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Graph Theoretic Methods in the Stability of Vehicle Formations

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Abstract— **This paper investigates the stabilization of vehicle formations using techniques from algebraic graph theory. The vehicles exchange information according to a pre-specified (undirected) communication graph, G. The feedback control is based only on relative information about vehicle states shared via the communication links. We prove that a linear stabilizing feedback always exists provided that G is connected. Moreover, we show how the rate of convergence to formation is governed by the size of the smallest positive eigenvalue of the Laplacian of G. Several numerical simulations are used to illustrate the results.**

I. Introduction

From minisatellites to drone planes, the need to control the coordinated motion of multiple autonomous vehicles has received increasing attention recently [2], [3], [4], [6], [7], [8], [13], [14], [16]. One of the main goals is to distribute the control activity as much as possible while still achieving a coordinated objective. The objective investigated in this paper is that of attaining a moving formation. That is, the goal of the vehicles is to achieve and maintain pre-specified relative positions and orientations with respect to each other. Each vehicle is provided information only from a subset of the group. The specific subset is given through the set of "neighbors" in the communication graph. This graph need not be related to the actual formation geometry.

The feedback scheme investigated is inspired by the motion of aggregates of individuals in nature. Flocks of birds and schools of fish achieve coordinated motions of large numbers of individuals without the use of a central controlling mechanism [10]. A computer graphics model to simulate flock behavior is presented in [11]. In a different context, a simple model is proposed in [15] that captures the observed motions of self-driven particles. These models employ feedback laws in which the motions of nearest neighbors are averaged. The notion of a communication graph is introduced in [2], and an averaging feedback law is proposed based on the flow of information.

Keeping with the information flow approach, a probabilistic model for communication losses is introduced in [4] where it is shown that if the probability of losing a link is not too low, the formation is still achieved. A discrete averaging law is used in [6] to achieve a common heading. There, the communication graphs are allowed to change, provided that they remain connected. In [7], [8], [9], [12], [14] the authors use artificial potentials to generate feedback laws. The resulting nonlinear feedback laws can be shown to stabilize the formation under various geometric consistency criteria.

In this paper we study the communication graph approach. We consider a general vehicle model and use state-space techniques to prove that stabilizability of a formation can be achieved, provided that the communication graph is connected. We show that the rate of convergence to formation is governed by the smallest positive eigenvalue of the Laplacian matrix of the communication graph. We also demonstrate how, for a fixed feedback gain matrix, convergence can be improved by choosing alternative communication graphs.

The paper is organized as follows. In section II we set up the basic model. The relevant graph theoretic definitions and results are collected in section III. The main results are proved in section IV. Numerical simulations are presented in section V.

II. Model

We assume given N vehicles with the same dynamics

$$\dot{x}_i = A_{veh}x_i + B_{veh}u_i \qquad i = 1\dots N \quad x_i \in \mathbb{R}^{2n}$$

where the entries of x_i represent n configuration variables for vehicle i and their derivatives.

We are also given a graph G which captures the communication links between vehicles (see next section for precise definitions of graph theoretic concepts). Each vertex represents a vehicle and two vertices are connected by an edge if the corresponding vehicles communicate directly with each other. We refer to such vehicles as "neighbors". For each vehicle i, $\mathbb{J}_i$ denotes the set of its neighbors.

In this model, each vehicle only knows its state relative to its neighbors. That is, u_i is a function of $x_j - x_i$ for each $j \in \mathbb{J}_i$.

The study will focus on the simplest such rule: use as input an average based on the neighbors' states. To make this more precise we make the following definitions.

Definition 2.1: A **formation** is a vector $h = h_p \otimes \begin{pmatrix}1\\0\end{pmatrix} \in \mathbb{R}^{2nN}$ (where $\otimes$ denotes the Kronecker product). The N vehicles are **in formation** h if there are vectors $q, w \in \mathbb{R}^n$ such that $(x_p)_i - (h_p)_i = q$, $(x_v)_i = w$, $i = 1\dots N$, where the

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subscript p refers to the position components of x_i, and the subscript v refers to the corresponding velocities.

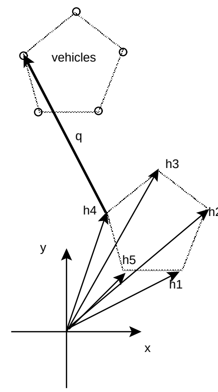

Fig. 1. Vehicles in formation

Goal: design an output feedback law that steers the vehicles to the desired formation.

Error output functions z_i are computed from an average of the relative displacement of the neighboring vehicles as follows

$$z_i = (x_i - h_i) - \frac{1}{|\mathbb{J}_i|} \sum_{j \in \mathbb{J}_i} (x_j - h_j) \qquad i = 1, \ldots, N.$$

As a result, the corresponding output vector z can be written as $z = L(x-h)$ where $L = L_G \otimes I_{2n}$ and L_G is the Laplacian matrix of the communication graph G (see section III).

Collecting the equations for all the vehicles into a single system we obtain

$$\begin{aligned} \dot{x} &= Ax + Bu \\ z &= L(x-h) \end{aligned}$$

with $A = I_N \otimes A_{veh}$, $B = I_N \otimes B_{veh}$.

We will show below that the vehicles are in formation if and only if $z = 0$.

With this formulation we pose the following:

Problem: Find matrices $F_1, \ldots, F_N$ such that if $F = \mathrm{diag}(F_1, \ldots, F_N)$ and

$$\dot{x} = Ax + BFL(x-h)$$

then $z \to 0$.

This is an output stabilization problem. The particular structure of the matrices A, B, and L offer opportunities for characterizing stabilizing matrices F in terms of specific properties of the communication graph G. We show below how the eigenvalues of the graph Laplacian L_G play a central role.

Given the block structure of the matrices A and B, we will look for F in the form $F = I_N \otimes F_{veh}$ (a "decentralized" control with the same feedback law for all vehicles).

Let U be a matrix such that $\widetilde{L}_G = U^{-1} L_G U$ is upper triangular. In particular, the eigenvalues of L_G are the diagonal entries of $\widetilde{L}_G$. A direct calculation using the special form of A, B, and F gives

$$(U^{-1} \otimes I_n)(A + BFL)(U \otimes I_n) = I_N \otimes A_{veh} + \widetilde{L}_G \otimes B_{veh} F_{veh}$$

The right hand side is block upper triangular. Its diagonal blocks are of the form:

$$A_{veh} + \lambda B_{veh} F_{veh}$$

where λ is an eigenvalue of L_G. There is one block for each eigenvalue. Therefore, the eigenvalues of $A + BFL$ are those of $A_{veh} + \lambda B_{veh} F_{veh}$ for λ an eigenvalue of L_G.

III. Spectral Graph Theory

For our purposes, a *graph* G consists of a finite set $\mathcal{V}$ of *vertices* and a set E of 2-element subsets of $\mathcal{V}$ to be referred to as *edges*. By construction, then, observe that G is undirected and has no loops or multiple edges.

We say a graph G is *connected* if for any vertices $i, j \in \mathcal{V}$, there exists a path of edges in G from i to j. If G is connected, then for any $i, j \in \mathcal{V}$, we define the *distance* between i and j to be the number of edges in a shortest path joining i and j. The *diameter* $\mathcal{D}$ of a connected graph G is the maximum distance between any two vertices of G.

Let G denote a graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. Let $\mathrm{Mat}_{\mathcal{V}}(\mathbb{R})$ denote the set of all matrices with real entries whose rows and columns are indexed by the vertices of G. By the *adjacency matrix* of G we mean the matrix $Q \in \mathrm{Mat}_{\mathcal{V}}(\mathbb{R})$ with entries

$$q_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in \mathcal{E}, \\ 0 & \text{otherwise} \end{cases} \qquad (i, j \in \mathcal{V}).$$

Because G is undirected, the matrix Q is symmetric. The *degree matrix* of G is the diagonal matrix $D \in \mathrm{Mat}_{\mathcal{V}}(\mathbb{R})$ with diagonal entries

$$d_{ii} = |\{j \in \mathcal{V} : \{i,j\} \in \mathcal{E}\}| \qquad (i \in \mathcal{V}).$$

The degree matrix encodes the number of vertices adjacent to each vertex. We will assume that the graph is connected and so the matrix D is invertible. The *Laplacian* of G is the matrix L_G defined by

$$L_G = I_N - D^{-1} Q,$$

where $N = |\mathcal{V}|$. This is, in general, different from the traditional Laplacian $\mathcal{L} = D - Q$ that is commonly used in the graph theory literature. In the case when G is k-regular, $D = kI_N$, so $L_G = k^{-1}\mathcal{L}$. It follows that an eigenvector of $\mathcal{L}$ with eigenvalue λ is also an eigenvector of L_G with eigenvalue λ/k. We refer to the set of eigenvalues of L_G (together with their multiplicities) as the Laplacian *spectrum* of G, and properties of these eigenvalues and their associated eigenvectors are collectively referred to as *spectral properties*.

Some relatively simple, but powerful, results about the spectrum of L_G are (see [1]):

1) All of the eigenvalues of L_G are nonnegative real numbers ≤ 2.

2) Zero is an eigenvalue of L_G.
3) The zero eigenvalue occurs with multiplicity one whenever the graph G is connected. In this case, the eigenvectors associated with the zero eigenvalue are all scalar multiples of the all one's vector. Thus, the null space of L_G is the same for all connected graphs.
4) If G is connected, then each nonzero eigenvalue λ of L_G satisfies
$$\lambda \geq \frac{1}{\mathcal{D} \sum_{i\in\mathcal{V}} d_{ii}},$$
where $\mathcal{D}$ denotes the diameter of G.

We include below as example a list of spectra of various well known classes of graphs.

Name	# ver.	Eigenvalues
Complete graph K_n	n	$0, \frac{n}{n-1}, \ldots, \frac{n}{n-1}$
Complete bipartite $K_{m,n}$	$m+n$	$0, 1, \ldots, 1, 2$
Path P_n	n	$1-\cos(\frac{\pi k}{n-1})$ $k=0,\ldots,n-1$
Cycle C_n	n	$1-\cos(\frac{2\pi k}{n})$ $k=0,\ldots,n-1$
n-cube Q_n	2^n	$\frac{2k}{n}$ $k=0,\ldots,n$

For additional graph theoretic terms and results see [5].

IV. Stabilizability

We show first that for the present decentralized feedback law to achieve formation stability, the individual vehicle dynamics must have a particular form. We start from the assumption that we have a second order model on each coordinate with acceleration as the input variable, and that the equations for each configuration variable are decoupled. To simplify the presentation we assume further that each coordinate satisfies the same dynamic equations. Except for re-scaling, the matrices A_{veh} and B_{veh} have the form

$$A_{veh} = I_n \otimes \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22} \end{pmatrix} \qquad B = I_n \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Observe that A_{veh} has the form $\mathrm{diag}\left(\left(\begin{smallmatrix} 0 & 1 \\ a_{21} & a_{22} \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 0 & 1 \\ a_{21} & a_{22} \end{smallmatrix}\right)\right)$. The results still hold if

$$A_{veh} = \mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ a_{21}^1 & a_{22}^1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ a_{21}^n & a_{22}^n \end{pmatrix}\right),$$

that is, if different 2×2 blocks are used for each configuration variable.

Proposition 4.1: If for every formation h there exists a stabilizing feedback matrix $F = I_N \otimes F_{veh}$ such that $L(x-h) \to 0$, then $a_{21} = 0$.

Proof: As mentioned above the feedback matrix F is such that $\lim_{t\to\infty} L(x(t)-h) \to 0$ (notice that h is constant by definition).

We use Kronecker products to simplify the calculations. We have $h = h_p \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $L = L_G \otimes I_{2n} = L_G \otimes I_n \otimes I_2$. We also write $x = x_p \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_v \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Everywhere the index p denotes the position components of the vehicles and the index v the velocity components. In particular, $\dot{x}_p = x_v$. Then $x - h = (x_p - h_p) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_v \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then,

$$\begin{aligned} L(x-h) &= (L_G \otimes I_n \otimes I_2)\left((x_p - h_p) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\ &+ (L_G \otimes I_n \otimes I_2)\left(x_v \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= ((L_G \otimes I_n)(x_p - h_p)) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &+ ((L_G \otimes I_n)x_v) \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

Since by hypothesis $L(x-h) \to 0$ we must have

$$(L_G \otimes I_n)(x_p - h_p) \to 0 \tag{1}$$
$$(L_G \otimes I_n)x_v \to 0 \tag{2}$$

By taking derivatives in the formula for $L(x-h)$, and since h is constant, we get

$$\begin{aligned} L(\dot{x} - \dot{h}) &= (L_G \otimes I_n)\dot{x}_p \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &+ (L_G \otimes I_n)\dot{x}_v \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= (L_G \otimes I_n)x_v \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &+ (L_G \otimes I_n)\dot{x}_v \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

The first term converges to 0 as $t \to \infty$ (by (2)). On the other hand, using the dynamic equations for the vehicles, we get:

$$L(\dot{x} - \dot{h}) = L(Ax + BFL(x-h)) = LAx + LBFL(x-h)$$

where

$$\begin{aligned} LAx &= (L_G \otimes I_{2n})(I_N \otimes A_{veh})\left(x_p \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_v \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= (L_G \otimes I_n)x_p \otimes \begin{pmatrix} 0 \\ a_{21} \end{pmatrix} + (L_G \otimes I_n)x_v \otimes \begin{pmatrix} 1 \\ a_{22} \end{pmatrix} \end{aligned}$$

Therefore, from (1) and (2), we get that LAx converges as $t \to \infty$. In fact, $\lim_{t\to\infty} LAx = a_{21}(L_G \otimes I_n)h_p \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Since $LBFL(x-h) \to 0$ we conclude that $(L_G \otimes I_n)\dot{x}_v$ converges, and so, it must converge to 0. Since

$$\lim_{t\to\infty} LAx = \lim_{t\to\infty} L\dot{x}_v = 0$$

by choosing a formation h_p such that $(L_G \otimes I_n)h_p \neq 0$, we obtain $a_{21} = 0$. □

Remark 4.2: The vehicles are in formation h if and only if $L(x-h) = 0$. To see this notice that the null space of $L = L_G \otimes I_{2n}$ is spanned by $\mathbf{1} \otimes \mathbf{e}_j$ where $\mathbf{e}_j$, $j = 1, \ldots, 2n$ are the standard basis vectors in $\mathbb{R}^{2n}$. Thus

$$\begin{aligned} L(x-h) = 0 \quad &\Leftrightarrow \quad x - h = \mathbf{1} \otimes \alpha \quad \text{for} \quad \alpha \in \mathbb{R}^{2n} \\ &\Leftrightarrow \quad (x_p)_i - (h_p)_i = q \quad (x_v)_i = w \end{aligned}$$

for $i = 1, \ldots, N$, where $\alpha = q \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + w \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The next result shows that if we can simultaneously stabilize $A_{veh} + \lambda B_{veh} F_{veh}$ for all eigenvalues λ, then the vehicles will converge to formation.

Proposition 4.3: Let $F_{veh} = I_n \otimes \begin{pmatrix} f_1 & f_2 \end{pmatrix}$. Suppose that the matrix $A_{veh} + \lambda B_{veh} F_{veh}$ is stable (Hurwitz) for each nonzero eigenvalue λ of the communication Laplacian L_G. Then $L(x-h) \to 0$.

Proof: As shown earlier the eigenvalues of $A + BFL$ are those of $A_{veh} + \lambda B_{veh} F_{veh}$ for each λ in the spectrum of L_G. Since 0 is an eigenvalue of L_G of multiplicity 1 (for connected graphs), then each eigenvalue of A_{veh} will also be an eigenvalue of $A + BFL$ with the same multiplicity. Our assumption is that all other eigenvalues of $A + BFL$ have negative real part.

The structure of the proof is as follows. First we expand the system to $\dot{y} = My$ using h_p as a new variable in a standard form. Then we show that a suitable subspace is M-invariant. Thirdly we show that the map induced on the quotient space is stable. Finally, we show that convergence in the quotient space means convergence to formation.

Since the desired formation is constant, the formation variable h_p satisfies $\dot{h}_p = 0$. We consider the extended system

$$\dot{x} = Ax + BFLx - BFL\left(I_{nN} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) h_p \quad (3)$$

$$\dot{h}_p = 0 \quad (4)$$

Notice that $h = \left(I_{nN} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) h_p$. We write the above equations as $\dot{y} = My$ where $y = \begin{pmatrix} x \\ h_p \end{pmatrix}$ and M is the $(3nN) \times (3nN)$ matrix given by

$$M = \begin{pmatrix} A + BFL & -BFL\left(I_{nN} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\ 0 & \mathbf{0}_{nN} \end{pmatrix}$$

Define the subspace $\mathcal{S}$ by

$$\mathcal{S} = \left\{ \begin{pmatrix} x \\ h_p \end{pmatrix} : Lx - L\left(h_p \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 0 \right\} =$$

$$= \left\{ \begin{pmatrix} x \\ h_p \end{pmatrix} : L(x-h) = 0 \right\}$$

Since the nullspace of L_G is spanned by the N-dimensional all one's vector $\mathbf{1}_N$, a basis of $\mathcal{S}$ is given by

$$\mathbf{B} = \left\{ \begin{pmatrix} \mathbf{1}_N \otimes \mathbf{e}_i \\ 0 \end{pmatrix} : \mathbf{e}_i \in \mathbb{R}^{2n}, i = 1, \ldots, 2n \right\} \quad (5)$$
$$\bigcup \left\{ \begin{pmatrix} \mathbf{e}_j \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \mathbf{e}_j \end{pmatrix} : \mathbf{e}_j \in \mathbb{R}^{nN}, j = 1, \ldots, nN \right\}$$

Claim: The space $\mathcal{S}$ is M-invariant.

An element of $\mathcal{S}$ has the form $y = \begin{pmatrix} \mathbf{1}_N \otimes \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} \beta \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \beta \end{pmatrix}$. Then, from Equations (3), (4) we get

$$My = \begin{pmatrix} \left(I_{nN} \otimes \begin{pmatrix} 0 & 1 \\ 0 & a_{22} \end{pmatrix}\right)\left(\mathbf{1}_N \otimes \alpha + \beta \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{1}_N \otimes A_{veh}\alpha \\ 0 \end{pmatrix} \in \mathbb{S}$$

This calculation also shows that the matrix of the restriction of the transformation induced by M on $\mathcal{S}$ is exactly $\begin{pmatrix} A_{veh} & 0 \\ 0 & \mathbf{0}_{nN} \end{pmatrix}$. The matrix M induces a linear transformation on the quotient space $\mathbb{R}^{2nN}/\mathcal{S}$ whose eigenvalues are those of $A_{veh} + \lambda B_{veh} F_{veh}$ for λ a **nonzero** eigenvalue of L_G. By assumption these eigenvalues have negative real parts. Therefore the quotient dynamics are stable. This implies that if $y = \begin{pmatrix} x \\ h_p \end{pmatrix}$ and $\dot{y} = My$ then $y + \mathcal{S} \to \mathcal{S}$ in the quotient space. From the definition of $\mathcal{S}$ this means that $L(x-h) \to 0$. □

We now want to show that stabilizing feedback matrices indeed exist.

Proposition 4.4: Given A, B as above, and a connected graph with Laplacian L_G, there exists F_{veh} such that $A_{veh} + \lambda B_{veh} F_{veh}$ is stable for each $\lambda \neq 0$ in the spectrum of L_G.

Proof: In fact, we will restrict the feedback matrices F to have the form $F = I_{nN} \otimes \begin{pmatrix} f_1 & f_2 \end{pmatrix}$. The problem reduces to showing that f_1 and f_2 may be chosen to stabilize $H_\lambda = \begin{pmatrix} 0 & 1 \\ 0 & a_{22} \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 \\ f_1 & f_2 \end{pmatrix}$ for all $\lambda \neq 0$ in the spectrum of L_G. The matrix H_λ has characteristic polynomial $p(x) = x^2 - (a_{22} + \lambda f_2)x - \lambda f_1$. So the matrix is stable if and only if

$$a_{22} + \lambda f_2 < 0 \qquad \lambda f_1 < 0. \quad (6)$$

Recall that all eigenvalues of L_G are nonnegative. Let λ_1 denote the minimum nonzero eigenvalue of L_G. Choose f_1 and f_2 so that $a_{22} + \lambda_1 f_2 < 0$ and $\lambda_1 f_1 < 0$. With this choice of F_{veh} the matrix H_λ is stable for all nonzero eigenvalues λ of L_G. □

The above proof also shows how the eigenvalues of L_G affect the rate of convergence to formation. The discriminant of the polynomial $p(x)$ is $(a_{22} + \lambda f_2)^2 + 4\lambda f_1$. Thus, for a fixed f_2, choosing f_1 so that

$$\frac{(a_{22} + \lambda f_2)^2}{4\lambda} < -f_1$$

for every nonzero eigenvalue λ of L_G guarantees complex (non-real) roots, thereby providing the fastest rate of convergence. Thus we have proved the following.

Proposition 4.5: For f_1 and f_2 as above, the rate of convergence to formation is $(a_{22} + \lambda_1 f_2)/2$.

This is illustrated in the numerical simulations below.

Remark 4.6: For a fixed number N of vehicles, stabilizing gains f_1 and f_2 can be chosen independently of the graph. To see this recall the inequalities

$$\frac{1}{\mathcal{D}\sum_{i\in\mathcal{V}} d_{ii}} \leq \lambda \leq 2$$

and note that for a graph with N vertices, $\mathcal{D} \leq N-1$ and $\sum_{i\in\mathcal{V}} d_{ii} \leq N(N-1)$.

V. Examples

We illustrate the results with various numerical simulations. First we assume that $a_{22} = 0$ so each coordinate is modelled as a double integrator. In all these examples the desired formation is specified as the vertices of a regular pentagon. Figure 2 shows convergence to formation using the same feedback matrix but two different graphs, the path P_5 and the complete graph K_5. The increase in λ_1 accounts for achieving the formation sooner. The formation drifts in space at a constant speed because the vehicles were in motion at the start. The common velocity vector is the average of all velocities.

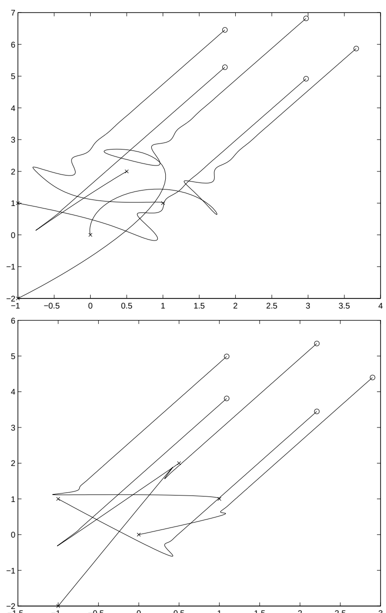

Fig. 2. Path on top ($\lambda_1 = 0.2929$), complete graph on bottom ($\lambda_1 = 1.25$)

In Figure 3 the same graph is used (the cycle C_5) but different feedback matrices.

Figure 4 illustrates the effect of the a_{22} term in the resulting formations. While these examples use the same 2×2 matrix for each controlled quantity in a single vehicle, the effect of different values for each of them should be clear from these pictures. The model still assumes that all the vehicles have the same dynamics. For $a_{22} = 0$ the vehicles achieve a constant velocity. For $a_{22} < 0$ the vehicles eventually stop. For $a_{22} > 0$ the vehicles uniformly accelerate.

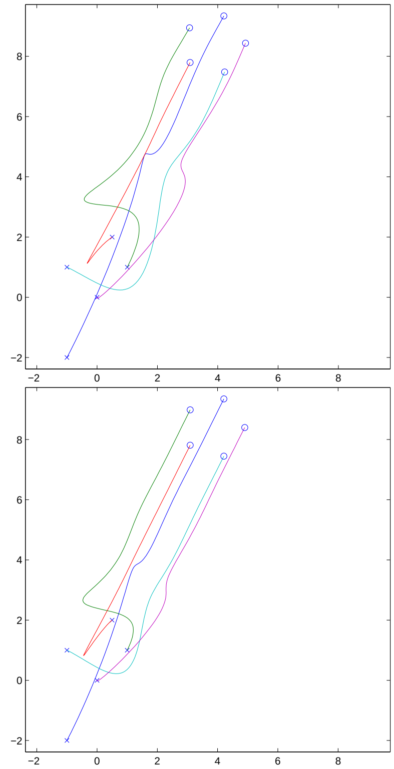

Fig. 3. Cycle with different feedback gains

Figure 5 shows simulations for a model including orientation of the vehicles. The communication graph is a 5-cycle and the formation is a regular pentagon with all vehicles orientated in the same direction. In the first plot the vehicles are given an initial velocity while in the second they start from rest. The final positions are indicated with circles.

VI. Conclusions

We have demonstrated the close connection between spectral graph theory and one of the current methods of control of vehicle formations. We have made explicit how to choose stabilizing feedback gains in terms of estimates for the eigenvalues of the Laplacian of the communication graph. Furthermore, we have derived an expression for the rate of convergence to formation that is a linear function of the smallest positive eigenvalue of the Laplacian.

For a fixed feedback gain matrix F, convergence to formation can be improved by modifying the communication graph in such a way as to increase the value of λ_1.

For simplicity we have restricted our analysis to undirected graphs. However, analogous results hold for directed graphs.

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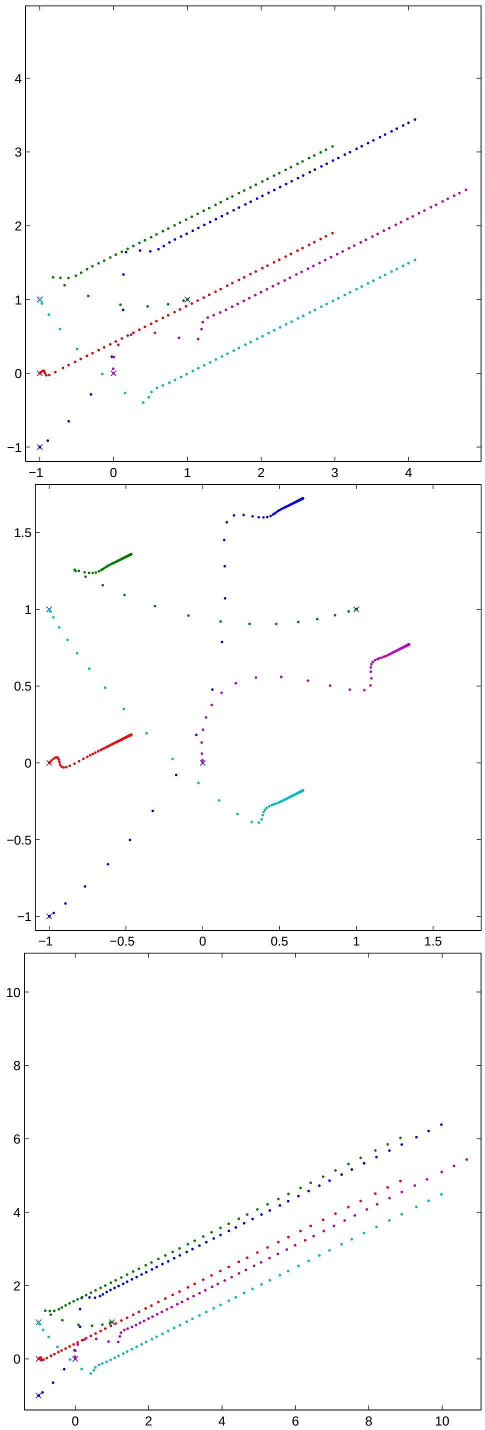

Fig. 4. Effect of a_{22}. From top to bottom $a_{22} = 0$, $a_{22} < 0$, $a_{22} > 0$

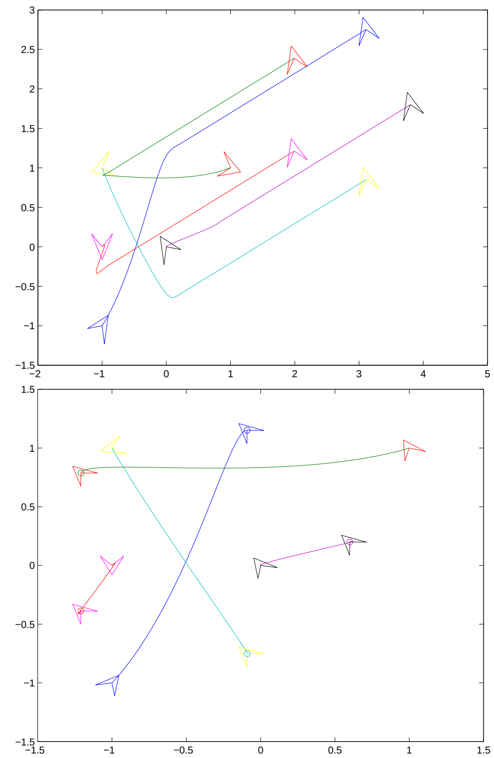

Fig. 5. Oriented vehicles

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