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Finite Frequency Property-Based Robust Control Analysis and Synthesis

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***Abstract*— This paper derives two linear matrix inequality (LMI) conditions for robust control analysis based on finite frequency properties (FFP) and proposes robust control synthesis based on FFP. Solvability conditions of the synthesis problem are derived in the form of LMI condition adding a rank condition. The conditions are still non-convex due to the existence of the rank condition. However, we can solve the problem by using a more efficient proposed method from the viewpoint of computation.**

I. INTRODUCTION

In the case of control system synthesis based on $\mathcal{H}_\infty$ control, for example, in order to improve a following performance of the control system to a step change in the reference input, we have to decrease a gain of the sensitivity function which is a transfer function from the reference input to the error output in broader low frequency band. Therefore, so far, a weighting function $W_S(j\omega)$ is chosen as shown in Figure **1** and the synthesis has been done so that the gain of sensitivity function $S(j\omega)$ may become smaller than that of $W_S(j\omega)^{-1}$ at all the point of frequency ω. However, it is necessary to select the weighting function appropriately when using this method, and the control performance depends on the weighting function greatly.

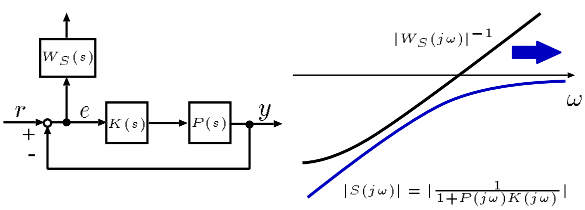

Fig. **1**. Loopshaping.

Then, this paper considers a synthesis method that improves the following control performance directly based on finite frequency properties (FFP) [1]–[3] without using the weighting function. Our basic idea is that we can expect an improvement of the following performance, for example, the settling time in terms of the step response, if the gain of sensitivity function $S(j\omega)$ becomes smaller than γ in the broader frequency band like the shaded portion in the left picture of Figure **2**, or if the gain of complementary sensitivity function $T(j\omega)$ becomes bigger than γ in the broader frequency band like the shaded portion in the right picture of Figure **2**. This paper derive a robust control analysis and synthesis condition based on FFP. Moreover, we propose a new design algorithm using convex relaxation in order to solve the synthesis problem.

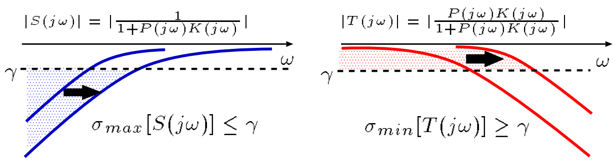

Fig. **2**. A basic idea based on FFP.

Notation: For a matrix A, its minimum singular value and its maximum singular value are denoted by $\sigma_{min}(A)$ and $\sigma_{max}(A)$, respectively. For matrices B and C, their Kronecker product is denoted by $B \otimes C$. The set of 2×2 Hermitian matrices are denoted by $\mathbf{H}$. A quadratic form function $\sigma : \mathbb{C} \times \mathbf{H} \to \mathbb{R}$ is defined by

$$\sigma(\lambda, \Pi) := \left[\begin{array}{c} \lambda \\ 1 \end{array}\right]^* \Pi \left[\begin{array}{c} \lambda \\ 1 \end{array}\right].$$

II. FFP-BASED ROBUST CONTROL ANALYSIS

We shall extend the control analysis condition [1], [2] of FFP to two corresponding **robust** control analysis conditions. Here the FFP mean generalized properties of the small gain condition ($\sigma_{max}[G(\lambda)] \le \gamma$), the high gain condition ($\sigma_{min}[G(\lambda)] \ge \gamma$), and the positive real condition ($G(\lambda)^* + G(\lambda) \ge 0$) in a finite frequency band where λ is the continuous time operator s or the discrete time operator z.

A. Analysis Problem Formulation

Consider an uncertain feedback system depicted in Figure **3** where $\Delta \in \boldsymbol{\Delta}$ is an uncertain matrix belonging to a known subset of complex matrices $\boldsymbol{\Delta}$ defined by

$$\begin{aligned} \boldsymbol{\Delta} := \{\, & \Delta = \mathsf{diag}(\delta_1 I_{n_1}, \ldots, \delta_s I_{n_s}, \Delta_1, \ldots, \Delta_f) : \\ & \delta_i \in \mathbb{R}\ {}^\forall i,\ \Delta_j \in \mathbb{C}^{p_j \times q_j}\ {}^\forall j,\ \|\Delta\| \le 1 \} \end{aligned} \quad (1)$$

and a state space representation of $\mathcal{H}(s)$ is given by

$$\left[\begin{array}{c} \dot{x} \\ z_1 \\ z_2 \end{array}\right] = \left[\begin{array}{ccc} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{array}\right] \left[\begin{array}{c} x \\ w_1 \\ w_2 \end{array}\right]. \quad (2)$$

Let the transfer function from input w_2 to output z_2 be denoted by $W_\Delta(s)$. Note that coefficients of a state space

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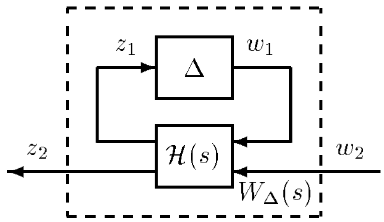

Fig. **3**. An uncertain system.

representation of $W_\Delta(s)$ are given by

$$\begin{bmatrix} \mathcal{A}_\Delta & \mathcal{B}_\Delta \\ \mathcal{C}_\Delta & \mathcal{D}_\Delta \end{bmatrix} := \begin{bmatrix} \mathcal{A} & \mathcal{B}_2 \\ \mathcal{C}_2 & \mathcal{D}_{22} \end{bmatrix} + \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{D}_{21} \end{bmatrix} \times (I - \Delta\mathcal{D}_{11})^{-1}\Delta \begin{bmatrix} \mathcal{C}_1 & \mathcal{D}_{12} \end{bmatrix} \quad (3)$$

where

$$\det(I - \Delta\mathcal{D}_{11}) \neq 0, \quad {}^\forall\Delta \in \boldsymbol{\Delta}$$

and $W_\Delta(s)$ can be defined on the imaginary axis.

We describe the following analysis problem and derive analysis conditions of FFP for the robust control system.

Analysis Problem: Derive conditions satisfying (i) or (ii): that is satisfying the FFP for robust control system where the transfer function matrix $W_\Delta(s)$ with Δ, $\Omega = \Omega^*$ and

$$\Xi := \Big\{ (\Phi, \Psi) \,|\, {}^\exists\alpha, \beta \in \mathbb{R},\ N \in \mathbb{C}^{2\times 2},\ \alpha \det(N) \neq 0$$

$$\Phi = N^* \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} N,\ \Psi = N^* \begin{bmatrix} -1 & \beta \\ \beta & 1 \end{bmatrix} N \Big\}, \quad (4)$$

$$\Gamma := \{\lambda \in \mathbb{C} \,|\, \sigma(\lambda, \Phi) = 0,\ \sigma(\lambda, \Psi) \geq 0\}$$

are given.

$$(i) \quad \det(\lambda I - A_\Delta) \neq 0, \quad {}^\forall\lambda \in \Gamma, \quad {}^\forall\Delta \in \boldsymbol{\Delta},$$
$$\begin{bmatrix} (\lambda I - \mathcal{A}_\Delta)^{-1}\mathcal{B}_\Delta \\ I \end{bmatrix}^* \Omega \begin{bmatrix} (\lambda I - \mathcal{A}_\Delta)^{-1}\mathcal{B}_\Delta \\ I \end{bmatrix} \leq 0,$$
$${}^\forall\lambda \in \Gamma, \quad {}^\forall\Delta \in \boldsymbol{\Delta}. \quad (5)$$

$$(ii) \quad {}^\exists\, Q_\Delta = Q_\Delta^* > 0, \quad P_\Delta = P_\Delta^* \quad \text{s.t.}$$
$$\begin{bmatrix} \mathcal{A}_\Delta & \mathcal{B}_\Delta \\ I & 0 \end{bmatrix}' L(P_\Delta, Q_\Delta) \begin{bmatrix} \mathcal{A}_\Delta & \mathcal{B}_\Delta \\ I & 0 \end{bmatrix} + \Omega \leq 0, \quad (6)$$
$$L(P_\Delta, Q_\Delta) := \Phi \otimes P_\Delta + \Psi \otimes Q_\Delta,\ (\Phi, \Psi) \in \Xi.$$

(i) and (ii) are equivalent conditions. Ω is shown as follows:

$$\Omega = \begin{bmatrix} \mathcal{C}_\Delta & \mathcal{D}_\Delta \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} \mathcal{C}_\Delta & \mathcal{D}_\Delta \\ 0 & I \end{bmatrix}.$$

The special cases of (5) and (6) can express the small gain condition, the high gain condition and the positive real condition by characterizing Π. Moreover, a continuous and discrete time system can be shown by characterizing Φ and the finite frequency range can be shown by characterizing Ψ.

B. Analysis Condition

Two sufficient analysis conditions of FFP for the robust control system are described in Theorem 1 [3] and Theorem 2. First, we consider P_Δ and Q_Δ as standard affine functions depending on the uncertain parameter Δ where P_Δ and Q_Δ are characterized as follows:

$$P_\Delta = P_0 + H'\tilde{\Delta}\Lambda H, \quad Q_\Delta = Q_0 + H'\tilde{\Delta}\tilde{\Lambda} H,$$
$$\tilde{\boldsymbol{\Delta}} := \{\tilde{\Delta} = \mathrm{diag}(\delta_1 I_{v_1}, \ldots, \delta_m I_{v_m}, \Delta_1, \ldots, \Delta_f) :$$
$$\delta_i \in \mathbb{R}\ {}^\forall i,\ \Delta_j \in \mathbb{C}^{p_j \times q_j}\ {}^\forall j,\ \|\tilde{\Delta}\| \leq 1\}. \quad (7)$$

H is a given constant matrix. Λ and $\tilde{\Lambda}$ are constant symmetric matrices with the commutative structure of $\tilde{\Delta}$.

Theorem 1: Let $W_\Delta(s)$, $(\Phi, \Psi) \in \Xi$ and Δ be given by (3), (4) and (1). Then finite frequency properties for the robust control system (3) hold if there exit matrices $P_0 = P_0^*$, $Q_0 = Q_0^*$, $\Theta_1 = \Theta_1^*$, $\Theta_2 = \Theta_2^*$ satisfying

$$(i) \quad F_1 L(P_0, Q_0) F_1' + \frac{1}{2} F_1 L(\Lambda H, \tilde{\Lambda} H)' F_2'$$
$$+ \frac{1}{2} F_2 L(\Lambda H, \tilde{\Lambda} H) F_1' + F_3 \Pi F_3' + F_4 \Theta_1 F_4' < 0,$$

$$F_1 := \begin{bmatrix} \mathcal{A}' & I \\ \mathcal{B}_1' & 0 \\ \mathcal{B}_2' & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ F_2 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix},\ F_3 := \begin{bmatrix} \mathcal{C}_2' & 0 \\ \mathcal{D}_{21}' & 0 \\ \mathcal{D}_{22}' & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$F_4 := \begin{bmatrix} \mathcal{C}_1' & \mathcal{A}'H' & H' & 0 & 0 & 0 \\ \mathcal{D}_{11}' & \mathcal{B}_1'H' & 0 & I & 0 & 0 \\ \mathcal{D}_{12}' & \mathcal{B}_2'H' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

$$(ii) \quad -\begin{bmatrix} Q_0 & H'\tilde{\Lambda}'/2 \\ \tilde{\Lambda} H/2 & 0 \end{bmatrix} + \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix}' \Theta_2 \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix} < 0.$$

$$(iii) \quad \begin{bmatrix} I \\ \nabla^* \end{bmatrix}^* \Theta_1 \begin{bmatrix} I \\ \nabla^* \end{bmatrix} \geq 0, \quad {}^\forall\nabla \in \boldsymbol{\nabla},$$
$$\boldsymbol{\nabla} := \{\ \mathrm{diag}(\Delta, \tilde{\Delta}, \tilde{\Delta}) \mid {}^\forall\Delta \in \Delta\ \}.$$

$$(iv) \quad \begin{bmatrix} I \\ \tilde{\Delta}^* \end{bmatrix}^* \Theta_2 \begin{bmatrix} I \\ \tilde{\Delta}^* \end{bmatrix} \geq 0, \quad {}^\forall\tilde{\Delta} \in \tilde{\boldsymbol{\Delta}}.$$

Proof: The proof is omitted for the space. ■

Next, we consider P_Δ and Q_Δ as rational functions depending on the uncertain parameter Δ where P_Δ and Q_Δ are characterized as follows:

$$P_\Delta := \begin{bmatrix} I \\ N_\Delta \end{bmatrix}^* P \begin{bmatrix} I \\ N_\Delta \end{bmatrix}, \quad P = P^*,$$
$$Q_\Delta := \begin{bmatrix} I \\ N_\Delta \end{bmatrix}^* Q \begin{bmatrix} I \\ N_\Delta \end{bmatrix}, \quad Q = Q^*,$$
$$N_\Delta := (I - \Delta\mathcal{D}_{11})^{-1}\Delta\mathcal{C}_1.$$

The following theorem is obtained by using the above rational parameter dependent functions.

Theorem 2: Let $W_\Delta(s)$, $(\Phi, \Psi) \in \Xi$ and Δ be given by (3), (4) and (1). Then finite frequency properties for the robust control system (3) hold if there exit matrices $P = P^*$, $Q = Q^*$, $\Theta_1 = \Theta_1^*$, $\Theta_2 = \Theta_2^*$ satisfying

$$(i) \quad \mathcal{F}_1 L(P, Q)\mathcal{F}_1' + \mathcal{F}_2 \Pi \mathcal{F}_2' + \mathcal{F}_3 \Theta_1 \mathcal{F}_3' < 0,$$

$$\mathcal{F}_1 := \begin{bmatrix} \mathcal{A}' & 0 & I & 0 \\ \mathcal{B}_1' & 0 & 0 & 0 \\ \mathcal{B}_2' & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \end{bmatrix}, \quad \mathcal{F}_2 := \begin{bmatrix} \mathcal{C}_2' & 0 \\ \mathcal{D}_{21}' & 0 \\ \mathcal{D}_{22}' & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{F}_3 := \begin{bmatrix} \mathcal{C}_1' & \mathcal{C}_1' & \mathcal{A}'\mathcal{C}_1' & 0 & 0 & 0 \\ \mathcal{D}_{11}' & 0 & \mathcal{B}_1'\mathcal{C}_1' & I & 0 & 0 \\ \mathcal{D}_{12}' & 0 & \mathcal{B}_2'\mathcal{C}_1' & 0 & 0 & 0 \\ 0 & \mathcal{D}_{11}' & 0 & 0 & I & 0 \\ 0 & 0 & \mathcal{D}_{11}' & 0 & 0 & I \end{bmatrix}.$$

$$(ii) \quad -Q + \begin{bmatrix} \mathcal{C}_1 & \mathcal{D}_{11} \\ 0 & I \end{bmatrix}' \mathit{\Theta}_2 \begin{bmatrix} \mathcal{C}_1 & \mathcal{D}_{11} \\ 0 & I \end{bmatrix} < 0.$$

$$(iii) \quad \begin{bmatrix} I \\ \nabla^* \end{bmatrix}^* \mathit{\Theta}_1 \begin{bmatrix} I \\ \nabla^* \end{bmatrix} \geq 0, \quad {}^\forall \nabla \in \boldsymbol{\nabla},$$

$$\boldsymbol{\nabla} := \{ \, \mathsf{diag}(\Delta, \Delta, \Delta) \mid {}^\forall \Delta \in \boldsymbol{\Delta} \, \}.$$

$$(iv) \quad \begin{bmatrix} I \\ \Delta^* \end{bmatrix}^* \mathit{\Theta}_2 \begin{bmatrix} I \\ \Delta^* \end{bmatrix} \geq 0, \quad {}^\forall \Delta \in \boldsymbol{\Delta}.$$

Proof: The proof is omitted for the space. ■

The infinite inequality conditions (iii) and (iv) in Theorem 1 and Theorem 2 can be eliminated by using the D-G scaling etc. as a special structure of Θ_1, Θ_2, $\mathit{\Theta}_1$ and $\mathit{\Theta}_2$ but this approach is conservative. However, it can easily check whether the robust control system satisfies FFP by solving LMI conditions with a computer.

C. Numerical Analysis Example

We confirm whether the analysis conditions in Theorem 1 and Theorem 2 become analysis conditions in consideration of robustness by using numerical examples.

> **Problem:** Consider a system having uncertainties of k and ζ given by (2) with
>
> $$W_\Delta(s) = \frac{k + \delta_k}{s^2 + 2(\zeta + \delta_\zeta)s + 1},$$
> $$k = 1, \ \zeta = 0.5, \ |\delta_k| \leq 0.2, \ |\delta_\zeta| \leq 0.2.$$
>
> Maximize a frequency ω_0 using Theorem 1 and Theorem 2 satisfying $\sigma_{min}[W_\Delta(s)] \geq \gamma$.

Since the above problem considers the continuous system and the high gain condition in low frequency, parameters in Theorem 1 and 2 are set as follows:

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \Psi = \begin{bmatrix} -1 & 0 \\ 0 & \omega_0^2 \end{bmatrix}, \Pi = \begin{bmatrix} -I & 0 \\ 0 & \gamma^2 I \end{bmatrix}.$$

The maximal frequency ω_0 satisfying $\sigma_{min}[W_\Delta(s)] \geq \gamma$ is calculated by using MATLAB. The results are obtained as follows:

$$(\gamma, \omega_0) = \begin{cases} (1/\sqrt{2}, \ 0.7414), \ (0.1, \ 2.8210) : \mathrm{Theorem1}, \\ (1/\sqrt{2}, \ 0.7413), \ (0.1, \ 2.8209) : \mathrm{Theorem2}. \end{cases}$$

In comparison with the above calculated results and values obtained by the bode diagrams in Figure **4**, we understand that the value in consideration of robustness is calculated. The FFP robust analysis conditions in Theorem 1 and Theorem 2 are valid.

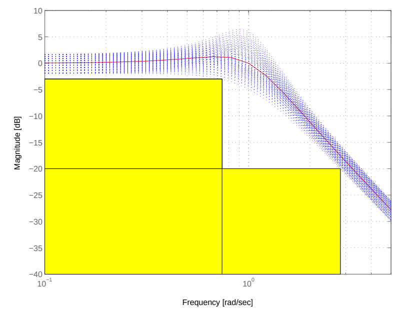

Fig. **4**. Bode diagrams.

III. FFP-Based Robust Control Synthesis

A solvability condition of a general FFP-based robust control synthesis problem is derived in the form of LMI conditions adding the rank condition.

A. Synthesis Problem Formulation

Consider the feedback system depicted in Figure **5** where a state space representation of a plant $G(s)$ is given by

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \\ \hdashline y \end{bmatrix} = \left[\begin{array}{ccc:c} A & B_1 & B_2 & B_3 \\ C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ \hdashline C_3 & D_{31} & D_{32} & 0 \end{array}\right] \begin{bmatrix} x \\ w_1 \\ w_2 \\ \hdashline u \end{bmatrix}$$

$$=: \left[\begin{array}{c:c} M_{11} & M_{12} \\ \hdashline M_{21} & M_{22} \end{array}\right] \begin{bmatrix} x \\ w_1 \\ w_2 \\ \hdashline u \end{bmatrix} =: M \begin{bmatrix} x \\ w_1 \\ w_2 \\ \hdashline u \end{bmatrix}, \quad (8)$$

and a state space representation of a general feedback controller $K(s)$ is given by

$$\begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix} =: \mathcal{K} \begin{bmatrix} x_c \\ y \end{bmatrix}. \quad (9)$$

Moreover, L and I_l are defined by

$$L := \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} := \left[\begin{array}{cc:cc} M_{11} & 0 & 0 & M_{12} \\ 0 & 0 & I & 0 \\ \hdashline 0 & I & 0 & 0 \\ M_{21} & 0 & 0 & 0 \end{array}\right],$$

$$I_l := \left[\begin{array}{ccc:c} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \\ \hdashline 0 & 0 & I & 0 \end{array}\right].$$

An appropriate size of the matrix L is chosen such that

$$\mathcal{M} = I_l(L_{11} + L_{12}\mathcal{K}L_{21})I_l \quad (10)$$

holds. In this case, the closed-loop system is described as follows:

$$\begin{bmatrix} \dot{x}_{cl} \\ z_1 \\ z_2 \end{bmatrix} = \mathcal{M} \begin{bmatrix} x_{cl} \\ w_1 \\ w_2 \end{bmatrix} =: \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix} \begin{bmatrix} x_{cl} \\ w_1 \\ w_2 \end{bmatrix} \quad (11)$$

where the state variable is defined by $x_{cl} := [x' \ \ x_c']'$.

Since the synthesis problem makes $\mathcal{M}$ become a variable, the analysis conditions (i) in Theorem 1 and

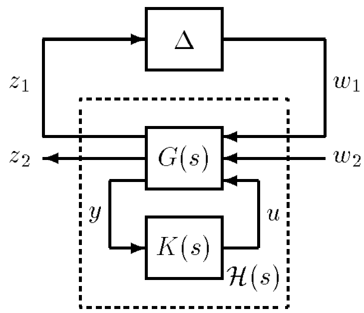

Fig. **5**. A general feedback control system.

Theorem 2 become non-convex conditions. Then the non-convex condition reduces to an LMI condition with a rank condition respectively such that a numerical computational method [4] is applicable.

> **Synthesis Problem:** Derive a necessary and sufficient condition where there exists the controller $K(s)$ defined by (9) satisfying Theorem 1 or Theorem 2.

B. Synthesis Condition

First we describe a basic lemma [5], [6] which will be used to derive the solvability condition of synthesis problem.

Lemma 1: A real symmetric matrix Υ is given by

$$\Upsilon := \begin{bmatrix} R & S \\ S' & U \end{bmatrix}$$

where $U > 0$. Then the following four statements are equivalent.

(a) $R - SU^{-1}S' < 0$.
(b) $\begin{bmatrix} S' & U' \end{bmatrix}'^{\perp} \Upsilon \begin{bmatrix} S' & U' \end{bmatrix}'^{\perp\prime} < 0$.
(c) There exists a real symmetric matrix W satisfying

$$\Upsilon < W, \quad W \geq 0, \quad \operatorname{rank} W = \operatorname{rank} U. \tag{12}$$

(d) There exists a real symmetric matrix V satisfying

$$X := \begin{bmatrix} R+V & S \\ S' & U \end{bmatrix} \geq 0, \quad V > 0, \tag{13}$$

$\operatorname{rank} X = \operatorname{rank} U$.

When we utilize the following condition

$$\Upsilon < W, \quad W \geq 0$$

except for the rank condition from (12), the statement (c) becomes a convex relaxation condition of the statement (a), equivalently, the statement (b) and the statement (d). We can obtain the same convex relaxation on the statement (d).

Using Lemma 1 to Theorem 1, we have the following theorem [3].

Theorem 3: Let a controlled object and $(\Phi, \Psi) \in \Xi$ be given by (8) and (4). Then the synthesis problem is solvable if and only if there exit $W \geq 0$, $U > 0$, R, S, K, $P_0 = P_0^*$, $Q_0 = Q_0^*$, $\Theta_1 = \Theta_1^*$, $\Theta_2 = \Theta_2^*$ satisfying analysis conditions (ii), (iii), (iv) in Theorem 1 and

$$(v) \quad \begin{bmatrix} R & S \\ S' & U \end{bmatrix} < W, \quad \operatorname{rank} W = \operatorname{rank} U$$

where R, S, U, T, J are defined by

$$R := T' \begin{bmatrix} L(P_0, Q_0) & \frac{1}{2}L(\Lambda H, \tilde{\Lambda})' & 0 & 0 \\ \frac{1}{2}L(\Lambda H, \tilde{\Lambda}) & 0 & 0 & 0 \\ 0 & 0 & \Pi & 0 \\ 0 & 0 & 0 & \Theta_1 \end{bmatrix} T,$$

$$S := \begin{bmatrix} -I & J \end{bmatrix}', \quad U := \mu I,$$

$$J := \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 & 0 & 0 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} & 0 & 0 \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} & 0 & 0 \\ H\mathcal{A} & H\mathcal{B}_1 & H\mathcal{B}_2 & 0 & 0 \\ H & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

$$T := \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

Proof: It is trivial from Lemma 1. ■

Solution of the synthesis problem to Theorem 2 is obtained by the following theorem.

Theorem 4: Let a controlled object and $(\Phi, \Psi) \in \Xi$ be given by (8) and (4). Then the synthesis problem is solvable if and only if there exit $\mathsf{W} \geq 0$, $\mathsf{U} > 0$, R, S, K, $P = P^*$, $Q = Q^*$, $\mathit{\Theta}_1 = \mathit{\Theta}_1^*$, $\mathit{\Theta}_2 = \mathit{\Theta}_2^*$ satisfying analysis conditions (ii), (iii), (iv) in Theorem 2 and

$$(v) \quad \begin{bmatrix} \mathsf{R} & \mathsf{S} \\ \mathsf{S}' & \mathsf{U} \end{bmatrix} < \mathsf{W}, \quad \operatorname{rank} \mathsf{W} = \operatorname{rank} \mathsf{U}$$

where R, S, U, $\mathcal{T}$, $\mathcal{J}$ are defined by

$$\mathsf{R} := \mathcal{T}' \begin{bmatrix} L(P, Q) & 0 & 0 \\ 0 & \Pi & 0 \\ 0 & 0 & \mathit{\Theta}_1 \end{bmatrix} \mathcal{T},$$

$$\mathsf{S} := \begin{bmatrix} -I & \mathcal{J} \end{bmatrix}', \quad \mathsf{U} := \mu I,$$

$$\mathcal{J} := \begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 & 0 & 0 \\ \mathcal{C}_1 & \mathcal{D}_{11} & \mathcal{D}_{12} & 0 & 0 \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_{22} & 0 & 0 \\ \mathcal{C}_1 & 0 & 0 & \mathcal{D}_{11} & 0 \\ \mathcal{C}_1\mathcal{A} & \mathcal{C}_1\mathcal{B}_1 & \mathcal{C}_1\mathcal{B}_2 & 0 & \mathcal{D}_{11} \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

$$\mathcal{T} := \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

Proof: It is trivial from Lemma 1. ■

The above solvability condition of the synthesis problem is derived in the form of LMI condition adding a rank condition. Note that the solvability condition of the robust well-posedness problem is also derived in the same form of the LMI condition adding a rank condition [5].

C. Design Method

In this subsection, we propose a new simple and effective design method in order to solve the synthesis conditions in Theorem 1, Theorem 2, the robust well-posedness problem [5], and so on.

1) Design Procedure: In order to solve the above problems, we generalize these problems to the following:

General Problem: Find a scalar $\mu > 0$ and matrices $W \geq 0$, R, S satisfying

$$\tilde{\Upsilon} := \begin{bmatrix} R & S \\ S' & \mu I_{\mathsf{n}} \end{bmatrix} < W := \begin{bmatrix} W_{11} & W_{12} \\ W_{12}' & wI_{\mathsf{n}} \end{bmatrix}, \tag{14}$$

$$\text{rank } (W) = \text{rank } (\mu I_{\mathsf{n}}) = \mathsf{n}. \tag{15}$$

The design procedure is described below. This approach is extension of the linearization algorithm in reference [7].

Design Procedure

1. Solve $\tilde{\Upsilon}$, W s.t. $\tilde{\Upsilon} < W$, $W \geq 0$ and $\mu > 0$.
2. $W_{11}^0 := W_{11}$, $W_{12}^0 := W_{12}$, $w^0 := w$ and let $j = 1$.
3. Fix $\mathcal{W}_{11} := W_{11}^{j-1}$, $\mathcal{W}_{12} := W_{12}^{j-1}$, $\tilde{w} := w^{j-1}$. Solve $\lambda_j := \min_{\mu>0, W \geq 0, R, S}$
$$\begin{cases} \mathsf{tr}[(\tilde{w}W_{11} + w\mathcal{W}_{11} - \tilde{w}\mathcal{W}_{11}) \\ -(W_{12}\mathcal{W}_{12}' + \mathcal{W}_{12}W_{12}' - \mathcal{W}_{12}\mathcal{W}_{12}')] \end{cases} \tag{16}$$
and let $\mathcal{W}_{11}^j := W_{11}$, $\mathcal{W}_{12}^j := W_{12}$, $w^j := w$.
4. If $\left|\frac{\lambda_j - \lambda_{j-1}}{\lambda_j}\right| \leq \tau$ for sufficiently small $\tau > 0$, then stop. Otherwise let $j \to j-1$ and go to **3.**

2) Explanation of the Design Procedure: We explain the above design procedure in the following explanation 1 and 2. First, we explains a numerical computational method to satisfy the rank condition step by step from the convex relaxation condition without the rank condition.

Explanation 1 of the Design Procedure

The following equation

$$W = \begin{bmatrix} I & W_{12}/w \\ 0 & I_{\mathsf{n}} \end{bmatrix} \begin{bmatrix} W_{11} - W_{12}W_{12}'/w & 0 \\ 0 & wI_{\mathsf{n}} \end{bmatrix} \times \begin{bmatrix} I & 0 \\ W_{12}'/w & I_{\mathsf{n}} \end{bmatrix} \tag{17}$$

holds. If $W_{11} - W_{12}W_{12}'/w = 0$, then the rank condition (15) holds. We can choose

$$\mathsf{tr}[wW_{11} - W_{12}W_{12}'] \tag{18}$$

as an objective function to be approached to 0 subject to LMI constraints and $\mathsf{tr}[wW_{11} - W_{12}W_{12}'] \geq 0$ due to $W \geq 0$ and $w > 0$. This is a simple and effective key idea of our method.

Explanation 2 of the Design Procedure

Next, a linear approximated function of a matrix product function $F(X) = XX'$ is obtained by the following:

$$(i) \quad \tilde{F}(X) = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial F}{\partial x_{ij}}\bigg|_{x_{ij}^0} (x_{ij} - x_{ij}^0) + F(X_0),$$

$$\begin{aligned}(ii) \quad \tilde{F}(X) &= X_0(X - X_0)' + (X - X_0)X_0' + X_0X_0' \\ &= X_0X' + XX_0' - X_0X_0'.\end{aligned}$$

(i) and (ii) are equivalent.

Proof: The (p, q) element of $F(X) = XX'$ is $\sum_{i=1}^{n} x_{pi}x_{qi}$ by using

$$F(X) = XX' = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} & \dots & x_{m1} \\ x_{12} & x_{22} & \dots & x_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \dots & x_{mn} \end{bmatrix}.$$

Hence, the (p, q) element of a linear approximated function is

$$\begin{aligned}\tilde{F}(X)_{pq} &= \sum_{i=1}^{n} x_{qi}^0(x_{pi} - x_{pi}^0) + \sum_{i=1}^{n} x_{pi}^0(x_{qi} - x_{qi}^0) + \sum_{i=1}^{n} x_{pi}^0x_{qi}^0 \\ &= \sum_{i=1}^{n} x_{pi}^0x_{qi} + \sum_{i=1}^{n} x_{pi}x_{qi}^0 - \sum_{i=1}^{n} x_{pi}^0x_{qi}^0.\end{aligned}$$

Moreover, the (p, q) element of (ii) becomes

$$\begin{aligned}\tilde{F}(X)_{pq} &= (X_0X' + XX_0' - X_0X_0')_{pq} \\ &= \sum_{i=1}^{n} x_{pi}^0x_{qi} + \sum_{i=1}^{n} x_{pi}x_{qi}^0 - \sum_{i=1}^{n} x_{pi}^0x_{qi}^0.\end{aligned}$$

Therefore, (i) and (ii) are equivalent. ■

Consequently, from the above equivalence, we can understand that the linearized function of (18) at the fixed point $(\tilde{w}, \mathcal{W}_{11}, \mathcal{W}_{12})$ becomes the function of (16).

IV. Numerical Synthesis Example

In order to confirm whether we can design a controller satisfying **robust** stability and finite frequency properties of a closed-loop system by using our proposed algorithm, this section shows a synthesis result of an easy numerical example with a finite feasible parameter space of a controller satisfying the robust stability in Figure **6**.

A. Problem Formulation of Numerical Synthesis Example

Problem: Consider a plant given by transfer function $P(s) = (s-1)/\{s^2 + (10+\delta)s - 1\}$ with an uncertainty: $|\delta| \leq 2$, and a designed PI controller given by $K(s) = p_0 + p_1/s$. Find the controller $K(s)$ with which a closed-loop system satisfies an internal stable, the complementary sensitivity function of the closed-loop system in Figure **1** satisfies a low gain condition $\sigma_{max}|G_{yr}(jw)| \leq 1.6$ for all the frequency domain, and ω_0 is maximized where $G_{yr}(s)$ satisfies the high gain condition $\sigma_{min}[G_{yr}(j\omega)] \geq$

0.55 in a finite frequency band $0 \leq \omega \leq \omega_0$ for the plant $P(s)$.

B. Analysis Result in the case of Given Controller Parameter (p_0, p_1)

First a true robust stability region of the closed-loop system is checked by using a plane of p_0 and p_1 in Figure **6** before the controller $K(s)$ is designed by using the proposed method in the previous section. Next a maximal value ω_0 of the frequency is estimated by Theorem 2, where the closed-loop system satisfies the low and high gain condition in the robust stability region for all ω such that $0 \leq \omega \leq \omega_0$.

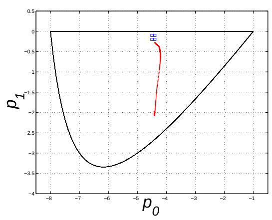

Fig. **6**. A robust stability region.

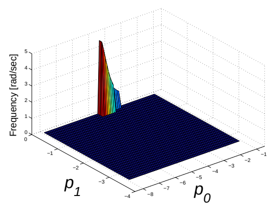

Fig. **7**. (p_0, p_1) v.s. ω_0.

Figure **7** indicates the maximal values ω_0 where the closed-loop system satisfies the design specification at fixed values of (p_0, p_1) respectively. The area of $\omega_0 = 0$ in Figure **7** means that the synthesis problem is infeasible. From Figure **7**, we can understand that the frequency band becomes broader when the values of p_0 and p_1 are small.

C. Synthesis Result

The PI controller is designed by our method using Theorem 4 where an observable canonical form of state space representation of the PI controller is utilized, and then the values of $(p_0, p_1) = (-4.3968, -0.2803)$ and $\omega_0 = 4.0$ are obtained. The thick curve line in Figure **6** also illustrates the movement of (p_0, p_1) while iterating to minimize the trace of objective function. We can understand that the values of (p_0, p_1) approach the small square region where the design specification is satisfied. Figure **8** shows $\sum_{i=1}^{2}[\mathrm{tr}(\tilde{w}_i W_{11i} + w_i \mathcal{W}_{11i} - W_{12i}\mathcal{W}'_{12i} - \mathcal{W}_{12i} W'_{12i})]$ as a function of the iteration numbers where $i = 1, 2$ means that there exist two conditions of low and high gain design specification. The properties of the designed closed system $G_{yr}(s)$ in Figure **1** are indicated by Figure **9** and Figure **10**. Figure **9** and Figure **10** illustrate bode diagrams and step responses of the closed system with arising the perturbations $|\delta| \leq 2$ respectively. We can understand that the design specification is satisfied and our proposed method is helpful.

V. Conclusion

This paper has derived the new analysis condition of the finite frequency properties for robust control systems. Moreover, in this paper, we have formulated the problem of the control synthesis based on the derived analysis condition. The solvability condition of the synthesis problem was derived in the form of LMI condition adding a rank condition. Finally, the effectiveness of this synthesis approach was confirmed through the numerical example solved by using the proposed design algorithm.

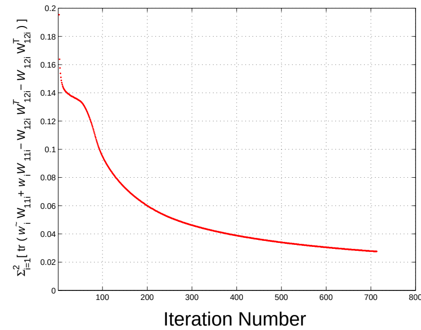

Fig. **8**. Iteration numbers v.s. values of trace.

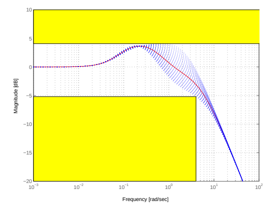

Fig. **9**. Bode diagrams.

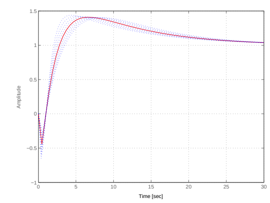

Fig. **10**. Step responses.

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References

[1] T. Iwasaki and S. Hara: LMI characterizations of finite frequency properties for linear time-invariant systems, *Proc. 31st SICE Symposium on Control Theory*, pp. 115-118 (2002)

[2] T. Iwasaki, S. Hara and H. Yamauchi: Dynamical system design from a control perspective: Finite frequency positive-realness approach, *IEEE Transactions on Automatic Control*, Vol. 48, No. 8, pp. 1337-1354 (2003)

[3] E. Nishio and T. Kiyama: Robust control analysis and synthesis based on finite frequency properties, *Proc. 2nd SICE Annual Conf.*, pp. 2704-2709 (2003)

[4] K. M. Grigoriadis and E. B. Beran: Alternating projection algorithms for linear matrix inequalities problems with rank constraints; *Advances in Linear Matrix Inequality Methods in Control* (L. El Ghaoui and S. Niculescu, Ed(s)), SIAM (2000)

[5] T. Kiyama, S. Toyora and S. Hara: A solvability condition of general robust control problems, *Proc. SICE Annual Conf.*, pp. 677-678 (2002)

[6] T. Kiyama: Sector condition-based saturating control analysis and synthesis; In consideration of the direct-feedthrough matrix from output to input of saturations, Systems, Control and Information, Vol. 47, No. 11, pp. 30-37 (2003)

[7] L. El Ghaoui, F. Oustry and M. AitRami: A cone complementarity linearization algorithm for static out-feedback and related problems, *IEEE Transactions on Automatic Control*, pp. 1171-1176 (1997)