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Piecewise Quadratic Lyapunov Functions for Piecewise Affine Time-Delay Systems

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***Abstract*— We investigate some particular classes of hybrid systems subject to a class of time delays; the time delays can be constant or time varying. For such systems, we present the corresponding classes of piecewise continuous Lyapunov functions.**

***Index Terms*— Lyapunov functions, hybrid systems, stability**

I. Introduction

Construction of Lyapunov functions is a fundamental problem in system theory — its importance stems from the fact that the internal stability of a system is concluded if an associated Lyapunov function is shown to exist. This paper concerns such a construction for a class of systems that are *hybrid* in the sense that the state trajectory evolution is governed by different dynamical equations over different polyhedral partitions X_i of the state-space X; i.e., the system is modelled by an ensemble of subsystems, each of which is a valid representation of the system over a set of such partitions. A motivating application for the study of such systems is described in [6].

Conceptually, perhaps the simplest solution is a *common quadratic* Lyapunov function, i.e. a quadratic function which is a global Lyapunov function for the subsystems comprising the hybrid system [3]. However, the construction of such a Lyapunov function is an $\mathcal{NP}$-hard problem even when the subsystems are linear time invariant [1]. Furthermore, the existence of such a function is, in principle, an overly restrictive requirement to deduce the stability [4, Section IV].

Conservatism introduced by a *global* Lyapunov function V can be reduced by searching for a set $\{V_i\}$ of *local* Lyapunov functions and by ensuring that the Lyapunov functions *match* in the sense that the values of Lyapunov functions V_i and V_j are equal when the state trajectory leaves a cell X_i and enters a cell X_j, where V_i is a local Lyapunov function in the cell X_i and V_j is a local Lyapunov function in the cell X_j (see [2] and [7]). In this context, an elegant result has been recently derived by [4] to construct Lyapunov functions when the subsystem dynamics are known to be affine time invariant; an independent interpretation of this result is given in [3]. For some practical applications, however, the piecewise affine structure must be modified to address modelling uncertainties and time delays [6]. For such systems, consequently, the stability conditions laid down by [4] get modified as we will demonstrate.

The paper is organized as follows. The notation and the key relevant concepts are introduced in Section II. The problems are formulated in Section III and the relevant prior art is described in Section IV. Our main results are presented in Section V and discussed in Section VI. The paper is concluded in Section VII. Formal proofs are presented in the Appendix.

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II. Preliminaries

The notation is introduced as and when necessary. Capital letter symbols, such as F and G, denote operators whereas small letter symbols, such as x and y, denote real signals which may possibly be vector valued or matrix valued. The set of all real (complex) numbers is denoted $\mathbb{R}$ ($\mathbb{C}$) and the set of all integers is denoted $\mathbb{Z}$. The notation $\doteq$ stands for 'defined as'. The inner product $\langle x, y\rangle \doteq \int_{-\infty}^{\infty} y(t)^T x(t)\; dt$. The Euclidean norm $\|x\| \doteq \sqrt{\langle x, x\rangle}$. The vector space of signals for which the Euclidean norm exists is denoted $\mathcal{L}_2^n$. The vector space $\mathcal{L}_2^n$ is generally referred to as $\mathcal{L}_2$. Fourier transform of x is denoted $\widehat{x}$. Conjugate transpose of a vector or matrix $(\cdot)$ is denoted $(\cdot)^*$; its transpose is denoted $(\cdot)^T$ and $\left((\cdot)^2\right)^T$ is denoted $(\cdot)^{2T}$. Given $z \in \mathbb{R}^{n\times n}$, $z \succeq 0$ implies that every element of z is nonnegative. The (i,j)-th element of a matrix $(\cdot)$ is denoted as either $(\cdot)_{i,j}$ or $(\cdot)_{ij}$, depending on the ease of reading. Time derivative of the signal x is denoted $\dot{x}$.

Definition 1 (Piecewise Affine Systems, [4]): The class $\mathcal{S}_H$ of hybrid systems is defined by a family of ordinary differential equations as:

$$\dot{x}(t) = A_i x(t) + a_i, \quad \forall\; x(t) \in X_i$$

where $A_i \in \mathbb{R}^{n\times n}, a_i \in \mathbb{R}^n$, and $\{X_i\}_{i\in I} \subset \mathbb{R}^n$ is a partition of the state-space into a finite number of closed, and possibly unbounded, polyhedral cells with pairwise disjoint interior. The set of cells that include the origin is denoted I_0, i.e. $a_i = 0,\ \forall\, i \in I_0$; its compliment is denoted I_1. □

Definition 2 (Piecewise Affine Time-Delay Systems $\mathcal{S}_{\tau c}$): The class $\mathcal{S}_{\tau c}$ of hybrid systems is defined by a family of retarded ordinary differential equations as:

$$\dot{x}(t) = A_i x(t) + A_{di} x(t-\tau) + a_i, \quad \forall\, x(t) \in X_i$$

where $A_i, A_{di} \in \mathbb{R}^{n\times n}, a_i \in \mathbb{R}^n$, $0 < \tau \in \mathbb{R}$ and $\{X_i\}_{i\in I} \subset \mathbb{R}^n$ is a partition of the state-space as in $\mathcal{S}$. The set of cells that include the origin is denoted I_0, i.e. $a_i = 0$, $\forall\, i \in I_0$; its compliment is denoted I_1. □

Definition 3 (Piecewise Affine Time-Delay Systems $\mathcal{S}_{\tau cL}$): The class $\mathcal{S}_{\tau cL}$ is obtained from the $\mathcal{S}_{\tau c}$ by replacing the term $A_{di}x(t-\tau)$ with the term $\sum_{\ell=1}^{L} A_{di\ell}x(t-\tau_\ell)$ where $A_{di\ell} \in \mathbb{R}^{n\times n}$, $0 < \tau_\ell \in \mathbb{R}$, and $0 < L \in \mathbb{Z}$. □

Definition 4 (Piecewise Affine Time-Delay Systems $\mathcal{S}_{\tau v}$): The class $\mathcal{S}_{\tau v}$ of hybrid systems is defined by a family of retarded ordinary differential equations as:

$$\dot{x}(t) = A_i x(t) + A_{di} x(t-\tau(t)) + a_i, \quad \forall\, x(t) \in X_i$$

where the time varying time delay is constrained as

$$0 \le \tau(t) \le h, \quad \dot{\tau}(t) \le d < 1 \qquad \forall t \in \mathbb{R},$$

for some $h, d \in \mathbb{R}$, $A_i, A_{di} \in \mathbb{R}^{n\times n}, a_i \in \mathbb{R}^n$, and $\{X_i\}_{i\in I} \subset \mathbb{R}^n$ is a partition of the state-space as in $\mathcal{S}$. The set of cells that include the origin is denoted I_0, i.e. $a_i = 0$, $\forall\, i \in I_0$; its compliment is denoted I_1. □

Definition 5 (Piecewise Affine Time-Delay Systems $\mathcal{S}_{\tau vL}$): The class $\mathcal{S}_{\tau vL}$ is obtained from the $\mathcal{S}_{\tau v}$ by replacing the term $A_{di}x(t-\tau(t))$ with the term $\sum_{\ell=1}^{L} A_{di\ell}x(t-\tau_\ell(t))$ where the time varying time delay is constrained as

$$0 \le \tau_\ell(t) \le h_\ell, \quad \dot{\tau}_\ell(t) \le d_\ell < 1 \qquad \forall t \in \mathbb{R},$$

$A_{di\ell} \in \mathbb{R}^{n\times n}$, $0 < \tau_\ell(t) \in \mathbb{R}$, and $0 < L \in \mathbb{Z}$. □

III. Problem Formulation

Problem 1: Determine a set of computationally tractable analytical conditions under which $\mathcal{S}_{\tau c}$ is stable. □

Problem 2: Determine a set of computationally tractable analytical conditions under which $\mathcal{S}_{\tau cL}$ is stable. □

Problem 3: Determine a set of computationally tractable analytical conditions under which $\mathcal{S}_{\tau v}$ is stable. □

Problem 4: Determine a set of computationally tractable analytical conditions under which $\mathcal{S}_{\tau vL}$ is stable. □

IV. Prior Art

An elegant result on the stability analysis of $\mathcal{S}_H$ is given by [4]. Briefly speaking, the development is as follows. Denote

$$\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}.$$

Let $\bar{E}_i = \begin{bmatrix} E_i \\ e_i \end{bmatrix}$, $\bar{F}_i = \begin{bmatrix} F_i \\ f_i \end{bmatrix}$, where $\begin{bmatrix} e_i \\ f_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\forall\, i \in I_0$, such that

$$\bar{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix} \succeq 0, \qquad \forall\, x \in X_i,\ i \in I;$$

$$\bar{F}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix}, \qquad \forall\, x \in X_i \cap X_j,\ \ i,j \in I. \tag{1}$$

Lemma 1 (Theorem 1, [4]): Consider symmetric matrices T, U_i, and W_i such that U_i and W_i have non negative entries while $P_i \doteq F_i^T T F_i$, for all $i \in I_0$, and $\bar{P}_j \doteq \bar{F}_j^T T \bar{F}_j$, for all $j \in I_1$, satisfy

$$A_i^T P_i + P_i A_i + E_i^T U_i E_i \ < \ 0 \tag{2}$$

$$P_i - E_i^T W_i E_i \ > \ 0 \tag{3}$$

$$\bar{A}_j^T \bar{P}_j + \bar{P}_j \bar{A}_j + \bar{E}_j^T U_j \bar{E}_j \ < \ 0 \tag{4}$$

$$\bar{P}_j - \bar{E}_j^T W_j \bar{E}_j \ > \ 0 \tag{5}$$

for all $i \in I_0$ and for all $j \in I_1$. Then, every piecewise continuous trajectory of $\mathcal{S}_H$ tends to zero exponentially. □

Remark 1: An independent interpretation, and a slight improvement, of this result is given in [3]. □

Remark 2: To ensure that the local Lyapunov functions match on the cell boundaries, [4] takes the predetermined matrices $\bar{F}_i$ and $\bar{F}_j$ as the given variables, the predetermination being as given by (1), and uses the elements of the matrix T as the free variables. Now, the condition (1) allows for a number of choices of $\bar{F}_i$ and $\bar{F}_j$ which might violate the matching condition, thereby incurring an unnecessarily high cost of computation. This can be avoided by working directly with the local Lyapunov functions P_i and P_j as the unknown variables and by stipulating that $P_i - P_j = 2\,\mathrm{herm}\,(F_{ij}K_{ij})$, $\forall\, i,j$ where the elements K_{ij} are known variables. □

V. Main Results

It is not possible to consider an aggregate state $\zeta(t) \doteq [x(t) \quad x(t-\tau)]^T$ and apply the arguments of [4] in a straightforward manner to the system of dynamical equations described in terms of ζ. This is so because, in general, it is difficult to deduce the cell containing $x(t-\tau)$ given that a particular cell contains $x(t)$ and, hence, it is difficult to state the correct matching conditions for the local Lyapunov functions. We now present solutions to Problem 1 and Problem 2. Denote

$$\bar{A}_{dj} \doteq \begin{bmatrix} A_{dj} & 0 \\ 0 & 0 \end{bmatrix}.$$

Lemma 2 (Solution to Problem 1): Consider symmetric matrices T, U_i and W_i such that U_i and W_i have non-negative entries while $P_i \doteq F_i^T T F_i$, for all $i \in I_0$, and $\bar{P}_j \doteq \bar{F}_j^T T \bar{F}_j$, for all $j \in I_1$, satisfy the following inequalities:

$$\begin{cases} \begin{bmatrix} H_i & \tau P_i & \tau A_i^T A_{di}^T R A_{di}^2 \\ \tau P_i & -\tau R & 0 \\ \tau A_{di}^{2T} R A_{di} A_i & 0 & \tau A_{di}^{2T} R A_{di}^2 - Q \end{bmatrix} < 0 \\ P_i - E_i^T W_i E_i > 0, \quad Q > 0, \quad R > 0 \end{cases} \tag{6}$$

$$\begin{cases} \begin{bmatrix} \bar{H}_j & \tau \bar{P}_j & \tau \bar{A}_j^T \bar{A}_{dj}^T \bar{R} \bar{A}_{dj}^2 \\ \tau \bar{P}_j & -\tau \bar{R} & 0 \\ \tau \bar{A}_{dj}^{2T} \bar{R} \bar{A}_{dj} \bar{A}_j & 0 & \tau \bar{A}_{dj}^{2T} \bar{R} \bar{A}_{dj}^2 - \bar{Q} \end{bmatrix} < 0 \\ \bar{P}_j - \bar{E}_j^T W_j \bar{E}_j > 0, \quad \bar{Q} > 0, \quad \bar{R} > 0 \end{cases} \tag{7}$$

for all $i \in I_0$ and all $j \in I_1$ where

$$\widetilde{A}_i \doteq A_i + A_{di}, \qquad \hat{A}_j \doteq \bar{A}_j + \bar{A}_{dj},$$
$$H_i \doteq \widetilde{A}_i^T P_i + P_i \widetilde{A}_i + Q + \tau A_i^T A_{di}^T R A_{di} A_i + E_i^T U_i E_i,$$
$$\bar{H}_j \doteq \hat{A}_j^T \bar{P}_j + \bar{P}_j \hat{A}_j + \bar{Q} + \tau \bar{A}_j^T \bar{A}_{dj}^T \bar{R} \bar{A}_{dj} \bar{A}_j + \bar{E}_j^T U_j \bar{E}_j.$$

Then, every piecewise continuous trajectory of $\mathcal{S}_{\tau c}$ tends to zero exponentially. □

Proof: See the proof in the Appendix section. ■

Remark 3: Lemma 1 may be derived as a special case of our Theorem 1 by setting $\tau = 0,\ A_{di} = 0, Q = 0$. This is so because the Lyapunov function used by [4] can be derived as a special of our Lyapunov function, given by (A.1), by setting the $V_2(\cdot)$ and $V_3(\cdot)$ terms to zero. □

Remark 4: A conservative delay-independent condition is formulated as follows:

$$\begin{cases} \begin{bmatrix} A_i^T P_i + P_i A_i + Q + E_i^T U_i E_i & P_i A_{di} \\ A_{di}^T P_i & -Q \end{bmatrix} < 0 \\ P_i - E_i^T W_i E_i > 0, \quad Q > 0 \end{cases} \tag{8}$$

$$\begin{cases} \begin{bmatrix} \bar{A}_j^T \bar{P}_j + \bar{P}_j \bar{A}_j + \bar{Q} + \bar{E}_j^T U_j \bar{E}_j & \bar{P}_j \bar{A}_{dj} \\ \bar{A}_{dj}^T \bar{P}_j & -\bar{Q} \end{bmatrix} < 0 \\ \bar{P}_j - \bar{E}_j^T W_j \bar{E}_j > 0, \quad \bar{Q} > 0 \end{cases} \tag{9}$$

for all $i \in I_0$ and $j \in I_1$. □

Remark 5: A further conservative condition, stated by the small gain theorem, is obtained by setting $Q = I$. □

Remark 6: A lower bound on the maximum delay τ^* for which the system $\mathcal{S}_\tau$ is stable can be obtained by checking whether the conditions laid down by Theorem 1 are satisfied as τ increases, starting with $\tau = 0$: the least value τ^* for which the conditions laid down by Theorem 1 are not satisfied, is a conservative estimate of the maximum delay τ under which the system $\mathcal{S}_\tau$ is stable. □

Example 1: Consider the following piecewise linear time-delay system $\dot{x}(t) = A_i x(t) + A_{di} x(t-\tau)$ with the cell decomposition expressed by $E_i x \succeq 0$,

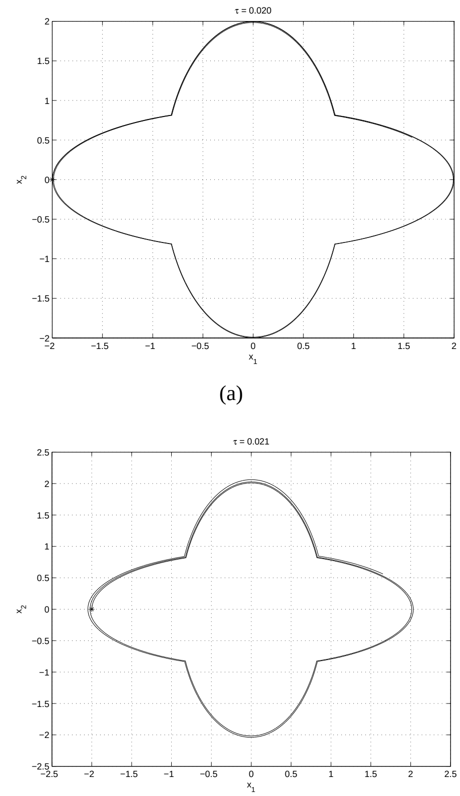
$$E_1 = -E_3 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad E_2 = -E_4 = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The system matrices are given by

$$A_1 = A_3 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix},$$
$$A_{d1} = A_{d3} = \begin{bmatrix} 0 & 5 \\ -1 & 0 \end{bmatrix}, \quad A_{d2} = A_{d4} = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}.$$

The system is reduced to Example 1 in [4] when $\tau = 0$. It can be verified from Eq. (8) that the system is not stable regardless of delay. By applying Lemma 2, the estimated delay margin is $\tau^* = 0.0142$. We can observe from simulations that the system becomes unstable with time-delay between 0.020 and 0.021 with initial value $x_0 = \begin{bmatrix} -2 & 0 \end{bmatrix}^T$. See Figure 1. □

Remark 7: By applying the delay-dependent condition in [5] and [8], the same procedure as in Lemma 2 yields



(a)

(b)

Fig. 1. State trajectories of the system in Example 1 with (a) $\tau = 0.020$, and (b) $\tau = 0.021$.

the condition

$$\begin{bmatrix} H_i & \tau P_i A_{di} A_i & \tau P_i A_{di}^2 \\ \tau A_i^T A_{di}^T P_i & -\tau Q & 0 \\ \tau A_{di}^{2T} P_i & 0 & -\tau R \end{bmatrix} < 0 \tag{10}$$

where $H_i = \widetilde{A}_i^T P_i + P_i \widetilde{A}_i + \tau Q + \tau R + E_i^T U_i E_i$. Application of the condition to the above example shows the estimated delay margin is $\tau^* = 0.0136$, which is more conservative than the conditions in Lemma 2. □

Theorem 1 (Solution to Problem 2): Consider symmetric matrices T, U_i and W_i such that U_i and W_i have nonnegative entries while $P_i \doteq F_i^T T F_i$ satisfy the condition (11) for all $i \in I_0$ where

$$X_\ell \doteq A_{di\ell}^T R_\ell A_{di\ell}, \quad \widetilde{A}_i \doteq A_i + \sum_{\ell=1}^{L} A_{di\ell},$$

$$H_i \doteq \widetilde{A}_i^T P_i + P_i \widetilde{A}_i + \sum_{\ell=1}^{L} Q + \sum_{\ell=1}^{L} \tau_\ell A_i^T X_\ell A_i + E_i^T U_i E_i.$$

The conditions for $j \in I_1$ is formulated similarly. Then, every piecewise continuous trajectory of $\mathcal{S}_{\tau c L}$ tends to zero exponentially. □

Proof: See the proof in the Appendix section. ■

Lemma 3 (Solution to Problem 3): Consider symmetric matrices T, U_i and W_i such that U_i and W_i have nonnegative entries while $P_i \doteq F_i^T T F_i$, for all $i \in I_0$,

$$
\begin{cases}
\begin{bmatrix}
H_i & \tau_1 P_i & \cdots & \tau_L P_i & \sum_{\ell=1}^{L}\tau_\ell A_i^T X_\ell A_{di1} & \cdots & \sum_{\ell=1}^{L}\tau_\ell A_i^T X_\ell A_{diL} \\
\tau_1 P_i & -\tau_1 R_1 & & 0 & 0 & \cdots & 0 \\
\vdots & & \ddots & & \vdots & \vdots & \vdots \\
\tau_L P_i & 0 & & -\tau_L R_L & 0 & \cdots & 0 \\
\sum_{\ell=1}^{L}\tau_\ell A_{di1}^T X_\ell A_i & 0 & \cdots & 0 & \sum_{\ell=1}^{L}\tau_\ell A_{di1}^T X_\ell A_{di1} - Q_1 & \cdots & \sum_{\ell=1}^{L}\tau_\ell A_{di1}^T X_\ell A_{diL} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sum_{\ell=1}^{L}\tau_\ell A_{diL}^T X_\ell A_i & 0 & \cdots & 0 & \sum_{\ell=1}^{L}\tau_\ell A_{diL}^T X_\ell A_{di1} & \cdots & \sum_{\ell=1}^{L}\tau_\ell A_{diL}^T X_\ell A_{diL} - Q_L
\end{bmatrix} < 0 \\
P_i - E_i^T W_i E_i > 0, \quad Q_\ell > 0, \quad R_\ell > 0, \quad \ell = 1, \cdots, L
\end{cases}
\tag{11}
$$

and $\bar{P}_j \doteq \bar{F}_j^T T \bar{F}_j$, for all $j \in I_1$, satisfy the following inequalities:

$$
\begin{cases}
\begin{bmatrix}
H_i & hP_i & hA_i^T A_{di}^T R A_{di}^2 \\
hP_i & -hR & 0 \\
hA_{di}^{2T} R A_{di} A_i & 0 & hA_{di}^{2T} R A_{di}^2 + (d-1)Q
\end{bmatrix} < 0 \\
P_i - E_i^T W_i E_i > 0, \quad Q > 0, \quad R > 0
\end{cases}
$$

$$
\begin{cases}
\begin{bmatrix}
\bar{H}_j & h\bar{P}_j & h\bar{A}_j^T \bar{A}_{dj}^T \bar{R} \bar{A}_{dj}^2 \\
h\bar{P}_j & -h\bar{R} & 0 \\
h\bar{A}_{dj}^{2T} \bar{R} \bar{A}_{dj} \bar{A}_j & 0 & h\bar{A}_{dj}^{2T} \bar{R} \bar{A}_{dj}^2 + (d-1)\bar{Q}
\end{bmatrix} < 0 \\
\bar{P}_j - \bar{E}_j^T W_i \bar{E}_j > 0, \quad \bar{Q} > 0, \quad \bar{R} > 0
\end{cases}
$$

for all $i \in I_0$ and all $j \in I_1$ where

$$
\begin{aligned}
\widetilde{A}_i &\doteq A_i + A_{di}, \qquad \hat{A}_j \doteq \bar{A}_j + \bar{A}_{dj}, \\
H_i &\doteq \widetilde{A}_i^T P_i + P_i \widetilde{A}_i + Q + hA_i^T A_{di}^T R A_{di} A_i + E_i^T U_i E_i, \\
\bar{H}_j &\doteq \hat{A}_j^T \bar{P}_j + \bar{P}_j \hat{A}_j + \bar{Q} + h\bar{A}_j^T \bar{A}_{dj}^T \bar{R} \bar{A}_{dj} \bar{A}_j + \bar{E}_j^T U_j \bar{E}_j.
\end{aligned}
$$

Then, every piecewise continuous trajectory of $\mathcal{S}_{\tau v}$ tends to zero exponentially. □

Proof: The proof follows on the lines of the proof of Lemma 2 by replacing τ by $\tau(t)$ in Eq. (A.1) and applying Leibniz rule. ■

Theorem 2 (Solution to Problem 3): Consider symmetric matrices T, U_i and W_i such that U_i and W_i have nonnegative entries while $P_i \doteq F_i^T T F_i$ satisfy the condition (13) for all $i \in I_0$. The conditions for $j \in I_1$ is formulated similarly. Then, every piecewise continuous trajectory of $\mathcal{S}_{\tau v L}$ tends to zero exponentially. □

Proof: The proof follows on the lines of the proof of Lemma 3 and Theorem 1. ■

VI. Discussion

An application of this theory is the design of an advanced hazard warning system for highway transportation safety. The problem of designing a decentralized advance hazard warning system for highway transportation systems entails the development of efficient switching controllers. It so turns out that the vehicle dynamics can be represented by a finite number of modes, each of which is represented by a low order transfer function and a constant time delay. The problem of highway safety analysis then gets translated into that of the stability analysis of a time delay hybrid system. Effectively, the mode changes partition the state space into cells that share, at most, only each other's boundaries, and the hybrid system has a piecewise affine form in each of the cells. A detailed case study is given in [6].

VII. Conclusion

We have derived classes of piecewise continuous Lyapunov functions for classes of time-delay hybrid systems inspired by a highway safety application described in [6]. Our Theorem 1 and Theorem 2 extend the well known [4, Theorem 1]. ■

VIII. Acknowledgment

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$$\begin{cases}\begin{bmatrix} H_i & h_1P_i & \cdots & h_LP_i & \sum_{\ell=1}^L h_\ell A_i^T X_\ell A_{di1} & \cdots & \sum_{\ell=1}^L h_\ell A_i^T X_\ell A_{diL} \\ h_1P_i & -h_1R_1 & & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & \vdots & \vdots \\ h_LP_i & 0 & & -h_LR_L & 0 & \cdots & 0 \\ \sum_{\ell=1}^L h_\ell A_{di1}^T X_\ell A_i & 0 & \cdots & 0 & \sum_{\ell=1}^L h_\ell A_{di1}^T X_\ell A_{di1} + (d_1-1)Q_1 & \cdots & \sum_{\ell=1}^L h_\ell A_{di1}^T X_\ell A_{diL} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{\ell=1}^L h_\ell A_{diL}^T X_\ell A_i & 0 & \cdots & 0 & \sum_{\ell=1}^L h_\ell A_{diL}^T X_\ell A_{di1} & \cdots & \sum_{\ell=1}^L h_\ell A_{diL}^T X_\ell A_{diL} + (d_L-1)Q_L \end{bmatrix} < 0 \\ P_i - E_i^T W_i E_i > 0, \quad Q_\ell > 0, \quad R_\ell > 0, \quad \ell = 1, \cdots, L \end{cases} \tag{13}$$

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APPENDIX. FORMAL PROOFS

A. Proof of Lemma 2

Consider the Lyapunov function

$$V(x,t,\tau) = V_1(x,t) + V_2(x,t,\tau) + V_3(x,t,\tau) \tag{A.1}$$

where

$$\begin{aligned} V_1(x,t) &\doteq x(t)^T P_i x(t), \\ V_2(x,t,\tau) &\doteq \int_{t-\tau}^t x(\xi)^T Q x(\xi) d\xi, \\ V_3(x,t,\tau) &\doteq \int_{-\tau}^0 \int_{t+\zeta}^t \Psi(\xi)^T A_{di}^T R A_{di} \Psi(\xi) d\xi\, d\zeta, \\ \Psi(\xi) &\doteq A_i\, x(\xi) + A_{di}\, x(\xi - \tau). \end{aligned}$$

The term V_3 is to account for the delay dependency. Let

$$\Pi \doteq \begin{bmatrix} H_i + \tau P_i R^{-1} P_i - E_i^T U_i E_i & \tau A_i^T A_{di}^T R A_{di}^2 \\ \tau A_{di}^{2T} R A_{di} A_i & -Q + \tau A_{di}^{2T} R A_{di}^2 \end{bmatrix}.$$

It can be easily verified that $V(x,t,\tau)$ is continuous in x and t, piecewise continuously differentiable in t, and

$$\alpha \|x(t)\| \le V(x,t,\tau) \le \beta \|x(t)\|, \quad \forall\, t \ge 0$$

for some $\alpha > 0$ and $\beta > 0$. Now, note that

$$0 < x(t)^T E_i^T U_i E_i\, x(t), \qquad \forall\, x(t) \in X_i, \tag{A.2}$$

$$-2a^T b \le \inf_{X>0} \left(a^T X a + b^T X^{-1} b\right),$$

$$\dot{x}(t) = \widetilde{A}_i x(t) - A_{di} \int_{t-\tau}^t \big(A_i x(\xi) + A_{di} x(\xi - \tau)\big) d\xi.$$

Hence, it may be verified, by using(6), (A.2) and Schur complement, that

$$\begin{aligned} \frac{\partial V}{\partial t} =\ & 2x(t)^T P_i \widetilde{A}_i x(t) - 2x(t)^T P_i A_{di} \int_{t-\tau}^t \Psi(\xi)\ d\xi \\ & + x(t)^T Q x(t) - x(t-\tau)^T Q x(t-\tau) \\ & + \tau \Psi(t)^T A_{di}^T R A_{di} \Psi(t) - \int_{t-\tau}^t \Psi(\xi)^T A_{di}^T R A_{di} \Psi(\xi) d\xi \\ \le\ & x(t)^T \big(\widetilde{A}_i^T P_i + P_i \widetilde{A}_i + Q + \tau P_i R^{-1} P_i\big) x(t) \\ & - x(t-\tau)^T Q x(t-\tau) + \tau \Psi(t)^T A_{di}^T R A_{di} \Psi(t) \\ =\ & \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \Pi \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \\ <\ & 0. \end{aligned}$$

Hence the proof. ∎

B. Proof of Theorem 1

Proof: Choosing the Lyapunov function

$$V(x,t,\tau) = V_1(x,t) + V_2(x,t,\tau) + V_3(x,t,\tau) \tag{A.3}$$

with

$$\begin{aligned} V_1(x,t) &\doteq x(t)^T P_i x(t), \\ V_2(x,t,\tau) &\doteq \sum_{\ell=1}^L \int_{t-\tau_\ell}^t x(\xi)^T Q_\ell x(\xi) d\xi, \\ V_3(x,t,\tau) &\doteq \sum_{\ell=1}^L \int_{-\tau_\ell}^0 \int_{t+\zeta}^t \Psi(\xi)^T A_{di\ell}^T R_\ell A_{di\ell} \Psi(\xi) d\xi\, d\zeta, \\ \Psi(\xi) &\doteq A_i\, x(\xi) + \sum_{\ell=1}^L A_{di\ell}\, x(\xi - \tau_\ell), \end{aligned} \tag{A.4}$$

the proof follows on the lines of the proof of Lemma 2. ∎