, several definitions, and some key results concerning dynamical systems that are necessary for developing the main results of this paper. Specifically, $\mathbb{R}$ denotes the reals and $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with Euclidean norm $\|\cdot\|$. Let $\mathcal{C}([a,b],\mathbb{R}^n)$ denote a Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. For a given real number $\tau \geq 0$ if $[a,b]=[-\tau,0]$ we let $\mathcal{C}=\mathcal{C}([-\tau,0],\mathbb{R}^n)$ and designate the norm of an element ϕ in $\mathcal{C}$ by $|||\phi||| = \sup_{\theta\in[-\tau,0]}\|\phi(\theta)\|$. If $\alpha,\beta\in\mathbb{R}$ and $x\in\mathcal{C}([\alpha-\tau,\alpha+\beta],\mathbb{R}^n)$, then for every $t\in[\alpha,\alpha+\beta]$, we let $x_t\in\mathcal{C}$ be defined by $x_t(\theta)=x(t+\theta)$, $\theta\in[-\tau,0]$. Furthermore, for $M\in\mathbb{R}^{m\times n}$, we write $M^{\rm T}$ to denote the transpose of M and $M\geq 0$ (resp., $M>0$) to denote the fact that the symmetric matrix M is nonnegative

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(resp., positive) definite. Let $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ denote a state space realization of a transfer function $G(s)$; that is, $G(s) = C(sI - A)^{-1}B + D$. The notation "$\overset{\min}{\sim}$" is used to denote a minimal realization. Finally, we write I_n to denote the $n \times n$ identity matrix and C^0 to denote continuous functions.

In this paper we represent dynamical systems $\mathcal{G}$ defined on the semi-infinite interval $[0, \infty)$ as a mapping between function spaces satisfying an appropriate set of axioms. For the following definition $\mathcal{U}$ is an input space and consists of bounded continuous U-valued functions on $[0, \infty)$. The set $U \subseteq \mathbb{R}^m$ contains the set of input values; that is, at any time t, $u(t) \in U$. The space $\mathcal{U}$ is assumed to be closed under the shift operator; that is, if $u \in \mathcal{U}$, then the function u_T defined by $u_T(t) = u(t+T)$ is contained in $\mathcal{U}$ for all $T \geq 0$. Furthermore, $\mathcal{Y}$ is an output space and consists of continuous Y-valued functions on $[0, \infty)$. The set $Y \subseteq \mathbb{R}^l$ contains the set of output values; that is, each value of $y(t) \in Y$, $t \geq 0$. The space $\mathcal{Y}$ is assumed to be closed under the shift operator; that is, if $y \in \mathcal{Y}$, then the function y_T defined by $y_T(t) = y(t+T)$ is contained in $\mathcal{Y}$ for all $T \geq 0$. Finally, $\mathcal{D}$ is a metric space with topology of uniform convergence and metric $\rho : \mathcal{D} \times \mathcal{D} \to [0, \infty)$. Hence, the notions of openness, convergence, continuity, and compactness that we use in the paper refer to the topology generated on $\mathcal{D}$ by the metric $\rho(\cdot, \cdot)$.

Definition 2.1 ([13]): A *stationary dynamical system* on $\mathcal{D}$ is the octuple $(\mathcal{D}, \mathcal{U}, U, \mathcal{Y}, Y, [0, \infty), s, q)$, where $s : [0, \infty) \times \mathcal{D} \times \mathcal{U} \to \mathcal{D}$ and $q : \mathcal{D} \times U \to Y$ are such that the following axioms hold:

- *i)* (Continuity): $s(\cdot, \cdot, u)$ is jointly continuous for all $u \in \mathcal{U}$.
- *ii)* (Consistency): $s(0, x_0, u) = x_0$ for all $x_0 \in \mathcal{D}$ and $u \in \mathcal{U}$.
- *iii)* (Determinism): $s(t, x_0, u_1) = s(t, x_0, u_2)$ for all $t \in [0, \infty)$, $x_0 \in \mathcal{D}$, and $u_1, u_2 \in \mathcal{U}$ satisfying $u_1(\tau) = u_2(\tau)$, $\tau \leq t$.
- *iv)* (Semi-group property): $s(\tau, s(t, x_0, u), u) = s(t + \tau, x_0, u)$ for all $x_0 \in \mathcal{D}$, $u \in \mathcal{U}$, and $\tau, t \in [0, \infty)$.
- *v)* (Read-out map): There exists $y \in \mathcal{Y}$ such that $y(t) = q(s(t, x_0, u), u(t))$ for all $x_0 \in \mathcal{D}$, $u \in \mathcal{U}$, and $t \geq 0$.

Henceforth, we denote the dynamical system $(\mathcal{D}, \mathcal{U}, U, \mathcal{Y}, Y, [0, \infty), s, q)$ by $\mathcal{G}$. Furthermore, we refer to $s(t, x_0, u)$, $t \geq 0$, as the *trajectory* or *state transition operator* of $\mathcal{G}$ corresponding to $x_0 \in \mathcal{D}$ and $u \in \mathcal{U}$. For a given trajectory $s(t, x_0, u)$, $t \geq 0$, we refer to $x_0 \in \mathcal{D}$ as the *initial condition* of $\mathcal{G}$. For the dynamical system $\mathcal{G}$ given by Definition 2.1, a function $r : U \times Y \to \mathbb{R}$ is called a *supply rate* [13] if it is locally integrable; that is, for all input-output pairs $u \in U$ and $y \in Y$ satisfying the dynamical system $\mathcal{G}$, $r(\cdot, \cdot)$ satisfies $\int_{t_1}^{t_2} |r(u(s), y(s))| \mathrm{d}s < \infty$, $t_1, t_2 \geq 0$.

Definition 2.2 ([13], [14]): A dynamical system $\mathcal{G}$ is *exponentially dissipative with respect to the supply rate* $r(u, y)$ if there exists a C^0 nonnegative-definite function $V_\mathrm{s} : \mathcal{D} \to \mathbb{R}$, called a *storage function*, and a scalar $\varepsilon > 0$ such that the *dissipation inequality*

$$e^{\varepsilon t} V_\mathrm{s}(x(t)) \leq e^{\varepsilon t_1} V_\mathrm{s}(x(t_1)) + \int_{t_1}^{t} e^{\varepsilon s} r(u(s), y(s)) \mathrm{d}s \quad (1)$$

is satisfied for all $t_1, t \geq 0$ and where $x(t) = s(t, x_0, u(t))$, $t \geq t_1$, with $x_0 \in \mathcal{D}$ and $u(t) \in U$. A dynamical system $\mathcal{G}$ is *dissipative with respect to the supply rate* $r(u, y)$ if there exists a C^0 nonnegative-definite function $V_\mathrm{s} : \mathcal{D} \to \mathbb{R}$ such that (1) is satisfied with $\varepsilon = 0$.

Next, consider a dynamical system Σ given by the octuple $(\hat{\mathcal{D}}, \mathcal{W}, U \times Y, \mathcal{Z}, Z, [0, \infty), \hat{s}, \hat{q})$, where $Z \subseteq \mathbb{R}^p$, $\mathcal{Z}$ is an output space which consists of continuous Z-valued functions on $[0, \infty)$, and consider the cascade interconnection of $\mathcal{G}$ and Σ as shown in Figure 1. We denote this interconnected dynamical system $(\mathcal{D} \times \hat{\mathcal{D}}, \mathcal{U}, U, \mathcal{Z}, Z, [0, \infty), [s^\mathrm{T}, \hat{s}^\mathrm{T}]^\mathrm{T}, \hat{q})$ by $\tilde{\mathcal{G}}$. For the following definition, let $\hat{Q} \in \mathbb{R}^{p \times p}$ and $\hat{Q} = \hat{Q}^\mathrm{T}$.

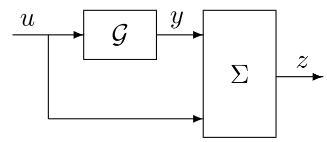

Fig. 1. Interconnection of $\mathcal{G}$ and Σ

Definition 2.3: A dynamical system $\mathcal{G}$ is $(\Sigma, \hat{Q})$-*exponentially dissipative* if there exists a C^0 nonnegative-definite function $\hat{V}_\mathrm{s} : \mathcal{D} \times \hat{\mathcal{D}} \to \mathbb{R}$, called a $(\Sigma, \hat{Q})$-*storage function* and a scalar $\varepsilon > 0$, such that the $(\Sigma, \hat{Q})$-*dissipation inequality*

$$e^{\varepsilon t} \hat{V}_\mathrm{s}(x(t), \hat{x}(t)) \leq e^{\varepsilon t} \hat{V}_\mathrm{s}(x(t_1), \hat{x}(t_1)) + \int_{t_1}^{t} e^{\varepsilon s} z^\mathrm{T}(s) \hat{Q} z(s) \mathrm{d}s \quad (2)$$

is satisfied for all $t, t_1 \geq 0$ and where $x(t) = s(t, x_0, u(t))$, $\hat{x}(t) = \hat{s}(t, \hat{x}_0, u(t), y(t))$, $t \geq t_1$, with $x_0 \in \mathcal{D}$, $\hat{x}_0 \in \hat{\mathcal{D}}$, $\hat{x}_0 = 0$, $u(t) \in U$, and $y(t) = q(x(t), u(t))$. A dynamical system $\mathcal{G}$ is $(\Sigma, \hat{Q})$-*dissipative* if there exists a C^0 nonnegative-definite function $\hat{V}_\mathrm{s} : \mathcal{D} \times \hat{\mathcal{D}} \to \mathbb{R}$ such that (2) is satisfied with $\varepsilon = 0$.

Remark 2.1: If $\mathcal{G}$ is $(\Sigma, \hat{Q})$-dissipative, where Σ is a linear dynamical system given by the transfer function $\hat{G}(s)$, then

$$\int_{-\infty}^{\infty} \left[\begin{array}{c} U(j\omega) \\ Y(j\omega) \end{array}\right]^\star \hat{G}^\star(j\omega) \hat{Q} \hat{G}(j\omega) \left[\begin{array}{c} U(j\omega) \\ Y(j\omega) \end{array}\right] \mathrm{d}\omega \geq 0, \quad (3)$$

where $U(s)$ and $Y(s)$, $s \in \mathbb{C}$, are the Laplace transforms of $u(t)$ and $y(t)$, respectively. Hence, $(\Sigma, \hat{Q})$-dissipativity is a time-domain analog to Integral Quadratic Constraints (IQCs) [15].

Remark 2.2: Let $p = l + m$ and let the dynamical system Σ be such that $z = q(\hat{x}, u, y) = [u^\mathrm{T} \; y^\mathrm{T}]^\mathrm{T}$. Furthermore, let $\hat{Q} = \left[\begin{array}{cc} R & S^\mathrm{T} \\ S & Q \end{array}\right]$, where $Q = Q^\mathrm{T} \in \mathbb{R}^{l \times l}$, $S \in \mathbb{R}^{l \times m}$, and $R = R^\mathrm{T} \in \mathbb{R}^{m \times m}$. In this case, $\mathcal{G}$ is $(\Sigma, \hat{Q})$-dissipative if and only if $\mathcal{G}$ is dissipative with respect to the quadratic supply rate $r(u, y) = y^\mathrm{T} Q y + 2 y^\mathrm{T} S u + u^\mathrm{T} R u$. Hence, $(\Sigma, \hat{Q})$-dissipativity provides a dynamic extension of dissipativity notions with respect to a quadratic supply rate.

The following result provides a sufficient condition for $(\Sigma, \hat{Q})$-dissipativity of $\mathcal{G}$ in the case where $\mathcal{G}$ and Σ are linear dynamical systems. Specifically, let $\mathcal{G}$ and Σ be given by transfer functions $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $\hat{G}(s) \sim$

$\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$, respectively, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, $D \in \mathbb{R}^{l\times m}$, $\hat{A} \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $\hat{B} \in \mathbb{R}^{\hat{n}\times(l+m)}$, $\hat{C} \in \mathbb{R}^{p\times\hat{n}}$ and $\hat{D} \in \mathbb{R}^{p\times(l+m)}$. In this case, the interconnection of $\mathcal{G}$ and Σ as shown in Figure 1 is given by the transfer function $\tilde{G}(s) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$, where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ \hat{B}_y C & \hat{A} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B \\ \hat{B}_y D + \hat{B}_u \end{bmatrix}, \quad (4)$$

$$\tilde{C} = [\hat{D}_y C \;\; \hat{C}], \qquad \tilde{D} = \hat{D}_u + \hat{D}_y D, \quad (5)$$

where $\hat{B}_u \in \mathbb{R}^{\hat{n}\times m}$, $\hat{B}_y \in \mathbb{R}^{\hat{n}\times l}$, $\hat{D}_u \in \mathbb{R}^{p\times m}$, and $\hat{D}_y \in \mathbb{R}^{p\times l}$ are such that $\hat{B} = [\hat{B}_u \;\hat{B}_y]$ and $\hat{D} = [\hat{D}_u \;\hat{D}_y]$.

Proposition 2.1: Consider the dynamical system $\mathcal{G}$ given by the transfer function $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, let $\hat{Q} \in \mathbb{R}^{p\times p}$, $\hat{Q} = \hat{Q}^{\mathrm{T}}$, and let Σ be a linear dynamical system given by the transfer function $\hat{G}(s) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$. Then, $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative if and only if there exists a nonnegative-definite matrix $\tilde{P} \in \mathbb{R}^{(n+\hat{n})\times(n+\hat{n})}$ and a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \varepsilon\tilde{P} & \tilde{P}\tilde{B} \\ \tilde{B}^{\mathrm{T}}\tilde{P} & 0 \end{bmatrix} \leq \begin{bmatrix} \tilde{C}^{\mathrm{T}} \\ \tilde{D}^{\mathrm{T}} \end{bmatrix} \hat{Q} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix}. \quad (6)$$

Furthermore, $\mathcal{G}$ is $(\Sigma, \hat{Q})$-dissipative if and only if there exists a nonnegative-definite matrix $\tilde{P} \in \mathbb{R}^{(n+\hat{n})\times(n+\hat{n})}$ such that (6) holds with $\varepsilon = 0$.

Proof. The proof is a direct consequence of the generalized Kalman-Yakubovich-Popov lemma [14]. □

Remark 2.3: Note that it follows from Propositon 2.1 that if $\tilde{G}(s) \overset{\min}{\sim} \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$, then $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative if and only if there exists a positive-definite matrix $\tilde{P}$ such that (6) holds.

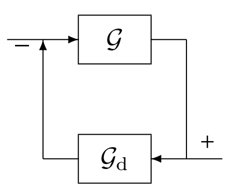

Fig. 2. Feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$

Next, we present a result on stability of feedback interconnection of dissipative dynamical systems. Specifically, consider the negative feedback interconnection of dynamical system $\mathcal{G}$ with a feedback system $\mathcal{G}_{\mathrm{d}}$ given by the octuple $(\mathcal{D}_{\mathrm{d}}, \mathcal{U}_{\mathrm{d}}, U_{\mathrm{d}}, \mathcal{Y}_{\mathrm{d}}, Y_{\mathrm{d}}, [0,\infty), s_{\mathrm{d}}, q_{\mathrm{d}})$. Note that with the feedback interconnection given in Figure 2, $u = -y_{\mathrm{d}}$ and $u_{\mathrm{d}} = y$. Hence, $U = Y_{\mathrm{d}}$ and $Y = U_{\mathrm{d}}$. Furthermore, consider a dynamical system Σ_{d} given by the octuple $(\hat{\mathcal{D}}, \mathcal{W}_{\mathrm{d}}, U_{\mathrm{d}} \times Y_{\mathrm{d}},\ Z_{\mathrm{d}}, Z, [0,\infty), \hat{s}_{\mathrm{d}}, \hat{q}_{\mathrm{d}})$, where $\hat{s}_{\mathrm{d}}(t, \hat{x}, u_{\mathrm{d}}, y_{\mathrm{d}}) = \hat{s}(t, \hat{x}_0, -y_{\mathrm{d}}, u_{\mathrm{d}})$ and $\hat{q}_{\mathrm{d}}(\hat{x}, u_{\mathrm{d}}, y_{\mathrm{d}}) = \hat{q}(\hat{x}, -y_{\mathrm{d}}, u_{\mathrm{d}})$. In addition, consider the interconnected dynamical system $\tilde{\mathcal{G}}_{\mathrm{d}}$ given by the octuple $(\mathcal{D}_{\mathrm{d}} \times \hat{\mathcal{D}}, \mathcal{U}_{\mathrm{d}}, U_{\mathrm{d}}, \mathcal{Z}_{\mathrm{d}}, Z,\ [0,\infty),\ [s_{\mathrm{d}}^{\mathrm{T}} \;\; \hat{s}_{\mathrm{d}}^{\mathrm{T}}], \hat{q}_{\mathrm{d}})$ (see Figure 3). The following definition is needed for the statement of the next result.

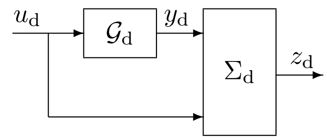

Fig. 3. Interconnection of $\mathcal{G}_{\mathrm{d}}$ and Σ_{d}

Definition 2.4: A dynamical system $\mathcal{G}$ with input-output pair (u, y) is *zero-state observable* if $u(t) \equiv 0$ and $y(t) \equiv 0$ implies $s(t, x_0, u) \equiv 0$.

For the statement of the next result let $\|\!| \cdot \|\!|_\sigma$ and $\|\!| \cdot \|\!|_\mu$ denote operator norms on $\mathcal{D}$ and $\mathcal{D}_{\mathrm{d}}$, respectively, and let $\gamma^+(x_0, x_{\mathrm{d}0}) = \cup_{t\geq 0}\{(s(t, x_0, u), s_{\mathrm{d}}(t, x_{\mathrm{d}0}, u_{\mathrm{d}}))\}$, with $u = -y_{\mathrm{d}}$ and $u_{\mathrm{d}} = y$, denote the positive orbit of the feedback system $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$. Furthermore, recall that $\gamma^+(x_0, x_{\mathrm{d}0})$ is *precompact* if $\gamma^+(x_0, x_{\mathrm{d}0})$ can be enclosed in the union of a finite number of ε-balls around elements of $\gamma^+(x_0, x_{\mathrm{d}0})$.

Theorem 2.1: Let $\hat{Q}, \hat{Q}_{\mathrm{d}} \in \mathbb{R}^{p\times p}$ be such that $\hat{Q} = \hat{Q}^{\mathrm{T}}$ and $\hat{Q}_{\mathrm{d}} = \hat{Q}_{\mathrm{d}}^{\mathrm{T}}$. Consider the feedback system consisting of the stationary dynamical systems $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$ with input-output pairs (u, y) and $(u_{\mathrm{d}}, y_{\mathrm{d}})$, respectively, and with $u_{\mathrm{d}} = y$ and $u = -y_{\mathrm{d}}$. Assume that $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$ are $(\Sigma, \hat{Q})$-dissipative and $(\Sigma_{\mathrm{d}}, \hat{Q}_{\mathrm{d}})$-dissipative with C^0 storage functions $V_{\mathrm{s}} : \mathcal{D} \times \hat{\mathcal{D}} \to \mathbb{R}$ and $V_{\mathrm{sd}} : \mathcal{D}_{\mathrm{d}} \times \hat{\mathcal{D}}_{\mathrm{d}} \to \mathbb{R}$, respectively, such that $V_{\mathrm{s}}(0,0) = 0$, $V_{\mathrm{sd}}(0,0) = 0$, and

$$\alpha(\|\!|x\|\!|_\sigma) \leq V_{\mathrm{s}}(x, \hat{x}),\ (x, \hat{x}) \in \mathcal{D} \times \hat{\mathcal{D}}, \quad (7)$$

$$\alpha_{\mathrm{d}}(\|\!|x_{\mathrm{d}}\|\!|_\mu) \leq V_{\mathrm{sd}}(x_{\mathrm{d}}, \hat{x}_{\mathrm{d}}),\ (x_{\mathrm{d}}, \hat{x}_{\mathrm{d}}) \in \mathcal{D}_{\mathrm{d}} \times \hat{\mathcal{D}}_{\mathrm{d}}, \quad (8)$$

where $\alpha, \alpha_{\mathrm{d}} : [0,\infty) \to [0,\infty)$ are class $\mathcal{K}_\infty$ functions. Furthermore, assume that for each initial condition $(x_0, x_{\mathrm{d}0}) \in \mathcal{D} \times \mathcal{D}_{\mathrm{d}}$, the positive orbit $\gamma^+(x_0, x_{\mathrm{d}0})$ of the feedback system $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$ is precompact. Finally, assume there exists a scalar $\sigma > 0$ such that $\hat{Q} + \sigma\hat{Q}_{\mathrm{d}} \leq 0$. Then the following statements hold:

i) The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$ is Lyapunov stable.

ii) If $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$ is Lyapunov stable and for every $x(0) \in \mathcal{D}$, $\|\!|x(t)\|\!|_\sigma \to 0$ as $t \to \infty$.

Proof. The proof follows from standard Lyapunov theory and invariant set arguments as applied to infinite-dimensional dynamical systems [11], [16]. □

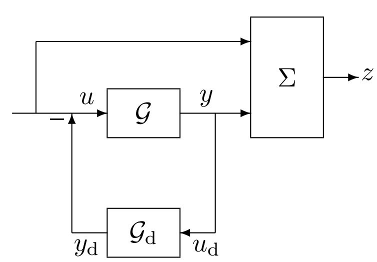

Fig. 4. Interconnection of $\mathcal{G}$, $\mathcal{G}_{\mathrm{d}}$, and Σ

Remark 2.4: Note that (7) and (8) are only sufficient conditions needed to prove Lyapunov stability for the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$. In the case of stability analysis of time-delay systems, (7) and (8) may be replaced by a weaker condition. See Remark 3.2 below.

Remark 2.5: In the case where Σ and $\Sigma_{\rm d}$ are such that $z = [u^{\rm T}\ y^{\rm T}]^{\rm T}$ and $z_{\rm d} = [-y_{\rm d}^{\rm T}\ u_{\rm d}^{\rm T}]^{\rm T}$, Theorem 2.1 specializes to Theorem 5.2 of [14].

III. Stability Theory for Time-Delay Dynamical Systems using Dissipativity Theory

In this section we consider linear time delay dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = Ax(t) + A_{\rm d}x(t-\tau), \quad x(\theta) = \phi(\theta), \quad -\tau \le \theta \le 0, \quad t \ge 0, \tag{9}$$

where $x(t) \in \mathbb{R}^n$, $t \ge 0$, $A \in \mathbb{R}^{n\times n}$, $A_{\rm d} \in \mathbb{R}^{n\times n}$, $\tau \ge 0$, and $\phi(\cdot) \in \mathcal{C} = \mathcal{C}([-\tau,0],\mathbb{R}^n)$ is a continuous vector valued function specifying the initial state of the system. Note that the state of (9) at time t is the *piece of trajectories* x between $t-\tau$ and t, or, equivalently, the *element* x_t in the space of continuous functions defined on the interval $[-\tau,0]$ and taking values in $\mathbb{R}^n$; that is, $x_t \in \mathcal{C}([-\tau,0],\mathbb{R}^n)$. Hence, $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau,0]$. Furthermore, since for a given time t the piece of the trajectories x_t is defined on $[-\tau,0]$, the uniform norm $|||x_t||| = \sup_{\theta\in[-\tau,0]} \|x(t+\theta)\|$ is used for the definitions of Lyapunov and asymptotic stability of (9). For further details see [4], [11].

Next, we rewrite (9) as a feedback system so that

$$\begin{aligned} \dot{x}(t) &= Ax(t) - A_{\rm d}u(t),\ x(0) = \phi(0),\ t \ge 0, && (10)\\ y(t) &= x(t), && (11)\\ y_{\rm d}(t) &= \mathcal{G}_{\rm d}(u_{\rm d}(t)), && (12) \end{aligned}$$

where $u(t) = y_{\rm d}(t)$, $u_{\rm d}(t) = y(t)$, and $\mathcal{G}_{\rm d} : \mathcal{C}([-\tau,\infty),\mathbb{R}^n) \to \mathcal{C}([0,\infty),\mathbb{R}^n)$ denotes a delay operator defined by $\mathcal{G}_{\rm d}(u_{\rm d}(t)) \triangleq u_{\rm d}(t-\tau)$. Note that (10)–(12) is a negative feedback interconnection of a linear finite-dimensional system $\mathcal{G}$ with transfer function $G(s) \sim \left[\begin{array}{c|c} A & -A_{\rm d} \\ \hline I_n & 0\end{array}\right]$ and the infinite-dimensional delay operator $\mathcal{G}_{\rm d}$. Hence, stability of (9) is equivalent to stability of the negative feedback interconnection of $G(s)$ and $\mathcal{G}_{\rm d}$. Next, we present a key result that shows that the delay operator $\mathcal{G}_{\rm d}$ is dissipative with respect to a quadratic supply rate. First, however, we show that the *input-output* operator $\mathcal{G}_{\rm d}$ can be characterized as a stationary dynamical system on $\mathcal{C}$. Specifically, let $\mathcal{U}_{\rm d} = \mathcal{C}([-\tau,\infty),\mathbb{R}^n)$, $\mathcal{Y}_{\rm d} = \mathcal{C}([0,\infty),\mathbb{R}^n)$, and $U_{\rm d} = Y_{\rm d} = \mathbb{R}^n$. Now, for every $\phi \in \mathcal{C}$, define $s_\theta : [0,\infty)\times\mathcal{C}\times\mathcal{U}_{\rm d} \to \mathcal{C}$ by

$$s_\theta(t,\phi,u_{\rm d}) = u_{\rm d}(t+\theta), \quad \theta \in [-\tau,0], \quad t \ge 0, \tag{13}$$

where $u_{\rm d}(\theta) = \phi(\theta)$, $\theta \in [-\tau,0]$. Finally, define $q_{\rm d} : \mathcal{C}\times U_{\rm d} \to Y_{\rm d}$ by

$$q_{\rm d}(s_\theta(t,\phi,u_{\rm d}),u_{\rm d}(t)) = s_{-\tau}(t,\phi,u_{\rm d}) = u_{\rm d}(t-\tau) = \mathcal{G}_{\rm d}(u_{\rm d}(t)). \tag{14}$$

Note that the octuple $(\mathcal{C},\mathcal{U}_{\rm d},U_{\rm d},\mathcal{Y}_{\rm d},Y_{\rm d},[0,\infty),s_\theta,q_{\rm d})$ satisfies Axioms i)–v) of Definition 2.1 which implies that the octuple $(\mathcal{C},\mathcal{U}_{\rm d},U_{\rm d},\mathcal{Y}_{\rm d},Y_{\rm d},[0,\infty),s_\theta,q_{\rm d})$ is a stationary dynamical system on $\mathcal{C}$. For notational convenience we refer to this dynamical system as $\mathcal{G}_{\rm d}$.

To show that $\mathcal{G}_{\rm d}$ is $(\Sigma_{\rm d},\hat{Q}_{\rm d})$-dissipative, let Σ denote a linear dynamical system given by the octuple $(\hat{\mathcal{D}},\mathcal{W},\mathbb{R}^n\times\mathbb{R}^n,\mathcal{Z},\mathbb{R}^{2\hat{p}},[0,\infty),\hat{s},\hat{q})$, where $\hat{\mathcal{D}} \subset \mathbb{R}^{2\hat{n}}$ and with transfer function $\hat{G}(s) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D}\end{array}\right]$, where

$$\begin{aligned} \hat{A} &= \text{block–diag}[A_1,A_1],\ \hat{B} = \text{block–diag}[B_1,B_1],\\ \hat{C} &= \text{block–diag}[C_1,C_1],\ \hat{D} = I_{2n} \end{aligned} \tag{15}$$

and where $A_1 \in \mathbb{R}^{\hat{n}\times\hat{n}}$ is Hurwitz, $B_1 \in \mathbb{R}^{\hat{n}\times n}$, and $C_1 \in \mathbb{R}^{\hat{p}\times\hat{n}}$. In this case, the dynamical system $\Sigma_{\rm d}$ is given by the transfer function $\hat{G}_{\rm d}(s) \sim \left[\begin{array}{c|c} \hat{A}_{\rm d} & \hat{B}_{\rm d} \\ \hline \hat{C}_{\rm d} & \hat{D}_{\rm d}\end{array}\right]$, where

$$\begin{aligned} \hat{A}_{\rm d} = \hat{A}, \quad \hat{B}_{\rm d} &= \begin{bmatrix} 0 & -B_1 \\ B_1 & 0\end{bmatrix},\\ \hat{C}_{\rm d} = \hat{C}, \quad \hat{D}_{\rm d} &= \begin{bmatrix} 0 & -I_n \\ I_n & 0\end{bmatrix}. \end{aligned} \tag{16}$$

Hence, the state space representation of the interconnection shown in Figure 3 is given by

$$\begin{aligned} y_{\rm d}(t) &= \mathcal{G}_{\rm d}(u_{\rm d}(t)), \quad s_\theta(0,\phi,u_{\rm d}) = \phi(\theta),\\ &\quad \theta \in [-\tau,0],\ t \ge 0, && (17)\\ \dot{x}_{{\rm d}_1}(t) &= A_1x_{{\rm d}_1}(t) - B_1y_{\rm d}(t), \quad x_{{\rm d}_1}(0) = 0, && (18)\\ \dot{x}_{{\rm d}_2}(t) &= A_1x_{{\rm d}_2}(t) + B_1u_{\rm d}(t), \quad x_{{\rm d}_2}(0) = 0, && (19)\\ \hat{z}_{{\rm d}_1}(t) &= C_1x_{{\rm d}_1}(t) - D_1y_{\rm d}(t), && (20)\\ \hat{z}_{{\rm d}_2}(t) &= C_1x_{{\rm d}_2}(t) + D_1u_{\rm d}(t). && (21) \end{aligned}$$

Lemma 3.1: Let $\hat{Q}_{\rm d} = \text{block-diag}[-Q,Q]$, where $Q \in \mathbb{R}^{\hat{p}\times\hat{p}}$. If $\phi(\theta) = 0$, $\theta \in [-\tau,0]$, then for every $u_{\rm d}(\cdot) \in \mathcal{U}_{\rm d}$,

$$\int_0^T \hat{z}_{\rm d}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d}(t){\rm d}t = \int_\theta^T \hat{z}_{{\rm d}_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{{\rm d}_2}(t){\rm d}t \ge 0,\ T > 0, \tag{22}$$

where $\theta = 0$, $T \in [0,\tau]$, and $\theta = T-\tau$, $T > \tau$.

Proof. Note that

$$\begin{aligned} x_{{\rm d}_1}(t) &= -\int_0^t e^{A_1(t-s)}B_1y_{\rm d}(s){\rm d}s, \quad t \ge 0,\\ \text{and}\quad x_{{\rm d}_2}(t) &= \int_0^t e^{A_1(t-s)}B_1u_{\rm d}(s){\rm d}s, \quad t \ge 0. \end{aligned}$$

Since $y_{\rm d}(t) = u_{\rm d}(t-\tau)$, $t \ge 0$ and $u_{\rm d}(\theta) = \phi(\theta) = 0$, $\theta \in [-\tau,0]$, it follows that $x_{{\rm d}_1}(t) = 0$, $t \in [0,\tau]$, and for all $t \ge \tau$,

$$x_{{\rm d}_1}(t) = -\int_\tau^t e^{A_1(t-s)}B_1u_{\rm d}(s-\tau){\rm d}s = -x_{{\rm d}_2}(t-\tau).$$

Hence, $\hat{z}_{{\rm d}_1}(t) = 0$, $t \in [0,\tau]$, and $\hat{z}_{{\rm d}_1}(t) = -\hat{z}_{{\rm d}_2}(t-\tau)$, $t > \tau$, which implies that

$$\begin{aligned} \int_0^T \hat{z}_{\rm d}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d}(t){\rm d}t &= \int_0^T [\hat{z}_{{\rm d}_2}^{\rm T}(t)Q\hat{z}_{{\rm d}_2}(t) - \hat{z}_{{\rm d}_1}^{\rm T}(t)Q\hat{z}_{{\rm d}_1}(t)]{\rm d}t\\ &= \int_{T-\tau}^T \hat{z}_{{\rm d}_2}^{\rm T}(t)Q\hat{z}_{{\rm d}_2}(t){\rm d}t \ge 0, \quad T \ge \tau. \end{aligned}$$

The case where $T \in [0,\tau]$ follows in a similar manner. □

Theorem 3.1: Consider the dynamical system $\mathcal{G}_{\rm d}$ defined by the octuple $(\mathcal{C},\mathcal{U}_{\rm d},U_{\rm d},\mathcal{Y}_{\rm d},Y_{\rm d},[0,\infty),s_\theta,q_{\rm d})$, where s_θ and $q_{\rm d}$ are given by (13) and (14), respectively. Next, let $\Sigma_{\rm d}$ be a linear dynamical system with transfer function

$\hat{G}_{\rm d}(s) \sim \left[\begin{array}{c|c} \hat{A}_{\rm d} & \hat{B}_{\rm d} \\ \hline \hat{C}_{\rm d} & \hat{D}_{\rm d} \end{array}\right]$, where $\hat{A}_{\rm d}$, $\hat{B}_{\rm d}$, $\hat{C}_{\rm d}$ and $\hat{D}_{\rm d}$ are given by (16), and let $\hat{Q}_{\rm d} = $ block-diag$[-Q, Q]$, where $Q \in \mathbb{R}^{\hat{p}\times\hat{p}}$, $Q > 0$. Then, $\mathcal{G}_{\rm d}$ is $(\Sigma_{\rm d}, \hat{Q}_{\rm d})$-dissipative. Furthermore,

$$V_{\rm sd}(\psi, \hat{x}_{\rm d_1}, \hat{x}_{\rm d_2}) = -\inf_{u_{\rm d}(\cdot)\in\mathcal{U}_{\rm d}, T\geq 0} \int_0^T \hat{z}_{\rm d}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d}(t){\rm d}t \tag{23}$$

is a $(\Sigma_{\rm d}, \hat{Q}_{\rm d})$-storage function for $\mathcal{G}_{\rm d}$ where the infimum in (23) is performed over all trajectories of $\tilde{\mathcal{G}}_{\rm d}$ with initial conditions $\phi(\cdot) = \psi(\cdot)$, $x_{\rm d_1}(0) = \hat{x}_{\rm d_1}$, and $x_{\rm d_2}(0) = \hat{x}_{\rm d_2}$.

Proof. It follows from (23) that

$$\begin{aligned} V_{\rm sd}(\psi, \hat{x}_{\rm d_1}, \hat{x}_{\rm d_2}) &= -\inf_{u_{\rm d}(\cdot)\in\mathcal{U}_{\rm d}, T\geq 0} \int_0^T \hat{z}_{\rm d}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d}(t){\rm d}t \\ &= \sup_{u_{\rm d}(\cdot)\in\mathcal{U}_{\rm d}, T\geq 0} \int_0^T [\hat{z}_{\rm d_1}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_1}(t) - \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t)]{\rm d}t. \end{aligned} \tag{24}$$

Hence, $V_{\rm sd}(\psi, \hat{x}_{\rm d_1}, \hat{x}_{\rm d_2}) \geq 0$, $\psi(\cdot) \in \mathcal{C}$, $\hat{x}_{\rm d_1}, \hat{x}_{\rm d_2} \in \mathbb{R}^n$. If $\psi(\theta) \equiv 0$, $\theta \in [-\tau, 0]$, $\hat{x}_{\rm d_1} = 0$, $\hat{x}_{\rm d_2} = 0$, then it follows from Lemma 3.1 that

$$\int_0^T \hat{z}_{\rm d}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d}(t){\rm d}t = \int_0^T \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t, \quad T \geq 0,$$

which implies that

$$V_{\rm sd}(0,0,0) = \sup_{u_{\rm d}(\cdot)\in\mathcal{U}_{\rm d}, T\geq 0} -\int_0^T \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t \leq 0.$$

Hence, since $V_{\rm sd}(0,0,0) \geq 0$, $V_{\rm sd}(0,0,0) = 0$. Next, note that for every $u_{\rm d}(t)$, $t \in [t_1, t_{\rm f}]$, and $T \in [t_1, t_{\rm f}]$,

$$\begin{aligned} -V_{\rm sd}(s_\theta(t_1,\psi,u_{\rm d}), x_{\rm d_1}(t_1), x_{\rm d_2}(t_2)) &\leq \int_{t_1}^{t_{\rm f}} \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t \\ &= \int_{t_1}^{T} \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t + \int_{T}^{t_{\rm f}} \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t. \end{aligned}$$

Hence,

$$\begin{aligned} -V_{\rm sd}(s_\theta(t_1,\psi,u_{\rm d}), x_{\rm d_1}(t_1), x_{\rm d_2}(t_1)) - \int_{t_1}^{T} \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t \leq \int_{T}^{t_{\rm f}} \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t, \end{aligned}$$

which implies that

$$\begin{aligned} -V_{\rm sd}(s_\theta(t_1,\psi,u_{\rm d}), x_{\rm d_1}(t_1), x_{\rm d_2}(t_2)) - \int_{t_1}^{T} \hat{z}_{\rm d_2}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d_2}(t){\rm d}t &\leq \inf_{u_{\rm d}(\cdot)\in\mathcal{U}_{\rm d}, t_{\rm f}\geq T} \int_{T}^{t_{\rm f}} \hat{z}_{\rm d}^{\rm T}(t)\hat{Q}_{\rm d}\hat{z}_{\rm d}(t){\rm d}t \\ &= -V_{\rm sd}(s_\theta(T,\psi,u_{\rm d}), x_{\rm d_1}(T), x_{\rm d_2}(T)), \end{aligned}$$

establishing the $(\Sigma_{\rm d}, \hat{Q}_{\rm d})$-dissipativity of $\mathcal{G}_{\rm d}$. □

Remark 3.1: In the case where $A_1 = 0$, $B_1 = 0$, and $C_1 = 0$, it can be shown that

$$V_{\rm sd}(\psi, x_{\rm d1}, x_{\rm d2}) = V_{\rm sd}(\psi) = \int_{-\tau}^{0} \psi^{\rm T}(\theta)Q\psi(\theta){\rm d}\theta. \tag{25}$$

Next, using Theorem 3.1, we present a sufficient condition on $G(s)$ that guarantees asymptotic stability of the negative feedback interconnection of the time delay dynamical system given by (9). For the following result we assume that $V_{\rm sd}(\cdot,\cdot,\cdot)$ given by (23) is continuously differentiable.

Theorem 3.2: Consider the linear time delay dynamical system given by (9). Let $\hat{Q} = $ block-diag$[Q, -Q]$, where $Q \in \mathbb{R}^{\hat{p}\times\hat{p}}$, $Q > 0$. Assume there exists a nonnegative definite matrix $\tilde{P} \in \mathbb{R}^{(n+\hat{n})\times(n+\hat{n})}$ and scalars $\varepsilon, \eta > 0$ such that (6) holds and $\tilde{P} \geq {\rm block\text{-}diag}[\eta I_n,\ 0_{\hat{n}\times\hat{n}},\ 0_{\hat{n}\times\hat{n}}]$, where

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & A_1 & 0 \\ B_1 & 0 & A_1 \end{bmatrix},\ \tilde{B} = \begin{bmatrix} -A_{\rm d} \\ B_1 \\ 0 \end{bmatrix},$$
$$\tilde{C} = \begin{bmatrix} 0 & C_1 & 0 \\ I_n & 0 & C_1 \end{bmatrix},\ \tilde{D} = \begin{bmatrix} I_n \\ 0 \end{bmatrix}. \tag{26}$$

Then the linear time delay dynamical system given by (9) is asymptotically stable for every $\tau \in [0, \infty)$.

Proof. It follows from Theorem 3.1 that $\mathcal{G}_{\rm d}$ is $(\Sigma_{\rm d}, \hat{Q}_{\rm d})$-dissipative with $(\Sigma_{\rm d}, \hat{Q}_{\rm d})$- storage function $V_{\rm sd}(\psi, x_{\rm d_1}, x_{\rm d_2})$, $\psi \in \mathcal{C}$, $x_{\rm d_1}, x_{\rm d_2} \in \mathbb{R}^{\hat{n}}$, given by (23). Next, it follows from Proposition 2.1 that $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative with $(\Sigma, \hat{Q})$-storage function $V_{\rm s}(\tilde{x}) = \tilde{x}^{\rm T}\tilde{P}\tilde{x}$, where $\tilde{x} = [x^{\rm T}, x_1^{\rm T}, x_2^{\rm T}]^{\rm T}$. Furthermore, note that $x = \psi(0)$ and as in the proof of Theorem 2.1, it can be shown that $\hat{x}_1(t) = x_{\rm d_1}(t)$, $\hat{x}_2(t) = x_{\rm d_2}(t)$, $t \geq 0$, and hence the state of the overall interconnection of $\mathcal{G}$, $\mathcal{G}_{\rm d}$, and Σ (see Figure 4) is given by $[\psi^{\rm T},\ \hat{x}^{\rm T}]^{\rm T}$ where $\hat{x} = [\hat{x}_1^{\rm T},\ \hat{x}_2^{\rm T}]^{\rm T}$. Next, using the Lyapunov-Krasovskii functional candidate $V(\psi, \hat{x}_1, \hat{x}_2) = V_{\rm s}(\psi(0), \hat{x}_1, \hat{x}_2) + V_{\rm sd}(\psi, \hat{x}_1, \hat{x}_2)$, it follows that

$$\dot{V}(x_t, \hat{x}_1(t), \hat{x}_1(t)) \leq -\varepsilon\tilde{x}^{\rm T}(t)P\tilde{x}(t) \leq -\varepsilon\eta x^{\rm T}(t)x(t). \tag{27}$$

Now, Lyapunov stability follows from standard arguments as applied to time delay systems (see Theorem 2.1 of [11, p. 132] for a similar proof). The proof of asymptotic stability is similar to that of Theorem 2.1 and hence is omitted. □

Remark 3.2: Note that if $V_{\rm s}(\tilde{x})$ and $V_{\rm sd}(\psi, x_{\rm d_1}, x_{\rm d_2})$ satisfy (7) and (8), then Theorem 3.2 follows from Theorem 2.1. However, in the case of time delay dynamical systems (7) and (8) can be replaced by a weaker condition

$$\eta\psi^{\rm T}(0)\psi(0) \leq V(\psi, \hat{x}_1, \hat{x}_2), \quad \psi \in \mathcal{C}, \quad \hat{x}_1, \hat{x}_2 \in \mathbb{R}^{\hat{n}}. \tag{28}$$

In this case, Lyapunov and asymptotic stability can be shown using the fact that $\|x(t)\| \leq \varepsilon$, $t \geq 0$, if and only if $|\!|\!| x_t |\!|\!| \leq \varepsilon$, $t \geq 0$.

Remark 3.3: Recall that the linear time delay dynamical system given by (9) is stable for all $\tau \in [0, \infty)$ if and only if [17] there exists $N : \jmath\mathbb{R} \to \mathbb{C}^{n\times n}$ such that $N(\jmath\omega) > 0$, $\omega \in \mathbb{R}$, and

$$G^*(\jmath\omega)N(\jmath\omega)G(\jmath\omega) - N(\jmath\omega) < 0, \qquad \omega \in \mathbb{R}. \tag{29}$$

Thus, if there exists $\tilde{P} \in \mathbb{R}^{(n+\hat{n})\times(n+\hat{n})}$ such that (6) holds, then it follows from Proposition 2.1 that $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative which implies (29) (see Remark

2.1) with $N(\jmath\omega) = G_1^*(\jmath\omega)QG_1(\jmath\omega)$, where $G_1(\jmath\omega) = C_1(\jmath\omega I_{\hat{n}} - A_1)^{-1}B_1 + I_n$, $\omega \in \mathbb{R}$. Hence, (6) is a sufficient condition for satisfying (29) and $G_1^*(\jmath\omega)QG_1(\jmath\omega)$ is a real rational approximation to $N(\jmath\omega)$ in (29).

Remark 3.4: In the case where $A_1 = 0$, $B_1 = 0$, and $C_1 = 0$, it follows from Theorem 3.2 that if there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + \varepsilon P + Q & -PA_{\mathrm{d}} \\ -A_{\mathrm{d}}^{\mathrm{T}}P & -Q \end{bmatrix} \leq 0, \qquad (30)$$

then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$ is asymptotically stable. Furthermore, it follows from Remark 3.1 that $V_{\mathrm{sd}}(\psi) = \int_{-\tau}^{0} \psi^{\mathrm{T}}(\theta)Q\psi(\theta)\mathrm{d}\theta$ and hence $V(\psi) = \psi^{\mathrm{T}}(0)P\psi(0) + \int_{-\tau}^{0} \psi^{\mathrm{T}}(\theta)Q\psi(\theta)\mathrm{d}\theta$ is a Lyapunov-Krasovskii functional for the linear time delay dynamical system (9). Thus, Theorem 3.2 provides a generalization to the sufficient conditions for linear time delay dynamical systems given in [9], [10].

IV. Illustrative Numerical Example

In this section, we provide a numerical example to illustrate the utility of the results developed in the paper. Consider the linear time delay dynamical system given by (9) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -5 & -2 \end{bmatrix},$$

$$A_{\mathrm{d}} = \begin{bmatrix} -0.0604 & 0.0060 & 0.3018 & 0 \\ 0.0060 & 0.0060 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1.2074 & 0 & -0.6037 & 0 \end{bmatrix}. \qquad (31)$$

Now, with $A_1 = -I_4$, $B_1 = I_4$, and $C_1 = [0_{4\times 4}\ I_4]^{\mathrm{T}}$, we can show that there exist positive definite matrices $\tilde{P}$ and Q such that (6) holds. Hence, it follows from Theorem 3.2 that the linear time-delay dynamical system given by (9) with A and A_{d} given by (31) is asymptotically stable for every $\tau \in [0, \infty)$. However, it can be shown that there does not exist positive-definite matrices P and Q such that (30) holds which shows that Theorem 3.2 provides less conservative sufficient conditions for stability analysis of time delay systems as compared to the standard sufficient conditions given in the literature (see, for example, [9], [10]).

V. Conclusion

In this paper, we extended the concepts of dissipativity and the exponential dissipativity to provide new sufficient conditions for guaranteeing asymptotic stability of a time delay dynamical system. Specifically, representing a time delay dynamical system as a negative feedback interconnection of a finite-dimensional linear dynamical system and an infinite-dimensional time delay operator, we showed that the time delay operator is dissipative. Finally, using stability of feedback interconnection results for dissipative systems, we developed new sufficient conditions for asymptotic stability of time delay dynamical systems. The overall approach provides an explicit framework for constructing Lyapunov-Krasovskii functionals as well as deriving new sufficient conditions for stability analysis of asymptotically stable time delay dynamical systems based on the dissipativity properties of the time delay operator.

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