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Robust Stability of Quasipolynomials: Vertex-Type Tests and Extensions

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***Abstract*— In this paper we study the robust stability of uncertain quasipolynomials, whose coef£cients may vary in a certain prescribed range. We consider speci£cally the co-called interval and diamond quasipolynomials, and our goal is to develop vertex- and edge type-conditions for these quasipolynomial families to be robustly stable independent of delay. Moreover, we also present a number of extensions to more general uncertain quasipolynomials, including both frequency-sweeping and vertex/edge type results.**

Keywords: Interval quasipolynomials, diamond quasipolynomials, robust stability, Kharitonov analysis.

I. Introduction

The effect of time delay on stability is a subject of recurring interest in the study of dynamical systems. This subject received considerable research attention in the last decade or so, and has undergone a notable development both conceptually and computationally; much of the progress during this period is documented in the recent monographs [15], [17], [18], [10]. Two particular stability notions, *delay-dependent* and *delay-independent stability*, respectively, have been the focus of these studies, and both time and frequency domain stability tests have been developed. Here by delay-independent stability of a system we mean that the system is stable for all nonnegative values of delay, and otherwise the system's stability is delay-dependent.

The aim of this paper is to study the *delay-independent* stability of time-delay systems when certain system parameters are only known to be within a prescribed range. Robust stability problems of this kind, as expected, are far more dif£cult. Indeed, while for a system with £xed parameters the problem has received a de£nitive answer, few results are available when the system is uncertain. In fact, based on the known facts in robust stability analysis [8] and the results for systems with incommensurate delays, it is not dif£cult to conclude that this problem will be equivalent to one of computing *structured singular values* [4] and can be shown to be NP-hard in general.

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In this paper we shall focus on polytopic uncertain quasipolynomials, and especially on its sub-families of interval and diamond quasipolynomials. It is worth noting that the robust stability of quasipolynomials has been a well-studied topic (see, e.g., [9], [14], [12], [10] and the references therein), though the results seem less well-developed. In particular, the stability of interval quasipolynomials has been under investigation for some time [2], [19]. In the earlier work [7], the authors showed that for uncertain quasipolynomials in these classes, readily computable stability conditions are available in the form of *frequency-sweeping tests*. In the present paper, we derive "vertex type" stability tests which seek to ascertain the stability of an entire family of quasipolynomials by checking certain special members in that family. Our results continue the spirit of the well-known Kharitonov approach [1], [3] and are of conceptual interest. We show that as in robust stability analysis for interval and diamond polynomials, the stability of the interval and diamond quasipolynomial families can be identi£ed with the stability of certain special vertex and edge members in the quasipolynomial families, where such vertex and edge quasipolynomials correspond to the vertex and edge polynomials of interval and diamond polynomials, respectively. The results thus complement in a natural manner the *frequency-sweeping conditions* obtained in [7].

II. Preliminaries

We consider the class of quasipolynomials given by

$$p\left(s;\ e^{-\tau_1 s},\ \cdots,\ e^{-\tau_m s}\right) = a_0(s) + \sum_{k=1}^{m} a_k(s) e^{-\tau_k s}, \quad (1)$$

where $\tau_k \geq 0$, and

$$\begin{aligned} a_0(s) &= s^n + \sum_{i=0}^{n-1} a_{0i} s^i, \\ a_k(s) &= \sum_{i=0}^{n-1} a_{ki} s^i, \qquad k = 1,\ \cdots,\ m. \end{aligned}$$

This quasipolynomial corresponds to the characteristic function of delay systems described by

$$y^{(n)}(t) + \sum_{i=0}^{n-1} \sum_{k=0}^{m} a_{ki} y^{(i)}(t - \tau_k) = 0, \qquad \tau_k \geq 0, \quad (2)$$

or more generally, those given in the state-space description

$$\dot{x}(t) = A_0\ x(t) + \sum_{k=1}^{r} A_k\ x(t - T_k), \qquad T_k \geq 0. \tag{3}$$

We study the stability properties of the quasipolynomial (1), and accordingly, the stability of the time-delay systems (2) and (3). The stability notion is stated formally as follows.

De£nition 2.1 *The quasipolynomial (1) is said to be stable if*

$$p\left(s;\ e^{-\tau_1 s},\ \cdots,\ e^{-\tau_m s}\right) \neq 0, \qquad \forall s \in \overline{\mathbb{C}}_+. \tag{4}$$

Here $\overline{\mathbb{C}}_+ := \{s:\ \Re(s) \geq 0\}$ *denotes the closed right half plane. It is said to be stable independent of delay if the condition (4) holds for all* $\tau_k \geq 0$.

We shall consider quasipolynomials with incommensurate, independent delays, by which we mean that in (1) the delay parameters $\tau_k,\ k = 1, \cdots, m$ are independent of each other. In this case, a necessary and suf£cient stability condition is available from, e.g., [11], [4], [10].

Lemma 2.1 *Let* $\tau_k, k = 1, \cdots, m$ *be independent delays. Then quasipolynomial (1) is stable independent of delay iff*
(i) $a_0(s)$ *is stable;*
(ii) $\sum_{k=0}^{m} a_{k0} \neq 0$*; and*
(iii)

$$\frac{\sum_{k=1}^{m} |a_k(j\omega)|}{|a_0(j\omega)|} < 1, \qquad \forall \omega > 0. \tag{5}$$

Assume now that the coef£cients of the quasipolynomial (1) vary in a prescribed set. More speci£cally, consider the uncertain quasipolynomial

$$p\left(s; e^{-\tau_1 s}, \cdots, e^{-\tau_m s};\ \alpha\right) = a_0(s, \alpha_0) + \sum_{k=1}^{m} a_k(s,\ \alpha_k) e^{-\tau_k s}, \tag{6}$$

where $\alpha_k\ \in\ \mathcal{Q}_k\ \subset\ \mathbb{R}^n,\ k\ =\ 0,\ \ \cdots,\ m$ represent the uncertain parameters. This uncertain quasipolynomial is said to be robustly stable independent of delay in the following sense.

De£nition 2.2 *The quasipolynomial family (6) is said to be robustly stable if for all* $\alpha_k \in \mathcal{Q}_k,\ k = 0, \cdots, m$,

$$p\left(s;\ e^{-\tau_1 s},\ \cdots,\ e^{-\tau_m s};\ \alpha\right) \neq 0, \qquad \forall s \in \overline{\mathbb{C}}_+. \tag{7}$$

It is robustly stable independent of delay if (7) holds $\forall \tau_k \geq 0$.

We shall assume that each uncertain vector α_k varies independently. Furthermore, we assume that each uncertain polynomial $a_k(s,\ \alpha_k)$ can be written as

$$\begin{aligned} a_0(s,\ \alpha_0) &= s^n + \sum_{i=0}^{n-1} \alpha_{0i} s^i, \\ a_k(s,\ \alpha_k) &= \sum_{i=0}^{n-1} \alpha_{ki} s^i, \qquad k = 1,\ \cdots,\ m \end{aligned} \tag{8}$$

where each $\alpha_{ki},\ k = 0,\ 1,\ \cdots,\ m$ is assumed to lie in a given interval $[\underline{\alpha}_{ki},\ \overline{\alpha}_{ki}]$, and the vector α_k belongs to a weighted ℓ_p de£ned in the following manner. De£ne

$$\alpha^*_{ki} := \frac{\overline{\alpha}_{ki} + \underline{\alpha}_{ki}}{2}, \gamma_{ki} := \frac{\overline{\alpha}_{ki} - \underline{\alpha}_{ki}}{2}, \Gamma_k := \mathrm{diag}\left(\gamma_{k1}, \cdots, \gamma_{kn}\right).$$

Then for any $p \in [1,\ \infty]$, the coef£cient vector α_k is characterized by the set

$$\mathcal{Q}_k^{(p)} := \{\alpha_k:\ \alpha_k = \alpha_k^* + \Gamma_k \delta_k,\ \|\delta_k\|_p \leq 1\},$$

where $\|\cdot\|_p$ denotes the ℓ_p-Hölder norm, that is,

$$\|\delta_k\|_p := \begin{cases} \left(\sum_{i=1}^{n} |\delta_{ki}|^p\right)^{1/p}, & 1 \leq p \leq \infty \\ \max_{1 \leq i \leq n} |\delta_{ki}|, & p = \infty. \end{cases}$$

As such, for $p = \infty,\ 1$, the uncertain polynomial $a_k(s,\ \alpha_k)$ de£nes the families of interval and diamond polynomials, respectively. We shall call the uncertain quasipolynomial (6) the interval and diamond quasipolynomials correspondingly.

An important observation is that for the uncertain quasipolynomial (6) to be robustly stable, it is both necessary and suf£cient that (i) the polynomial $a_0(s,\ \alpha_0)$ is robustly stable, (ii) $\sum_{k=0}^{m} \underline{\alpha}_{k0} > 0$, and (iii)

$$\frac{\sum_{k=1}^{m} \max_{\alpha_k \in \mathcal{Q}_k^{(p)}} |a_k(j\omega,\ \alpha_k)|}{\min_{\alpha_0 \in \mathcal{Q}_0^{(p)}} |a_0(j\omega)|} < 1, \qquad \forall \omega > 0. \tag{9}$$

This is clear from Lemma 2.1. These conditions form the basis for the frequency-sweeping conditions in [7], and will likewise play an important role in our present development.

Assume, without loss of generality, that n is an even integer; an analogous analysis applies when n is odd. Let $q \in [1, \infty]$ satisfy the relation $(1/p) + (1/q) = 1$. For each $\omega \in [0, \infty)$, de£ne

$$\begin{aligned} X_{k,q}(\omega): &= \left(\gamma_{k0}^q + (\gamma_{k2}\omega^2)^q + (\gamma_{k4}\omega^4)^q + \cdots\right)^{1/q}, \\ Y_{k,q}(\omega): &= \left(\gamma_{k1}^q + (\gamma_{k3}\omega^2)^q + (\gamma_{k5}\omega^4)^q + \cdots\right)^{1/q}, \\ R_k(\omega): &= \alpha^*_{k0} - \alpha^*_{k2}\omega^2 - \alpha^*_{k4}\omega^4 + \cdots, \\ I_k(\omega): &= \alpha^*_{k1} - \alpha^*_{k3}\omega^2 + \alpha^*_{k5}\omega^4 + \cdots \end{aligned}$$

Note that for $q = \infty$,

$$\begin{aligned} X_{k,\infty}(\omega) &= \max\left\{\gamma_{k0},\ \gamma_{k2}\omega^2,\ \gamma_{k4}\omega^4,\ \cdots\right\}, \\ Y_{k,\infty}(\omega) &= \max\left\{\gamma_{k1},\ \gamma_{k3}\omega^2,\ \gamma_{k5}\omega^4,\ \cdots\right\}. \end{aligned}$$

It also follows that $a_k(j\omega,\ \alpha_k^*) = R_k(\omega) + j\omega I_k(\omega)$.

III. Vertex and Edge Results

In this section we derive vertex- and edge-type conditions which seek to test the robust stability of a quasipolynomial family in terms of its vertex and edge quasipolynomials. While compared to the frequency-sweeping conditions obtained in [7], these results may not be favorable computationally, they remain to be of interest and constitute generalizations to the well-known Kharitonov type robust stability conditions.

A. Interval Quasipolynomials

We consider £rst the interval quasipolynomials. This class corresponds to the case that all the polynomials $a_k(s,\ \alpha_k)$ are each interval polynomials, that is, $\alpha_k \in [\underline{\alpha}_{ki},\ \overline{\alpha}_{ki}]$, or alternatively,

$$\mathcal{Q}_k^{(\infty)} := \{\alpha_k :\ \alpha_k = \alpha_k^* + \Gamma_k\delta_k,\ \|\delta_k\|_\infty \le 1\}. \tag{10}$$

For each $k = 0,\ 1,\ \cdots,\ m$, de£ne the four Kharitonov vertex polynomials

$$\begin{aligned}
K_{k,1}(s): &= \underline{\alpha}_{k0} + \underline{\alpha}_{k1}s + \overline{\alpha}_{k2}s^2 + \overline{\alpha}_{k3}s^3 + \cdots \\
K_{k,2}(s): &= \overline{\alpha}_{k0} + \overline{\alpha}_{k1}s + \underline{\alpha}_{k2}s^2 + \underline{\alpha}_{k3}s^3 + \cdots \\
K_{k,3}(s): &= \overline{\alpha}_{k0} + \underline{\alpha}_{k1}s + \underline{\alpha}_{k2}s^2 + \overline{\alpha}_{k3}s^3 + \cdots \\
K_{k,4}(s): &= \underline{\alpha}_{k0} + \overline{\alpha}_{k1}s + \overline{\alpha}_{k2}s^2 + \underline{\alpha}_{k3}s^3 + \cdots
\end{aligned}$$

and four edge polynomials

$$\begin{aligned}
K_{k,1}(s,\lambda): &= \lambda K_{k,1}(s) + (1-\lambda)K_{k,2}(s) \\
K_{k,2}(s,\lambda): &= \lambda K_{k,2}(s) + (1-\lambda)K_{k,3}(s) \\
K_{k,3}(s,\lambda): &= \lambda K_{k,3}(s) + (1-\lambda)K_{k,4}(s) \\
K_{k,4}(s,\lambda): &= \lambda K_{k,4}(s) + (1-\lambda)K_{k,1}(s)
\end{aligned}$$

It is well-known that the interval polynomial $a_k(s,\ \alpha_k)$ will be stable whenever $\alpha_{kn} \neq 0$ and the four vertex polynomials are stable. In the present setting, of relevance is the stability of the interval polynomial $a_0(s,\ \alpha_0)$.

We shall £rst state a suf£cient stability condition in terms of the vertex polynomials.

Theorem 3.1 *Let $\tau_k \ge 0$, $k = 1,\ \cdots,\ m$ be independent delays. Then the interval quasipolynomial (6), with $\alpha_k \in \mathcal{Q}_k^{(\infty)}$, $k = 0,\ 1,\ \cdots,\ m$, is robustly stable independent of delay if*
(i) The polynomials $K_{0,i_0}(s)$ are stable;
(ii) $\underline{\alpha}_{00} > \sum_{k=1}^{m} \max\{|\underline{\alpha}_{k0}|,\ |\overline{\alpha}_{k0}|\}$;
(iii) The 4^{m+1} quasipolynomials

$$K_{0,i_0}(s) + \sum_{k=1}^{m} K_{k,i_k}(s)e^{-\tau_k s}$$

are stable independent of delay, where $i_k \in \{1,\ 2,\ 3,\ 4\}$, $k = 0,\ 1,\ \cdots,\ m$.

It is worth noting that vertex-type conditions similar to Theorem 3.1 have been previously stated in [2] and [19], though each appears to contain some ¤aws. We remark that the key restriction in Theorem 3.1 is imposed by the condition (ii), which rendered the theorem a suf£cient but not necessary condition. For interval quasipolynomials that do satisfy the condition (ii), however, the result becomes both necessary and suf£cient for robust stability. More generally, if the condition is not satis£ed, we have the following necessary and suf£cient condition which requires checking also edge quasipolynomials.

Theorem 3.2 *Let $\tau_k \ge 0$, $k = 1,\ \cdots,\ m$ be independent delays. Then the interval quasipolynomial (6), with $\alpha_k \in \mathcal{Q}_k^{(\infty)}$, $k = 0,\ 1,\ \cdots,\ m$, is robustly stable independent of delay if*
(i) The polynomials $K_{0,i_0}(s)$ are stable;
(ii) $\sum_{k=0}^{m} \underline{\alpha}_{k0} > 0$;
(iii) The 4^{m+1} one-parameter quasipolynomials

$$K_{0,i_0}(s,\lambda) + \sum_{k=1}^{m} K_{k,i_k}(s)e^{-\tau_k s}$$

are stable independent of delay, where $\lambda \in [0,\ 1]$, and $i_k \in \{1,\ 2,\ 3,\ 4\}$, $k = 0,\ 1,\ \cdots,\ m$.

Note that whenever the condition $\sum_{k=0}^{m} \underline{\alpha}_{k0} > 0$ coincides with

$$\underline{\alpha}_{00} > \sum_{k=1}^{m} \max\{|\underline{\alpha}_{k0}|,\ |\overline{\alpha}_{k0}|\},$$

for example, when $\overline{\alpha}_{k0} \le 0$ for $k = 1,\ \cdots,\ m$, then Theorem 3.2 can be strengthened to Theorem 3.1, that is, to a necessary and suf£cient vertex test. Note also that in view of Lemma 2.1, the theorem can also be stated as follows.

Corollary 3.1 *Let $\tau_k \ge 0$, $k = 1,\ \cdots,\ m$ be independent delays. Then the interval quasipolynomial (6), with $\alpha_k \in \mathcal{Q}_k^{(\infty)}$, $k = 0,\ 1,\ \cdots,\ m$, is robustly stable independent of delay iff*
(i) The polynomials $K_{0,i_0}(s)$, $i_0 \in \{1,\ 2,\ 3,\ 4\}$, are stable;
(ii) $\sum_{k=0}^{m} \underline{\alpha}_{k0} > 0$; and
(iii) For $\lambda \in [0,\ 1]$ and for all $i_k \in \{1,\ 2,\ 3,\ 4\}$, $k = 0,\ 1,\ \cdots,\ m$,

$$\frac{\sum_{k=1}^{m} |K_{k,i_k}(j\omega)|}{|K_{0,i_0}(j\omega,\lambda)|} < 1, \qquad \forall\omega > 0. \tag{11}$$

B. Diamond Quasipolynomials

For the diamond quasipolynomial family, each polynomial $a_k(s,\ \alpha_k)$ de£nes a diamond polynomial, whose coef£cients vary in the parameter set

$$\mathcal{Q}_k^{(1)} := \{\alpha_k :\ \alpha_k = \alpha_k^* + \Gamma_k\delta_k,\ \|\delta_k\|_1 \le 1\}. \tag{12}$$

For simplicity, we shall consider only uniformly weighted diamond polynomials. This means that for each $k = 0,\ 1,\ \cdots,\ m$, $\gamma_{k1} = \cdots = \gamma_{kn} = \gamma_k$. At each $\omega > 0$, denote the value set of $a_k(j\omega,\ \alpha_k)$ by $V_k^D(\omega)$, i.e.,

$$V_k^D(\omega) = \left\{a_k(j\omega,\ \alpha_k) :\ \alpha_k \in \mathcal{Q}_k^{(1)}\right\}.$$

It is known that $V_k^D(\omega)$ forms a frequency-dependent diamond with its center at $a_k(j\omega,\ \alpha_k^*)$. For $\omega \in (0,\ 1]$, the four vertices of the diamond are the polynomials $e_{k,1}(s)$, $e_{k,2}(s)$, $e_{k,3}(s)$, and $e_{k,4}(s)$, while for $\omega \in (1,\ \infty)$, the vertices are $e_{k,5}(s)$, $e_{k,6}(s)$, $e_{k,7}(s)$, and $e_{k,8}(s)$, where

$$\begin{gathered}
e_{k,1}(s) = a_k(s,\ \alpha_k^*) - \gamma_k,\ e_{k,2}(s) = a_k(s,\alpha_k^*) + \gamma_k \\
e_{k,3}(s) = a_k(s,\alpha_k^*) - \gamma_k s,\ e_{k,4}(s) = a_k(s,\alpha_k^*) + \gamma_k s, \\
e_{k,5}(s) = a_k(s,\alpha_k^*) - \gamma_k s^{n-2},\ e_{k,6}(s) = a_k(s,\alpha_k^*) + \gamma_k s^{n-2}, \\
e_{k,7}(s) = a_k(s,\alpha_k^*) - \gamma_k s^{n-1},\ e_{k,8}(s) = a_k(s,\alpha_k^*) + \gamma_k s^{n-1}.
\end{gathered}$$

These are the extremes of the eight edge polynomials

$$\begin{aligned}
e_{k,1}(s,\lambda) &= a_k(s,\ \alpha_k^*) - \lambda\gamma_k - (1-\lambda)\gamma_k s,\\
e_{k,2}(s,\lambda) &= a_k(s,\alpha_k^*) + \lambda\gamma_k - (1-\lambda)\gamma_k s,\\
e_{k,3}(s,\lambda) &= a_k(s,\alpha_k^*) + \lambda\gamma_k + (1-\lambda)\gamma_k s,\\
e_{k,4}(s,\lambda) &= a_k(s,\alpha_k^*) - \lambda\gamma_k + (1-\lambda)\gamma_k s,\\
e_{k,5}(s,\lambda) &= a_k(s,\alpha_k^*) - \lambda\gamma_k s^{n-1} - (1-\lambda)\gamma_k s^{n-2},\\
e_{k,6}(s,\lambda) &= a_k(s,\alpha_k^*) + \lambda\gamma_k s^{n-1} - (1-\lambda)\gamma_k s^{n-2},\\
e_{k,7}(s,\lambda) &= a_k(s,\alpha_k^*) + \lambda\gamma_k s^{n-1} + (1-\lambda)\gamma_k s^{n-2},\\
e_{k,8}(s,\lambda) &= a_k(s,\alpha_k^*) - \lambda\gamma_k s^{n-1} + (1-\lambda)\gamma_k s^{n-2}.
\end{aligned}$$

It is well-known that the diamond polynomial $a_0(s,\ \alpha_0)$ is robustly stable if and only if its eight vertex polynomials are stable. We have the following result.

Theorem 3.3 *Let $\tau_k \geq 0,\ k=1,\ \cdots,\ m$ be independent delays. Assume that for each $k=0,\ 1,\ \cdots,\ m$, $\gamma_{k1} = \cdots = \gamma_{kn} = \gamma_k$. Then the diamond quasipolynomial (6), with $\alpha_k \in \mathcal{Q}_k^{(1)}$, $k=0,\ 1,\ \cdots,\ m$, is robustly stable independent of delay iff*
(i) The polynomials $e_{0,i_0}(s)$, $i_0 \in \{1,\ \cdots,\ 8\}$, are stable;
(ii) $\sum_{k=0}^{m} \underline{\alpha}_{k0} > 0$;
(iii) For $\lambda \in [0,\ 1]$ and for all $i_k \in \{1,\ \cdots,\ 8\}$, $k = 0,\ 1,\ \cdots,\ m$,

$$\frac{\sum_{k=1}^{m} |e_{k,i_k}(j\omega)|}{|e_{0,i_0}(j\omega,\lambda)|} < 1, \qquad \forall\omega > 0. \tag{13}$$

In view of Lemma 2.1, we may also state Theorem 3.3 alternatively as follows.

Corollary 3.2 *Let $\tau_k \geq 0,\ k=1,\ \cdots,\ m$ be independent delays. Assume that for each $k=0,\ 1,\ \cdots,\ m$, $\gamma_{k1} = \cdots = \gamma_{kn} = \gamma_k$. Then the diamond quasipolynomial (6), with $\alpha_k \in \mathcal{Q}_k^{(1)}$, $k=0,\ 1,\ \cdots,\ m$, is robustly stable independent of delay iff*
(i) The polynomials $e_{0,i_0}(s)$, $i_0 \in \{1,\ \cdots,\ 8\}$, are stable;
(ii) $\sum_{k=0}^{m} \underline{\alpha}_{k0} > 0$;
(iii) The 8^{m+1} one-parameter quasipolynomials

$$e_{0,i_0}(s,\lambda) + \sum_{k=1}^{m} e_{k,i_k}(s) e^{-\tau_k s}$$

are stable independent of delay for $\lambda \in [0,\ 1]$, where $i_k \in \{1,\ \cdots,\ 8\}$, $k=0,\ 1,\ \cdots,\ m$.

As in the case of interval diamond polynomials, Theorem 3.3 and Corollary 3.2 follow from the recognition that the maximum of $|a_k(j\omega,\ \alpha_k)$ is achieved on one of the vertices $e_{k,i_k}(j\omega)$, while the minimum of $|a_0(j\omega,\ \alpha_0)$ occurs on one of the edges $e_{0,i_0}(j\omega,\lambda)$. These results can be extended to more general cases where the diamond polynomials need not be uniformly weighted. Indeed, they can be extended to quasipolynomials with a polytopic uncertainty description. This is pursued in the next section.

IV. Extensions

Our purpose in this section is to provide a number of extensions to more general uncertain quasipolynomials than those in the interval and diamond families. We consider more general uncertainty descriptions and seek both frequency-sweeping and vertex type results. A generalization is also made to a special class of multivariate polynomials.

A. Uncertain Coef£cients in ℓ_p Balls

Our £rst generalization is sought after for a case where uncertain coef£cients are characterized by general ℓ_p norms. Consider the uncertain quasipolynomial (6) with the coef£cients described by

$$\mathcal{Q}_k := \{\alpha_k :\ \alpha_k = \alpha_k^* + \Gamma_k\delta_k,\ \|\delta_k\| \leq 1\}, \tag{14}$$

where

$$\|\delta_k\| := \max\{\|\delta_k^e\|_{p_1},\ \|\delta_k^o\|_{p_2}\}. \tag{15}$$

In other words, we assume that the real and imaginary parts of the coef£cient polynomials vary independently of each other. Note that uncertain polynomials with independent real and imaginary parts are studied in [20]. Note also that in this case a similar frequency-sweeping condition for the robust stability of $a_0(s,\ \alpha_0)$ can be obtained in much the same spirit as in [6].

Lemma 4.1 *Suppose that $a_0(s,\ \alpha_0^*)$ is stable. Then, the uncertain polynomial $a_0(s,\ \alpha_0)$ with $\mathcal{Q}_0$ given by (14-15) is robustly stable if and only if $\underline{\alpha}_{00} > 0$ and*

$$\min\left\{\frac{X_{0,q_1}(\omega)}{|R_0(\omega)|},\ \frac{Y_{0,q_2}(\omega)}{|I_0(\omega)|}\right\} < 1, \qquad \forall\omega > 0. \tag{16}$$

The following result gives a frequency-sweeping condition similar to those developed in [7].

Theorem 4.1 *Let τ_k, $k=1,\ \cdots,\ m$ be independent delays. De£ne*

$$\overline{\rho}_k(\omega) := \sqrt{(|R_k(\omega)| + X_{k,q_1}(\omega))^2 + \omega^2(|I_k(\omega)| + Y_{k,q_2}(\omega))^2},$$
$$\underline{\rho}_0(\omega) := \sqrt{M_R^2(\omega) + \omega^2 M_I^2(\omega)}\ ,$$

where

$$M_R(\omega) = \begin{cases} |R_0(\omega)| - X_{0,q_1}(\omega) & \text{if } |R_0(\omega)| > X_{0,q_1}(\omega)\\ 0 & \text{if } |R_0(\omega)| \leq X_{0,q_1}(\omega)\end{cases}$$
$$M_I(\omega) := \begin{cases} |I_0(\omega)| - Y_{0,q_2}(\omega) & \text{if } |I_0(\omega)| > Y_{0,q_2}(\omega)\\ 0 & \text{if } |I_0(\omega)| \leq Y_{0,q_2}(\omega)\end{cases}$$

Then the uncertain quasipolynomial (6), with $a_k(s,\ \alpha_k)$ given by (8) and (14-15), is robustly stable independent of delay iff
(i) The uncertain polynomial $a_0(s,\ \alpha_0)$ is stable;
(ii) $\sum_{k=0}^{m} \underline{\alpha}_{k0} > 0$;
(iii)

$$\frac{\sum_{k=1}^{m} \overline{\rho}_k(\omega)}{\underline{\rho}_0(\omega)} < 1, \qquad \forall\omega > 0. \tag{17}$$

B. Polytopic Uncertainty

Extensions may also be found to quasipolynomials with polytopic uncertainties. For this purpose, let the family of polynomials $a_k(s,\ \alpha_k)$ be the convex hull

$$\mathcal{P}_k := \text{conv}\left\{p_{k1}(s),\ \cdots,\ p_{k,l_k}(s)\right\}. \tag{18}$$

In other words, $a_k(s,\ \alpha_k)$ can be expressed as the convex combination of the *generating polynomials* $p_{kj}(s)$:

$$a_k(s,\ \alpha_k) = \sum_{j=1}^{l_k} \lambda_j p_{kj}(s), \tag{19}$$

where

$$\sum_{j=1}^{l_k} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1,\ \cdots,\ l_k,$$

and

$$p_{0j}(s) = s^n + \sum_{i=0}^{n-1} p_{0i}^{(j)} s^i, \tag{20}$$

$$p_{kj}(s) = \sum_{i=0}^{n-1} p_{ki}^{(j)} s^i,\ k = 1,\ 2,\ \cdots,\ m. \tag{21}$$

We note that both the interval and diamond polynomials fall as special cases of this polytopic class. Note also that the stability of $a_0(s,\ \alpha_0)$ can be checked using the so-called *edge theorem* and other tools (see, e.g., [1], [3]). We provide below both frequency-sweeping and edge type results for the corresponding quasipolynomials.

Theorem 4.2 *Let $\tau_k \geq 0,\ k = 1,\ \cdots,\ m$ be independent delays. De£ne*

$$\rho_{ij}(\omega) := \begin{cases} \frac{|\Im\{p_{0i}(-j\omega)p_{0j}(j\omega)\}|}{|p_{0i}(j\omega)-p_{0j}(j\omega)|} \\ \quad \text{if } \Re\{p_{0i}(-j\omega)p_{0j}(j\omega)\} < \\ \quad\quad \min\{|p_{0i}(j\omega)|^2,\ |p_{0j}(j\omega)|^2\}, \\ \min\{|p_{0i}(j\omega)|,\ |p_{0j}(j\omega)|\} \quad \text{otherwise.} \end{cases}$$

Then the uncertain quasipolynomial (6), with $a_k(s,\ \alpha_k)$ given by (19-21), is robustly stable independent of delay iff
(i) The polytopic polynomial $a_0(s,\ \alpha_0)$ is robustly stable;
(ii) $\sum_{k=0}^{m} \min_{1\leq j\leq l_k} p_{k0}^{(j)} > 0$;
(iii)

$$\frac{\sum_{k=1}^{m} \max_{1\leq j\leq l_k} |p_{kj}(j\omega)|}{\min_{1\leq i<j\leq l_0} \rho_{ij}(\omega)} < 1, \qquad \forall\omega > 0. \tag{22}$$

Note that the numerator in (22) can be easily computed. The computation of the denominator can be more demanding, which requires $l_0(l_0-1)/2$ computations of $\rho_{ij}(\omega)$.

We may also state this result alternatively in terms of the vertices of $a_k(j\omega,\ \alpha_k)$ and the edges of $a_0(j\omega,\ \alpha_0)$.

Corollary 4.1 *Let $\tau_k \geq 0,\ k = 1,\ \cdots,\ m$ be independent delays. Then the uncertain quasipolynomial (6), with $a_k(s,\ \alpha_k)$ given by (19-21), is robustly stable independent of delay iff*
(i) The edge polynomials $\lambda p_{0i}(s) + (1-\lambda)p_{0j}(s)$ are stable for all $1 \leq i \leq j \leq l_0$;
(ii) $\sum_{k=0}^{m} \min_{1\leq j\leq l_k} p_{k0}^{(j)} > 0$;
(iii) The quasipolynomials

$$\lambda p_{0i}(s) + (1-\lambda)p_{0j}(s) + \sum_{k=1}^{m} p_{k,j_k}(s)e^{-\tau_k s}$$

are stable for all $\lambda \in [0,\ 1]$, $1 \leq i < j \leq l_0$, and $1 \leq j_k \leq l_k$.

C. Multivariate Polynomials

It is straightforward to extend the preceding results to a special class of multivariate polynomials which are also known as *disc polynomials* [3]. This class of multivariate polynomials are described as

$$p\left(s;\ z_1,\ \cdots,\ z_m\right) = a_0(s) + \sum_{k=1}^{m} a_k(s)z_k. \tag{23}$$

The multivariate polynomial $p\left(s;\ z_1,\ \cdots,\ z_m\right)$ is said to be stable if

$$p\left(s;\ z_1,\ \cdots,\ z_m\right) \neq 0,\ \forall s \in \overline{\mathbb{C}}_+,\ z_k \in \mathbb{D}^c,\ k = 1,\ \cdots,\ m.$$

A necessary and suf£cient condition for the stability of $p\left(s;\ z_1,\ \cdots,\ z_m\right)$ is available from, e.g., [3], [16], which can also be seen rather trivially from [4].

Lemma 4.2 *The multivariate polynomial (23) is stable if and only if*
(i) $a_0(s)$ is stable;
(ii)

$$\frac{\sum_{k=1}^{m} |a_k(j\omega)|}{|a_0(j\omega)|} < 1, \qquad \forall\omega \geq 0. \tag{24}$$

Clearly, the sole difference between Lemma 4.2 and Lemma 2.1 lies at the frequency $\omega = 0$. It is thus unsurprising that the preceding results can all be extended readily to this class of multivariate polynomials with uncertain coef£cients.

V. Concluding Remarks

In this paper we have studied the robust stability of uncertain quasipolynomials, speci£cally those in the families of interval and diamond quasipolynomials. We addressed speci£cally the notion of robust stability independent of delay. For each of these quasipolynomial families, we showed that simple edge-type test can be used to checked the robust stability, which is both necessary and suf£cient. We also provided a vertex-type stability condition for interval quasipolynomials. This condition, while only a suf£cient condition in general, will become necessary and suf£cient under some rather mild restriction. Our results extend the well-known Kharitonov-type analysis to quasipolynomials, and complement the frequency-sweeping conditions developed elsewhere.

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