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Synthesis of Controllers for Non-minimum Phase and Unstable Systems Using Non-sequential MIMO Quantitative Feedback Theory

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Abstract **– Considered in this paper are multi-input multi-output (MIMO) systems with non-minimum phase (NMP) zeros and unstable poles, where some of the unstable poles are located to the right of, or close to, the NMP zeros. In the single-input single-output (SISO) case such systems pose serious difficulties in controller synthesis for performance and stability. In spite of the added degrees of freedom the MIMO case also poses difficulties, as has been shown in the stabilization of the X-29 aircraft. When using the MIMO QFT technique, the design procedure proceeds by decomposing the MIMO design problem into multiple multi-input single-output (MISO) design problems, with the equivalent SISO plants q_{ii}, where $\left(P^{-1}\right)_{ii} = 1/q_{ii}$, employed in each MISO design. Developed is a transformation scheme that can be used to condition the equivalent SISO plants so that the difficult problem of NMP zeros lying to the left of, or close to, unstable poles may be avoided. Examples illustrate the use of the proposed transformation.**

I. INTRODUCTION

In non-sequential multi-input multi-output Quantitative Feedback Theory (NS MIMO QFT) the $n \times n$ design problem is first converted into multiple multi-input single-output (MISO) design problems, with n single-input single-output (SISO) equivalent plants, the q_{ii}. The robust performance (RP) and robust stability (RS) specifications on the original $n \times n$ system are also translated into appropriate RP and RS specifications on these n MISO problems. Then the MISO QFT method is executed for each MISO design problem. If those n MISO design problems can be successfully completed, then it follows from the Schauder's fixed-point mapping theorem ([1]) that MIMO closed-loop uncertain system's RP specifications are satisfied. Moreover, from a NS MIMO QFT robust stability theorem ([4]), if the MIMO system also satisfies a necessary and sufficient existence condition the MIMO closed-loop uncertain system is guaranteed to be robustly stable. The basic QFT design procedure assumes a diagonal controller $G(s)$ and a fully populated prefilter $F(s)$.

For a SISO control design problem, when the plant has unstable poles lying to the right of, or close to, non-minimum phase (NMP) zeros, i.e. right-half plane (RHP) dipoles, the SISO QFT design problem may not be solvable. The reason for this is that in frequency domain designs a NMP zero restricts the maximum allowable loop cross-over frequency (ω_c) whereas an unstable pole calls for a minimum ω_c. Consequently, when the NMP zero is closer to the imaginary axis than an unstable pole, there may not exist a linear time invariant (LTI) controller that can stabilize the uncertain SISO system.

In MIMO plants, due to the directional dependency of the poles and zeros, the role played by RHP poles and zeros is not the same as that in SISO plants. Consequently, a MIMO plant with RHP dipoles will not always present a difficult design problem. However, due to the consideration of the SISO equivalent plants, MIMO plants with RHP dipoles do present a difficult design problem in NS MIMO QFT, as this kind of MIMO plant typically leads to some of the equivalent plants also having a RHP dipole for all plants in the uncertain plant family. In a NS MIMO QFT design, if one of the equivalent SISO plants has a RHP dipole for all plants in the family, it may not be possible to successfully complete that equivalent MISO QFT design problem. This necessarily means that the NS MIMO QFT procedure will not yield a robustly stable MIMO design and obviously limits the ability of NS MIMO QFT to deal with systems possessing RHP dipoles. Furthermore, a minimum phase (MP) and stable MIMO plant can lead to some SISO equivalent plants being unstable, and a NMP and stable MIMO plant can lead to some SISO equivalent plants being NMP and unstable. This further reduces the class of systems to which NS MIMO QFT can be successfully applied.

These limitations on the class of system to which NS MIMO QFT can be applied partly arise from the employment of a diagonal controller. However, it is entirely possible to use a fully populated controller in the design. As shown in this paper, it is possible to exploit the design freedom provided by a fully populated controller to transform the equivalent SISO plants from an apparent un-stabilizable situation to a stabilizable one, thus facilitating a successful NS MIMO QFT design.

II. MATRIX TRANSFORMATION

Described in this section is a procedure for determining a set of matrices M and N to yield desirable pole-zero

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locations in the equivalent SISO plants to facilitate a successful NS MIMO QFT design. Only one feasible approach is considered here, although there are number of other approaches that may be pursued.

As depicted in Fig. 1, two non-singular matrices M, N are introduced such that:

$$(I + MM^{-1}PNN^{-1}G)T = PNN^{-1}GF .$$

It follows that,

$$(I + M^{-1}PNN^{-1}GM)M^{-1}T = M^{-1}PNN^{-1}GMM^{-1}F .$$

The last equality can now be written as $(I + P_1G_1)T_1 = P_1G_1F_1$, where $P_1 = M^{-1}PN$, $G_1 = N^{-1}GM$, $T_1 = M^{-1}T$ and $F_1 = M^{-1}F$. Note that the above equation is in exactly the same form as the standard equation with P, G, F replaced by P_1, G_1, F_1. Consequently, one could derive all the required MIMO QFT design equations with these new variables. The following sections consider the employment of the transformation matrices M and N for 2×2 MIMO plants.

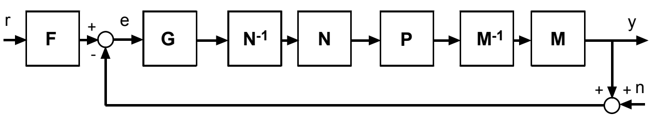

Fig. 1 Plant Transformation

Plant Set-up. Consider a 2×2 MIMO plant of the form $P = \left[z_{ij} / p_{ij} \right]$, where z_{ij} and p_{ij} are polynomials of s. The determinant of P is,

$$\det\left[P\right] = \frac{z_{11}\cdot z_{22}\cdot p_{12}\cdot p_{21} - z_{12}\cdot z_{21}\cdot p_{11}\cdot p_{22}}{p_{11}\cdot p_{22}\cdot p_{12}\cdot p_{21}} = \frac{k\cdot Z(s)}{\Phi(s)},$$

where $Z(s)$ is the zero polynomial of the 2×2 plant and $\Phi(s)$ is the pole polynomial of the 2×2 plant.

In NS MIMO QFT the MIMO design problem is decomposed into n MISO design problems with the equivalent SISO plants q_{ii}, where $\left(P^{-1}\right)_{ij} = 1/q_{ij}$. The corresponding equivalent plant matrix $Q = \left[q_{ij} \right]$ is given by:

$$Q = \begin{bmatrix} \dfrac{p_{22}\det P}{z_{22}} & \dfrac{p_{12}\det P}{z_{12}} \\ \dfrac{p_{21}\det P}{z_{21}} & \dfrac{p_{11}\det P}{z_{11}} \end{bmatrix} = \begin{bmatrix} \dfrac{kZ(s)p_{22}}{z_{22}\Phi(s)} & \dfrac{kZ(s)p_{12}}{z_{12}\Phi(s)} \\ \dfrac{kZ(s)p_{21}}{z_{21}\Phi(s)} & \dfrac{kZ(s)p_{11}}{z_{11}\Phi(s)} \end{bmatrix}$$

Selecting the Transformation Matrices. In the design procedure it is possible to use either the N or M matrix, or both the N and M matrices, with the latter more complex due to the added degrees of freedom.

1) *Applying only the N matrix transformation.* Suppose $N = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}$, where N^{-1} exists and n_{ij} are constants or polynomials of s. Defining $P_{1n} = P\cdot N$, the resulting Q matrix is:

$$Q_{1n} = \begin{bmatrix} q_{n11} & q_{n12} \\ q_{n21} & q_{n22} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{(n_{11}n_{22} - n_{12}n_{21})(z_{11}z_{22}p_{12}p_{21} - z_{12}z_{21}p_{11}p_{22})}{(n_{12}z_{21}p_{22} - n_{22}z_{22}p_{21})p_{11}p_{12}} & -\dfrac{(n_{11}n_{22} - n_{12}n_{21})(z_{11}z_{22}p_{12}p_{21} - z_{12}z_{21}p_{11}p_{22})}{(n_{12}z_{11}p_{12} - n_{22}z_{12}p_{11})p_{22}p_{21}} \\ -\dfrac{(n_{11}n_{22} - n_{12}n_{21})(z_{11}z_{22}p_{12}p_{21} - z_{12}z_{21}p_{11}p_{22})}{(n_{11}z_{21}p_{22} - n_{21}z_{22}p_{21})p_{11}p_{12}} & \dfrac{(n_{11}n_{22} - n_{12}n_{21})(z_{11}z_{22}p_{12}p_{21} - z_{12}z_{21}p_{11}p_{22})}{(n_{11}z_{11}p_{12} - n_{21}z_{12}p_{11})p_{22}p_{21}} \end{bmatrix}.$$

Let $\theta_1 = n_{12}z_{21}p_{22} - n_{22}z_{22}p_{21}$ and $\theta_2 = n_{11}z_{11}p_{12} - n_{21}z_{12}p_{11}$. The objective is to select the n_{ij} such that any RHP dipoles in the diagonal entries of Q_{1n} disappear. In order to accomplish this, one can select the target q_{nii} to be as follows:

(a) q_{n11} is MP but unstable, q_{n22} is NMP but stable and all the zeros of $(n_{11}n_{22} - n_{12}n_{21})$ are MP, if there are no unstable poles in q_{22}.

(b) q_{n11} is NMP but stable, q_{n22} is MP but unstable and all the zeros of $(n_{11}n_{22} - n_{12}n_{21})$ are MP, if there are no unstable poles in q_{11}.

(c) Furthermore, one could have q_{nii} to be NMP and unstable as long as the NMP zero is lying to the right of, and far away from, the unstable poles. This allows $(n_{11}n_{22} - n_{12}n_{21})$ to have NMP zeros which lie to the right of q_{nii}'s unstable poles.

Usually, cases (a) and (b) are preferred since they are easy to stabilize by applying a large gain to the unstable plant and a small gain to the NMP plant. Regardless of which case one employs, the matrix N cannot possess any NMP zeros that are the same as the unstable poles of the MIMO system, as these NMP zeros of N will show up as the controller's NMP zeros, since $G = N\cdot G_{1n}$. This results in RHP pole-zero cancellations between G and the original P and hence a loss of internal stability.

Remark: In cases where both q_{n11} and q_{n22} have a RHP dipole, one needs to remove the unstable pole from either q_{n11} or q_{n22}. There are two possible ways to resolve this situation. One is to produce a NMP zero in

$(n_{11}n_{22} - n_{12}n_{21})$ that cancels the unstable pole. However, this results in RHP pole-zero cancellations between G and the original P. The other way is to assign a common denominator for n_{12} and n_{22} such that this common denominator can cancel the unstable pole in q_{n11} or q_{n22}. However, by doing so $(n_{11}n_{22} - n_{12}n_{21})$ will also have that common denominator (the unstable pole). Thus, an attempt in this direction is futile. This suggests that to alleviate the problem associated with the unstable pole one needs to use both M and N matrices.

Now, assume $z_{ji}p_{jj} = (a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)$ and $z_{jj}p_{ji} = (b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)$ in θ_i. When selecting the n_{ij} in θ_1 and θ_2, one can take $n_{ij} = c_u s^u + c_{u-1}s^{u-1} + \cdots + c_0$ and $n_{jj} = d_v s^v + d_{v-1}s^{v-1} + \cdots + d_0$, and use the following formulae:

$$\begin{aligned}\theta_i &= \pm n_{ij} z_{ji} p_{jj} \mp n_{jj} z_{jj} p_{ji} \\ &= \pm\left(c_u s^u + c_{u-1}s^{u-1} + \cdots + c_0\right)\left(a_n s^n + \cdots + a_0\right) \\ &\quad \mp\left(d_v s^v + d_{v-1}s^{v-1} + \cdots + d_0\right)\left(b_m s^m + \cdots + b_0\right)\end{aligned}.$$

This can be put in the form $\theta_i = \alpha(s - r_1)(s - r_2)\cdots(s - r_t)$, where the r_i's are the roots of the polynomial θ_i.

In order to produce the desired q_{nii}, some r_i's can be specified to be the same as the NMP zeros and/or unstable poles of the MIMO system and the rest of them the free variables that can be arbitrarily chosen but have to be stable. For example, in case (b) one may need to assign r_1 of θ_2 to cancel the NMP zero such that q_{n22} is MP but unstable. The total number of roots, t, is the biggest number among $n+u$, $v+m$ and the number of the specified roots. This is also the highest order for the polynomial θ_i.

Normally, one has to search for a set of solutions for the n_{ij}'s in θ_i from constant, order 1-polynomials, and then on to order 2 polynomials, etc. and at the same time to select those free stable r_i's such that the determinant of the N matrix satisfies the remaining requirements. An easier way is to assume that the free r_is are known and select the orders, u and v, to be high enough such that the number of n_{ij} coefficients in θ_i matches the number of the polynomial θ_i's coefficients, $t+1$. Then solve the variables c_i's and d_i's by Gauss-elimination in terms of r_is. After that, one can specify those free r_i's such that the determinant of the N matrix satisfies the requirement. Sometimes, one can just arbitrarily select those free r_i's as a first attempt.

The trick is that whenever the order of both n_{ij}'s in θ_i is increased by one, the order t of the polynomial θ_i is only increased by one so that a sufficient number of independent coefficients appear to match a desired polynomial. In particular, one obtains two more variables by adopting this procedure. Thus, it is possible to set up enough variables for Gauss-elimination.

2) *Applying only the M matrix transformation.* Using the M matrix transformation is similar to the use of the N matrix. Suppose $M^{-1} = \begin{bmatrix} \hat{m}_{11} & \hat{m}_{12} \\ \hat{m}_{21} & \hat{m}_{22} \end{bmatrix}$ exists for some M, where the $\hat{m}_{ij}$ are constants or polynomials of s. With the transformation $P_{1m} = M^{-1} \cdot P$ the resulting Q matrix is:

$$Q_{1m} = \begin{bmatrix} \dfrac{(\hat{m}_{11}\hat{m}_{22} - \hat{m}_{12}\hat{m}_{21})(z_{11}z_{22}p_{12}p_{21} - z_{12}z_{21}p_{11}p_{22})}{(\hat{m}_{21}z_{12}p_{22} - \hat{m}_{22}z_{22}p_{12})\,p_{11}p_{21}} & -\dfrac{(\hat{m}_{11}\hat{m}_{22} - \hat{m}_{12}\hat{m}_{21})(z_{11}z_{22}p_{12}p_{21} - z_{12}z_{21}p_{11}p_{22})}{(\hat{m}_{11}z_{12}p_{22} - \hat{m}_{12}z_{22}p_{12})\,p_{11}p_{21}} \\ -\dfrac{(\hat{m}_{11}\hat{m}_{22} - \hat{m}_{12}\hat{m}_{21})(z_{11}z_{22}p_{12}p_{21} - z_{12}z_{21}p_{11}p_{22})}{(\hat{m}_{21}z_{11}p_{21} - \hat{m}_{22}z_{21}p_{11})\,p_{12}p_{22}} & \dfrac{(\hat{m}_{11}\hat{m}_{22} - \hat{m}_{12}\hat{m}_{21})(z_{11}z_{22}p_{12}p_{21} - z_{12}z_{21}p_{11}p_{22})}{(\hat{m}_{11}z_{11}p_{21} - \hat{m}_{12}z_{21}p_{11})\,p_{12}p_{22}} \end{bmatrix}.$$

We then follow the same procedure as described for the N matrix to select the M matrix with the controller $G = G_{1m} \cdot M^{-1}$.

3) *Using both N and M transformation matrices.* When the use of a single matrix fails, one can use both M and N. This approach is more powerful and more cumbersome. The main difference between using a single matrix and both matrices is that no p_{ij} in the denominator is left unaffected in the diagonal entries of Q. In some cases where q_{n11} and q_{n22} (or, q_{m11} and q_{m22}) are both unstable, using both matrices appears to be the only way to achieve desirable equivalent plants. The reason is that one is not able to cancel the unstable pole in all the diagonal entries of Q by selecting entries of a single matrix. However, the complexity of using both matrices grows as the number of entries and the number of coefficients increases. Hence, although one can take all entries of M and N to be polynomials in s and then try to select their coefficients, a simpler approach is to arbitrarily assign one of the matrices to be a specified constant matrix and then try to find another s-polynomial matrix.

Using the transformation $P_{1mn} = M^{-1} \cdot P \cdot N$, the resulting Q matrix is:

$$Q_{1mn} = \begin{bmatrix} q_{mn11} & q_{mn12} \\ q_{mn21} & q_{mn22} \end{bmatrix},$$

where,

$$q_{mn11} = \frac{\left(n_{11}n_{22} - n_{12}n_{21}\right)\left(\hat{m}_{11}\hat{m}_{22} - \hat{m}_{12}\hat{m}_{21}\right)\left(z_{11}z_{22}p_{12}p_{21} - z_{12}z_{21}p_{11}p_{22}\right)}{\begin{pmatrix} n_{12}\hat{m}_{21}z_{11}p_{22}p_{12}p_{21} + n_{12}\hat{m}_{22}z_{21}p_{11}p_{22}p_{12} \\ +n_{22}\hat{m}_{21}z_{12}p_{11}p_{22}p_{21} + n_{22}\hat{m}_{22}z_{22}p_{11}p_{12}p_{21} \end{pmatrix}},$$

$$q_{mn22} = \frac{\left(n_{11}n_{22} - n_{12}n_{21}\right)\left(\hat{m}_{11}\hat{m}_{22} - \hat{m}_{12}\hat{m}_{21}\right)\left(z_{11}z_{22}p_{12}p_{21} - z_{12}z_{21}p_{11}p_{22}\right)}{\begin{pmatrix} n_{11}\hat{m}_{11}z_{11}p_{22}p_{12}p_{21} + n_{11}\hat{m}_{12}z_{21}p_{11}p_{22}p_{12} \\ +n_{21}\hat{m}_{11}z_{12}p_{11}p_{22}p_{21} + n_{21}\hat{m}_{12}z_{22}p_{11}p_{12}p_{21} \end{pmatrix}}.$$

Again, the aim is to select n_{ij} and $\hat{m}_{ij}$ such that the RHP dipoles in the diagonal entries of Q disappear. The resulting controller transfer function matrix (TFM) is $G = N \cdot G_{1mn} \cdot M^{-1}$.

Following the procedures outlined in the preceding section, we can find a set of M and N matrices such that the transformed Q matrix is more suitable for design. It should be noted that we have given only one possible way of finding M and N matrices. Many more M and N sets are possible. Our continuing research will explore other ways of prescribing these transformation matrices.

III. EXAMPLES

In this section two concocted examples are given to illustrate the proposed method.

Example 1. In this example we consider a control problem for a 2×2 LTI uncertain system possessing a RHP dipole. The uncertain 2×2 plant is given by:

$$P = \begin{bmatrix} \dfrac{k}{s+a} & \dfrac{-k}{s+a} \\ \dfrac{k}{s+a} & \dfrac{k}{s-2} \end{bmatrix}, \text{ where } 1 \le k \le 2 \text{ and } 1 \le a \le 2.$$

We want to design a controller G and a prefilter F such that for all plants in the family the closed-loop system is robustly stable and the closed-loop transfer function matrix satisfies a RP specification, $\left|b_{ij}(j\omega)\right| \ge \left|t_{ij}(j\omega)\right| \ge \left|a_{ij}(j\omega)\right|$ when $i = j$ and $\left|b_{ij}(j\omega)\right| \ge \left|t_{ij}(j\omega)\right| \ge 0$ when $i \ne j$ (Values listed in Table I). First we choose the specific plant corresponding to $k = 1$, and $a = 1$ as the nominal plant. Thus,

$$P_0 = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{-1}{s+1} \\ \dfrac{1}{s+1} & \dfrac{1}{s-2} \end{bmatrix} \text{ and } Det(P_0) = \frac{(2s-1)}{(s-2)(s+1)^2}.$$

Note that there is a RHP dipole in each member of the plant family, since $Det(P) = \dfrac{k^2(2s+a-2)}{(s+a)^2(s-2)}$. Consequently, the standard NS MIMO QFT design problem is impossible or extremely difficult to solve. The proposed design method alleviates this difficulty in the design using NS MIMO QFT.

Step 1: We determine M and N matrices based on the nominal plant so that the resulting two SISO equivalent nominal plants do not possess any RHP dipoles. If we apply the NS MIMO QFT to the nominal plant P_0, the resulting Q-matrix is:

$$Q_0 = \begin{bmatrix} \dfrac{(2s-1)}{(s+1)^2} & \dfrac{(2s-1)}{(s+1)(s-2)} \\ \dfrac{-(2s-1)}{(s+1)(s-2)} & \dfrac{(2s-1)}{(s+1)(s-2)} \end{bmatrix}.$$

Note that the entry q_{011} is NMP but stable and the entry q_{022} possesses a RHP dipole, making the design problem difficult to solve.

Table I. Robust Performance Specification for Example 1

ω	0.001	0.01	0.1	1	2
$\lvert b_{11}(j\omega)\rvert$	20	20	15	7	4
$\lvert a_{11}(j\omega)\rvert$	-70	-50	-30	-10	-10
$\lvert b_{22}(j\omega)\rvert$	.01	.01	.0001	.00002	.00002
$\lvert a_{22}(j\omega)\rvert$	-.91	-.91	-.9999	-1	-1
$\lvert b_{12}(j\omega)\rvert$	25	25	25	5	5
$\lvert b_{21}(j\omega)\rvert$	-24	-40	-50	-60	-70

ω	5	10	20	40	80
$\lvert b_{11}(j\omega)\rvert$	0	-5	-10	-15	-20
$\lvert a_{11}(j\omega)\rvert$	-15	-20	-25	-30	-35
$\lvert b_{22}(j\omega)\rvert$	.00005	.00005	.00005	-5	-10
$\lvert a_{22}(j\omega)\rvert$	-1	-3	-6	-16	-21
$\lvert b_{12}(j\omega)\rvert$	-6	-6	-6	-11	-16
$\lvert b_{21}(j\omega)\rvert$	-80	-90	-90	-90	-90

ω	160	320	600	1200	2400
$\lvert b_{11}(j\omega)\rvert$	-25	-32	-37	-42	-47
$\lvert a_{11}(j\omega)\rvert$	-40	-46	-55	-62	-70
$\lvert b_{22}(j\omega)\rvert$	-15	-16	-18	-20	-20
$\lvert a_{22}(j\omega)\rvert$	-22	-25	-29	-31	-36
$\lvert b_{12}(j\omega)\rvert$	-21	-21	-21	-26	-26
$\lvert b_{21}(j\omega)\rvert$	-90	-90	-90	-90	-90

Units: rad/sec; dB

Suppose we let M be the identity matrix and apply only the N matrix transformation to the system such that

$P_{1n} = P \times N$. All the entries of N can be constants or polynomials of s and N^{-1} must exist. With this choice the nominal equivalent plant matrix (Q-matrix) is:

$$Q_{0n} = \begin{bmatrix} \frac{(n_{11}n_{22} - n_{12}n_{21})(2s-1)}{((n_{12}+n_{22})s - 2n_{12} + n_{22})(s+1)} & \frac{-(n_{11}n_{22} - n_{12}n_{21})(2s-1)}{(n_{12}-n_{22})(s+1)(s-2)} \\ \frac{-(n_{11}n_{22} - n_{12}n_{21})(2s-1)}{((n_{11}+n_{21})s - 2n_{11} + n_{21})(s+1)} & \frac{(n_{11}n_{22} - n_{12}n_{21})(2s-1)}{(n_{11}-n_{21})(s+1)(s-2)} \end{bmatrix}.$$

Considering the above TFM, we wish to select n_{11} and n_{22} such that q_{0n11} is NMP and stable and select n_{11} and n_{21} such that q_{0n22} is MP and unstable. Consequently, the following constraints are enforced:

- $(n_{11}n_{22} - n_{12}n_{21}) = f(s)$, where $f(s)$ is stable.
- $\theta_1 = (n_{12} + n_{22})s - 2n_{12} + n_{22}$, where θ_1 is stable.
- $\theta_2 = (n_{11} - n_{21}) = (2s-1)$, where it cancels the NMP zero.

Let $n_{11} = as + b, n_{21} = cs + d \Rightarrow a - c = 2, b - d = -1$. Choose $a = d = 3$, $b = 2$, $c = 1$. Necessarily, $n_{12} + n_{22} > 0$ and $n_{22} - 2n_{12} > 0$. Choose $n_{12} = 1$, $n_{22} = 3$. Therefore $N = \begin{bmatrix} 3s+2 & 1 \\ s+3 & 3 \end{bmatrix}$ and $Det(N) = 8s + 3$. This leads to the following Q-matrix which importantly has no RHP dipoles in either diagonal entry:

$$Q_{0n} = \begin{bmatrix} \frac{(8s+3)(2s-1)}{(4s+1)(s+1)} & \frac{0.5(8s+3)(2s-1)}{(s+1)(s-2)} \\ \frac{-(8s+3)}{(s+1)(2s+1)} & \frac{(8s+3)}{(s+1)(s-2)} \end{bmatrix}.$$

Step 2: We now apply the chosen M and N to our original problem and transform it into a new design problem. Using the transformation $P_1 = M^{-1} \times P \times N$ the transformed plant is:

$$P_1 = \begin{bmatrix} \frac{k(2s-1)}{s+a} & \frac{-2k}{s+a} \\ \frac{k(4s^2 + (a-1)s + 3a - 4)}{(s+a)(s-2)} & \frac{k(4s - 2 + 3a)}{(s+a)(s-2)} \end{bmatrix},$$

where $1 \le k \le 2$ and $1 \le a \le 2$. It remains to design a diagonal controller $G_1 = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}$ and prefilter $F_1 = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$ such that for all P_1 the system is robustly stable and its closed-loop TFM satisfies the performance specification in Table I. The performance specification for the transformed problem is the same as the performance specification for the original problem because $T_1 = M^{-1}T$ and $M = I$. It is worth noting that in this transformed NS MIMO QFT design problem, the nominal equivalent SISO plants are free from any RHP dipoles. Thus, one can easily stabilize the nominal plant.

Step 3: We now perform the NS MIMO QFT design on the transformed design problem. Since the design problem asks for the minimum magnitude off-diagonal elements of the closed-loop TFM, we let $f_{12} = 0, f_{21} = 0$. Due to the transformation of the design problem loop 1 can be easily stabilized using a small gain controller and loop 2 can be easily stabilized using a large gain controller. The chosen controller TFM is $G_1 = \mathrm{diag}\left[\frac{0.12}{s+1}, \frac{8.965 \times 10^9}{(s + 8.15 \times 10^4)}\right]$. The chosen prefilter TFM is $F_1 = \mathrm{diag}\left[3.7, \frac{106.67(s+230)}{(s+18)(s+1420)}\right]$.

With the RP specifications and the RS specification satisfied, the closed-loop system is guaranteed to be robustly stable (see Theorem 1 in [4]). Note that, whilst this control system serves to demonstrate the application of design process, it might be possible to reduce the bandwidth of the controller by employing a higher order controller.

Step 4: Inverse Transformation and Verification. We now assess the effect of the final design on the original system. The real controller for our original uncertain plant is given by,

$$G = N \times G_1 = \begin{bmatrix} \frac{3(3s+2)}{25(s+1)} & \frac{8.965 \times 10^9}{(s+81500)} \\ \frac{3(s+3)}{25(s+1)} & \frac{2.6895 \times 10^{10}}{(s+81500)} \end{bmatrix}.$$

The prefilter for our original plant is given by $F = F_1$ since $M = I$. We now calculate the closed-loop transfer function matrices for checking stability. The MIMO poles were calculated for different plants in the plant family and it was verified that all poles have negative real part. Moreover, there is no pole-zero cancellations between P and G. Thus, the uncertain plant is internally stabilized by the controller G.

Example 2. Consider the following 2×2 LTI plant with NMP transmission zero and unstable pole:

$$P_0 = \begin{bmatrix} \frac{1}{(s+1)(s+5)} & 0 \\ 0 & \frac{(s-1)}{(s-2)(s+3)} \end{bmatrix},$$

where $Q_0 = P_0$ and $Det(P_0) = \frac{s-1}{(s+1)(s+5)(s-2)(s+3)}$.

By employing transformation matrices we can transform the plant TFM such that neither of the SISO equivalent plants possess a RHP dipole. This plant cannot be handled by just the N matrix. Thus, we arbitrarily choose $M^{-1}=\begin{bmatrix}2 & 5\\ 3 & 1\end{bmatrix}$ resulting in:

$$P_{0m}=\begin{bmatrix}2 & 5\\ 3 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{(s+1)(s+5)} & 0\\ 0 & \frac{(s-1)}{(s-2)(s+3)}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{2}{(s+1)(s+5)} & \frac{5(s-1)}{(s-2)(s+3)}\\ \frac{3}{(s+1)(s+5)} & \frac{(s-1)}{(s-2)(s+3)}\end{bmatrix}.$$

We then select the N matrix so that q_{0mn11} is NMP but stable and q_{0mn22} is MP but unstable. This gives,

$$P_{0mn}=P_{0m}\times N=M^{-1}\times P_0\times N,$$

and $$Q_{0mn}=\left[q_{0mnij}\right],$$

where,

$$q_{0mn11}=\frac{-13(s-1)(n_{11}n_{22}-n_{12}n_{21})}{\mathrm{n}_{22}\mathrm{s}^3+(3\mathrm{n}_{12}+5\mathrm{n}_{22})\mathrm{s}^2+(3\mathrm{n}_{12}-\mathrm{n}_{22})\mathrm{s}-18\mathrm{n}_{12}-5\mathrm{n}_{22}},$$

$$q_{0mn22}=\frac{-13(s-1)(n_{11}n_{22}-n_{12}n_{21})}{5\mathrm{n}_{21}\mathrm{s}^3+(2\mathrm{n}_{11}+25\mathrm{n}_{21})\mathrm{s}^2+(2\mathrm{n}_{11}-5\mathrm{n}_{21})\mathrm{s}-12\mathrm{n}_{11}-25\mathrm{n}_{21}}.$$

Now we enforce the following constraints:

– $(n_{11}n_{22}-n_{12}n_{21})=f(s)$, where $f(s)$ is stable

– $\theta_1=\mathrm{n}_{22}\mathrm{s}^3+(3\mathrm{n}_{12}+5\mathrm{n}_{22})\mathrm{s}^2+(3\mathrm{n}_{12}-\mathrm{n}_{22})\mathrm{s}-18\mathrm{n}_{12}-5\mathrm{n}_{22}$, where θ_1 is stable

– $\theta_2=5\mathrm{n}_{21}\mathrm{s}^3+(2\mathrm{n}_{11}+25\mathrm{n}_{21})\mathrm{s}^2+(2\mathrm{n}_{11}-5\mathrm{n}_{21})\mathrm{s}-12\mathrm{n}_{11}-25\mathrm{n}_{21}=(s-1)(\mathrm{s}-2)g(s)$

The selected N is:

$$N=\begin{bmatrix}1.01s^2+0.12s-1.13 & -0.8095s^2-5s-4.1905\\ -2.03s+4.06 & 3.4286s+10.2857\end{bmatrix},$$

where $Det(N)=1.8195s^3+3.9367s^2+9.1533s+5.3905$
$=1.8195(s+0.7456)(s+0.7090+1.8630\mathrm{i})(s+0.7090-1.8630\mathrm{i})$.

With q_{0mn11} NMP and stable and q_{0mn22} MP and unstable, a small gain is chosen to stabilize q_{0mn11} and a large gain to stabilize q_{0mn22}. The resulting controller is:

$$G_{0mn}=\mathrm{diag}\left[\frac{-0.33}{(s+3)(s+4)},\frac{40}{(s+5)(s+6)}\right].$$

Thus,

$$G_0=\begin{bmatrix}\frac{-97.8066s^4-1287.4s^3-5888.7s^2-10714s-6011.9}{(s+3)(s+4)(s+5)(s+6)} & \frac{-34.0465s^4-445.1895s^3-2006.5s^2-3558.8s-1955.5}{(s+3)(s+4)(s+5)(s+6)}\\ \frac{412.7718\mathrm{s}^3+4126.4\mathrm{s}^2+13588\mathrm{s}+14731}{(s+3)(s+4)(s+5)(s+6)} & \frac{140.4935\mathrm{s}^3+1401.6\mathrm{s}^2+4552.5\mathrm{s}+4736.2}{(s+3)(s+4)(s+5)(s+6)}\end{bmatrix}$$

Analysis confirms that the closed-loop system with controller G_0 and plant P_0 is internally stable, with closed-loop TFM and sensitivity TFM stable and no RHP pole-zero cancellations between P_0 and G_0.

IV. CONCLUSIONS

In this paper, a transformation scheme is proposed to facilitate the synthesis of controllers for NMP and unstable systems using NS MIMO QFT. Through the use of transformation matrices it was shown that one can obtain a new set of equivalent q_{ii} with desired stable and/or MP structure. In effect the scheme helps to eliminate the presence of spurious RHP dipoles that may appear in the equivalent SISO plants, the q_{ii}, and hence makes the design feasible using NS MIMO QFT. A systematic procedure to prescribe the transformation matrices is provided. Different from the standard NS MIMO QFT, the transformation scheme leads to a fully populated controller. Thus, it is possible to systematically design a robust controller for a MIMO system with RHP dipoles using NS MIMO QFT. The proposed transformations apply in many situations making NS MIMO QFT a viable design method for a much larger class of problems than has been possible thus far.

V. REFERENCES

[1] Isaac Horowitz, "Quantitative synthesis of uncertain multiple input-output feedback system", International Journal of Control, 30(1), 1979, pp81-106.

[2] C.H. Houpis, and, S. J. Rasmussen, *Quantitative Feedback Theory: Fundamentals and Applications*, Marcel Dekker, New York; 1999.

[3] O. Yaniv, *Quantitative feedback design of linear and nonlinear control system*, Kluwer Academic Publisher, Norwell, MA; 1999.

[4] M. L. Kerr, S. Jayasuriya, and S. F. Asokanthan, "On Stability in Non-Sequential MIMO QFT Designs", ASME Journal of Dynamic Systems, Measurement, and Control (Accepted).