

# A Decentralized Output Feedback Controller for a Class of Large-Scale Interconnected Nonlinear Systems

Prabhakar R. Pagilla and Yongliang Zhu

**Abstract**—The decentralized output feedback control problem for a class of large-scale interconnected nonlinear systems is considered. The nonlinear interconnection function of each subsystem is assumed to satisfy a quadratic constraint on the entire state of the large-scale system. A decentralized estimated state feedback controller and a decentralized observer are designed for each subsystem. Sufficient conditions, for each subsystem, under which the proposed controller and observer can achieve exponential stabilization of the overall large-scale system are developed. Simulation results on a numerical example are given to verify the proposed design.

## I. INTRODUCTION

Large-scale interconnected systems can be found in such diverse fields as electrical power systems, space structures, manufacturing processes, transportation, and communication. An important motivation for the design of decentralized schemes is that the information exchange between subsystems of a large-scale system is not needed; thus, the individual subsystem controllers are simple and use only locally available information. Decentralized control of large-scale systems has received considerable interest in the systems and control literature. A large body of literature in decentralized control of large-scale systems can be found in [1]. In [2], a survey of early results in decentralized control of large scale systems was given. Decentralized control schemes that can achieve desired robust performance in the presence of uncertain interconnections can be found in [3], [4], [5]. A decentralized control scheme for robust stabilization of a class of nonlinear systems using the Linear Matrix Inequalities (LMI) framework was proposed in [6].

In many practical situations, complete state measurements are not available at each individual subsystem for decentralized control; consequently, one has to consider decentralized feedback control based on measurements only or design decentralized observers to estimate the state of individual subsystems that can be used for estimated state feedback control. There has been a strong research effort in literature towards development of decentralized control schemes based on output feedback via construction of decentralized observers. Early work in this area can be found in [7], [3], [1]. Subsequent work in [8], [9], [10], [11] has focused on the decentralized output feedback problem for a number of special class of nonlinear systems. The design of an observer-based output feedback controller is a challenging problem for nonlinear systems. It is well

known that the separation principle may not be applicable to nonlinear systems [12]. In [13], the decentralized controller and observer design problems were formulated in the LMI framework for large-scale systems with nonlinear interconnections that are quadratically bounded. Autonomous linear decentralized observer-based output feedback controllers for all subsystems were obtained. The existence of a stabilizing controller and observer depended on the feasibility of solving an optimization problem in the LMI framework; further, for a solution to exist, this formulation also required, for each subsystem, that the number of control inputs must be equal to the dimension of the state.

In this paper, we consider a class of large-scale system with quadratically bounded nonlinear interconnections; the same class of large-scale systems were also considered in [13]. We design a decentralized controller and observer that can achieve global exponential stabilization under two sufficient conditions. The tools used in the paper are related to concepts from the literature on the distance to uncontrollability (observability) of a pair of matrices  $(A, B)$  ( $(C, A)$ ) [14], [15], [16]. As opposed to the LMI framework, our design does not require as many control inputs as the number of state variables for each subsystem; further, the proposed design results in computable sufficient conditions for each subsystem as opposed to solving an optimization problem for the overall large-scale system. Insights into the problem are provided by considering various special cases which are practically relevant. Simulation results on a numerical example are given to validate the proposed decentralized design.

The rest of the paper is organized as follows. In Section II, the class of considered large-scale systems is given with a discussion of the problem and related results available in literature. Some important results from literature and a definition that will be used in the developments of the paper are given in Section III. In Section IV, the proposed decentralized controller/observer structure is given; sufficient conditions under which exponential stabilization are derived. Simulation results on an example are given in Section V. Section VI summarizes the paper and highlights some future research topics on the problem.

## II. PROBLEM FORMULATION

The following notation is used.  $\mathbb{R}$  represents the set of real numbers.  $\lambda_{\min}(M)$ ,  $\lambda_{\max}(M)$ ,  $M^H$ , and  $M^\top$  denote the minimum eigenvalue, the maximum eigenvalue, the complex conjugate transpose, and the transpose of the matrix  $M$ , respectively.  $M > 0$  ( $\geq 0$ ) denotes that the matrix

This work was supported by NSF under the CAREER Grant CMS-9982071. The authors are with the School of Mechanical and Aerospace Engineering, Oklahoma State University, Stillwater, OK 74078, USA. Corresponding author e-mail: pagilla@ceat.okstate.edu

$M$  is symmetric positive definite (symmetric positive semi-definite).  $\sigma_{\min}(M) \triangleq \sqrt{\lambda_{\min}(M^H M)}$ . The spectral norm of the matrix  $M$  is denoted by  $\|M\|$ . The identity matrix is denoted by  $I$ . The term  $\text{diag}(M_1, \dots, M_n)$  denotes a block diagonal matrix with  $M_1$  to  $M_n$  as its diagonal blocks.

The following class of large-scale interconnected nonlinear systems is considered:

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + h_i(t, x), \quad x_i(t_0) = x_{i0}, \quad (1a)$$

$$y_i(t) = C_i x_i(t) \quad (1b)$$

where  $x_i \in \mathbb{R}^{n_i}$ ,  $u_i \in \mathbb{R}^{m_i}$ ,  $y_i \in \mathbb{R}^{l_i}$ ,  $h_i \in \mathbb{R}^{n_i}$ ,  $t_0$ , and  $x_{i0}$  are the state, input, output, nonlinear interconnection function, initial time, and initial state of the  $i$ -th subsystem. System (1) consists of  $N$  subsystems, that is,  $i = 1 : N$ . The interconnections are assumed to be piecewise-continuous functions in both variables, and satisfy the quadratic constraints [13]

$$h_i^T(t, x) h_i(t, x) \leq \alpha_i^2 x^T H_i^T H_i x \quad (2)$$

where  $\alpha_i > 0$  are interconnection bounds,  $H_i$  are bounding matrices, and  $x^T = [x_1^T, x_2^T, \dots, x_N^T]$  is the state of the overall system. It is assumed that  $\alpha_i$  and  $\|H_i\|$  are known. Also, it is assumed that,  $(A_i, B_i)$  is controllable and  $(C_i, A_i)$  is observable. Without loss of generality, it is assumed that  $A_i$  is stable, that is, all eigenvalues of  $A_i$  have negative real parts. The overall system (1) can be rewritten as

$$\dot{x}(t) = A_D x(t) + B_D u(t) + h(t, x), \quad x(t_0) = x_0, \quad (3a)$$

$$y(t) = C_D x(t) \quad (3b)$$

where  $A_D = \text{diag}(A_1, \dots, A_N)$ ,  $B_D = \text{diag}(B_1, \dots, B_N)$ ,  $C_D = \text{diag}(C_1, \dots, C_N)$ ,  $u^T = [u_1^T, \dots, u_N^T]$ ,  $y^T = [y_1^T, \dots, y_N^T]$ , and  $h^T = [h_1^T, \dots, h_N^T]$ . The nonlinear interconnections  $h(t, x)$  are bounded as follows:

$$h^T(t, x) h(t, x) \leq x^T \Gamma x \quad (4)$$

where  $\Gamma = \sum_{i=1}^N \alpha_i^2 H_i^T H_i$ . The pair  $(A_D, B_D)$  is controllable and the pair  $(C_D, A_D)$  is observable, which is the direct result of each subsystem being controllable and observable.

Since the system (3) is linear with nonlinear interconnections, a common question to ask is under what conditions can we design a decentralized linear controller and a decentralized linear observer that will stabilize the system in the presence of bounded interconnections. Towards solving this problem, one can consider the following linear decentralized controller and observer:

$$u(t) = K_D \hat{x}(t), \quad (5)$$

$$\dot{\hat{x}}(t) = A_D \hat{x}(t) + B_D u(t) + L_D (y(t) - C_D \hat{x}(t)) \quad (6)$$

where  $K_D = \text{diag}(K_1, \dots, K_N)$  and  $L_D = \text{diag}(L_1, \dots, L_N)$  are the controller gain matrix and the observer gain matrix, respectively. Rewriting (3) and (6) in the coordinates  $x(t)$  and  $\hat{x}(t)$ , where  $\tilde{x}(t) \triangleq x(t) - \hat{x}(t)$  is the estimation error, the closed-loop dynamics is

$$\dot{x}(t) = (A_D + B_D K_D) x(t) - B_D K_D \tilde{x}(t) + h(t, x),$$

$$\dot{\tilde{x}}(t) = (A_D - L_D C_D) \tilde{x}(t) + h(t, x). \quad (7)$$

Two broad methods are used to design observer-based decentralized output feedback controllers for large-scale systems: (1) Design local observer and controller for each subsystem independently, and check the stability of the overall closed-loop system. In this method, the interconnection in each subsystem is regarded as an unknown input [7], [9]. (2) Design the observer and controller by posing the output feedback stabilization problem as an optimization problem. The optimization approach using LMIs can be found in [13]. We give a brief overview of this approach. It is assumed that  $H_i$  is known. The controller gain  $K_D$  and the observer gain  $L_D$  are obtained from the following minimization problem, provided it is feasible.

Minimize  $\sum_{i=1}^N \gamma_i$  subject to

$$\tilde{P}_1 > 0, \quad \tilde{P}_2 > 0,$$

$$\begin{bmatrix} A_D^T \tilde{P}_1 + \tilde{P}_1 A_D + M_D^T + M_D & -M_D & \tilde{P}_1 & H_1^T & \dots & H_N^T \\ -M_D & A_D^T \tilde{P}_2 + \tilde{P}_2 A_D - C_D^T N_D^T - N_D C_D & \tilde{P}_2 & 0 & \dots & 0 \\ \tilde{P}_1 & & \tilde{P}_2 & -I & 0 & \dots & 0 \\ H_1 & & 0 & 0 & -\gamma_1 I & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ H_N & & 0 & 0 & 0 & \dots & -\gamma_N I \end{bmatrix} < 0 \quad (8)$$

where  $\gamma_i = 1/\alpha_i^2$ , and

$$\tilde{P}_1 B_D K_D = M_D, \quad \tilde{P}_2 L_D = N_D. \quad (9)$$

Under the assumption that the above optimization problem, (8), is feasible, the gain matrices  $K_D$  and  $L_D$  were extracted as follows:

$$K_D = B_D^{-1} \tilde{P}_1^{-1} M_D \quad \text{and} \quad L_D = \tilde{P}_2^{-1} N_D. \quad (10)$$

The LMI formulation given above requires the invertibility of the input matrix  $B_D$ ; that is, it requires as many independent control inputs as the number of state variables in each subsystem. Although the formulation as an optimization problem using the LMI framework is quite elegant from a numerical perspective, as it not only computes the gains but also maximizes the interconnection bounds, it can be applied only to a restrictive class of systems in which the input matrix is invertible which is more restrict than matching condition. Further, obtaining block diagonal positive definite solutions,  $\tilde{P}_1$  and  $\tilde{P}_2$ , from the optimization problem (8) is itself a challenging problem; the solutions need to be block diagonal for the computed  $K_D$  and  $L_D$  to be block diagonal; otherwise, one does not get a decentralized solution.

Notice that, because of the nature of the interconnection,  $h_i(t, x)$ , in some cases, system (1) may not be stabilizable even with full-state feedback control. For example,

$$\dot{x}_1 = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A_1} x_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B_1} u_1 + \gamma_1 \begin{bmatrix} x_{11} - x_{12} \\ x_{21} \end{bmatrix}, \quad (11)$$

$$y_1 = [1 \quad 0] x_1. \quad (12)$$

If  $\gamma_1 = 1$ , then the first state of  $x_1$ ,  $x_{11}$ , has the dynamics  $\dot{x}_{11} = x_{11}$ , which is unstable and we lose controllability of the system. One cannot design a controller to stabilize the system (11) with the given interconnection, although  $(A_1, B_1)$  is controllable. From the example, it is clear that the structure and bounds of the interconnections will

affect controllability of subsystems. The same holds true for observability of the system. So, there must be some conditions on the system matrices and the interconnections under which the controllability and observability properties are preserved.

The objective of this paper is to design a totally decentralized observer-based linear controller that robustly regulates the state of the overall system without any information exchange between subsystems. In this paper, we reduce the problem of decentralized exponential stabilization of the large-scale system via output feedback into the existence of symmetric positive definite solutions of two Algebraic Riccati Equations (AREs). Further, we develop sufficient conditions for the existence of symmetric positive definite solutions; the conditions are developed using the concepts of distance to controllability and distance to observability [14] and their related results.

### III. PRELIMINARIES

In this section, a definition and some results relevant to the Algebraic Riccati Equation (ARE) which will be used in the subsequent developments of the paper are introduced.

*Definition 1:* The real number  $\delta(M, N)$  is defined as

$$\delta(M, N) \triangleq \min_{\omega \in \mathbb{R}} \sigma_{\min} \begin{bmatrix} i\omega I - M \\ N \end{bmatrix} \quad (13)$$

where  $i = \sqrt{-1}$ ,  $M \in \mathbb{R}^{n \times n}$ ,  $N \in \mathbb{R}^{p \times n}$ .

See [17] for a discussion of the number  $\delta$ .

*Lemma 1:* [17] Consider the Algebraic Riccati Equation

$$A^\top P + PA + PRP + Q = 0. \quad (14)$$

If  $R = R^\top \geq 0$ ,  $Q = Q^\top > 0$ ,  $A$  is Hurwitz, and the associated Hamiltonian matrix  $H = \begin{bmatrix} A & R \\ -Q & -A^\top \end{bmatrix}$  is hyperbolic, i.e.,  $H$  has no eigenvalues on the imaginary axis, then there exists a unique  $P = P^\top > 0$  to be the solution of the ARE (14).

*Lemma 2:* [17], [18] Let  $\gamma \geq 0$  and define

$$H_\gamma = \begin{bmatrix} A & I \\ C^\top C - \gamma^2 I & -A^\top \end{bmatrix}.$$

Then  $\gamma < \delta(A, C)$  if and only if  $H_\gamma$  is hyperbolic.

*Lemma 3:* [19] Assume  $A, Q_2, R \in \mathbb{R}^{n \times n}$ ,  $Q_2 = Q_2^\top$  and  $R = R^\top > 0$ . If  $P_2 = P_2^\top > 0$  satisfies

$$A^\top P_2 + P_2 A + P_2 R P_2 + Q_2 = 0$$

and  $Q_1 = Q_1^\top$  such that  $Q_1 \leq Q_2$ . Then there exists a  $P_1 = P_1^\top > 0$  such that  $P_1 \geq P_2$ , and

$$A^\top P_1 + P_1 A + P_1 R P_1 + Q_1 = 0.$$

### IV. THE PROPOSED DECENTRALIZED OUTPUT FEEDBACK CONTROLLER

Consider the following linear decentralized controller and observer for the  $i$ -th subsystem:

$$u_i(t) = K_i \hat{x}_i(t), \quad (15)$$

$$\dot{\hat{x}}_i(t) = A_i \hat{x}_i(t) + B_i u_i(t) + L_i (y_i(t) - C_i \hat{x}_i(t)) \quad (16)$$

where  $K_i$  and  $L_i$  are the controller and observer gain matrices. Substituting these into the system, (1), we obtain

$$\dot{x}_i(t) = (A_i + B_i K_i) x_i(t) - B_i K_i \tilde{x}_i(t) + h_i(t, x), \quad (17)$$

$$\dot{\tilde{x}}_i(t) = (A_i - L_i C_i) \tilde{x}_i(t) + h_i(t, x) \quad (18)$$

where  $\tilde{x}_i = x_i - \hat{x}_i$ . For simplicity define the following:

$$A_{Bi} = B_i K_i \text{ and } A_{Ci} = L_i C_i. \quad (19)$$

Consider the following Lyapunov function candidate:

$$V(x, \tilde{x}) = \sum_{i=1}^N \left( x_i^\top P_i x_i + \tilde{x}_i^\top \tilde{P}_i \tilde{x}_i \right). \quad (20)$$

The time derivative of  $V(x, \tilde{x})$  along the trajectories of (17) and (18), after some simplification, is given by

$$\dot{V}(x, \tilde{x}) = \sum_{i=1}^N \left\{ x_i^\top [(A_i + A_{Bi})^\top P_i + P_i (A_i + A_{Bi})] x_i + \tilde{x}_i^\top [(A_i - A_{Ci})^\top \tilde{P}_i + \tilde{P}_i (A_i - A_{Ci})] \tilde{x}_i \right. \\ \left. - \underbrace{\tilde{x}_i^\top A_{Bi}^\top P_i x_i - x_i^\top P_i A_{Bi} \tilde{x}_i}_{\text{cross terms}} + \underbrace{h_i^\top P_i x_i + x_i^\top P_i h_i}_{\text{nonlinear}} + \underbrace{h_i^\top \tilde{P}_i \tilde{x}_i + \tilde{x}_i^\top \tilde{P}_i h_i}_{\text{nonlinear}} \right\}. \quad (21)$$

Using the inequality

$$X^\top Y + Y^\top X \leq X^\top X + Y^\top Y, \quad X, Y \in \mathbb{R}^{m \times n} \quad (22)$$

for terms with under braces in (21), we obtain

$$\tilde{x}_i^\top (-A_{Bi})^\top P_i x_i + x_i^\top P_i (-A_{Bi}) \tilde{x}_i \leq \tilde{x}_i^\top A_{Bi}^\top A_{Bi} \tilde{x}_i + x_i^\top P_i P_i x_i, \quad (23a)$$

$$h_i^\top P_i x_i + x_i^\top P_i h_i \leq h_i^\top h_i + x_i^\top P_i P_i x_i, \quad (23b)$$

$$h_i^\top \tilde{P}_i \tilde{x}_i + \tilde{x}_i^\top \tilde{P}_i h_i \leq h_i^\top h_i + \tilde{x}_i^\top \tilde{P}_i \tilde{P}_i \tilde{x}_i. \quad (23c)$$

Each interconnection function,  $h_i(t, x)$  satisfies

$$h_i^\top(t, x) h_i(t, x) \leq \alpha_i^2 x^\top H_i^\top H_i x \leq \alpha_i^2 v_i x^\top x, \quad (24)$$

where  $v_i = \lambda_{\max}(H_i^\top H_i)$ . We also have

$$\sum_{i=1}^N 2h_i^\top h_i \leq \sum_{i=1}^N 2\alpha_i^2 v_i (x_1^\top x_1 + \dots + x_N^\top x_N) = \gamma^2 \sum_{i=1}^N x_i^\top x_i \quad (25)$$

where  $\gamma^2 \triangleq \sum_{i=1}^N 2\alpha_i^2 v_i$ . Using (23) and (25) in (21) we have

$$\dot{V}(x, \tilde{x}) \leq \sum_{i=1}^N \left\{ x_i^\top \left[ (A_i + A_{Bi})^\top P_i + P_i (A_i + A_{Bi}) + 2P_i P_i + \gamma^2 I \right] x_i \right. \\ \left. + \tilde{x}_i^\top \left[ (A_i - A_{Ci})^\top \tilde{P}_i + \tilde{P}_i (A_i - A_{Ci}) + A_{Bi}^\top A_{Bi} + \tilde{P}_i \tilde{P}_i \right] \tilde{x}_i \right\}. \quad (26)$$

Choose the following gain matrices:

$$K_i = -(B_i^\top B_i)^{-1} B_i^\top P_i, \quad L_i = \varepsilon_i \tilde{P}_i^{-1} C_i^\top / 2, \quad \varepsilon_i > 0. \quad (27)$$

Substituting the gains into (26), we have

$$\dot{V}(x, \tilde{x}) \leq \sum_{i=1}^N \left\{ x_i^\top \left[ A_i^\top P_i + P_i A_i + 2P_i (I - B_i (B_i^\top B_i)^{-1} B_i^\top) P_i + \gamma^2 I \right] x_i \right. \\ \left. + \tilde{x}_i^\top \left[ A_i^\top \tilde{P}_i + \tilde{P}_i A_i + \tilde{P}_i \tilde{P}_i + \tilde{Q}_{i1} - \varepsilon_i C_i^\top C_i \right] \tilde{x}_i \right\}. \quad (28)$$

where  $\tilde{Q}_{i1} \triangleq K_i^\top B_i^\top B_i K_i$ . From the above, we have the following result. For some  $\eta_i > 0$  and  $\tilde{\eta}_i > 0$ , if there exist positive definite solutions to the AREs

$$A_i^\top P_i + P_i A_i + 2P_i (I - B_i (B_i^\top B_i)^{-1} B_i^\top) P_i + \gamma^2 I + \eta_i I = 0, \quad (29)$$

$$A_i^\top \tilde{P}_i + \tilde{P}_i A_i + \tilde{P}_i \tilde{P}_i + \tilde{Q}_{i1} + \tilde{\eta}_i I - \varepsilon_i C_i^\top C_i = 0, \quad (30)$$

then

$$\dot{V}(x, \tilde{x}_i) \leq - \sum_{i=1}^N \left[ \eta_i x_i^\top x_i + \tilde{\eta}_i \tilde{x}_i^\top \tilde{x}_i \right] \quad (31)$$

where  $\varepsilon_i$  and  $\tilde{\eta}_i$  are chosen such that  $\tilde{Q}_{i1} + \tilde{\eta}_i I - \varepsilon_i C_i^\top C_i > 0$ . As a result, the problem reduces to the following: If

there are positive definite solutions to the AREs (29) and (30), then  $V(x, \tilde{x})$  is a Lyapunov function; that is,  $V(x, \tilde{x})$  is positive and  $\dot{V}(x, \tilde{x})$  is negative.

*Remark 1:* The control gain matrix  $K_i$  given by (27) requires that  $B_i^\top B_i$  is invertible;  $B_i^\top B_i$  is invertible if  $B_i$  has full column rank, which is always possible.

*Remark 2:* If  $A_i$  is not stable, then we can stabilize  $A_i$  by changing  $K_i$  and  $L_i$  given by (27), to the following:

$$K_i = -(B_i^\top B_i)^{-1} B_i^\top P_i - \bar{K}_i, \quad L_i = \frac{1}{2} \varepsilon_i \tilde{P}_i^{-1} C_i^\top + \bar{L}_i \quad (32)$$

where  $\bar{K}_i$  and  $\bar{L}_i$  are pre-feedback gains such that  $A_i^c \triangleq A_i - B_i \bar{K}_i$  and  $A_i^o \triangleq A_i - \bar{L}_i C_i$  are Hurwitz. In such a case,  $A_i$  in (29) and (30) must be replaced by  $A_i^c$  and  $A_i^o$ , respectively.

Notice that we cannot design the controller and observer independently, that is, the separation principle does not hold; the ARE (30) depends on the control gain matrix  $K_i$ . It is well known that the separation principle generally does not hold for nonlinear systems. But it should be noted that the above reduction procedure has yielded the following: one can design the controller gain independent of the observer and further, only the first ARE, (29), explicitly depends on the interconnection bounds.

The problem now reduces to the following: What are the conditions under which there exist positive definite solutions to the AREs (29) and (30). In the following, two sufficient conditions will be derived.

#### A. Sufficient conditions

We first consider the ARE (29). The associated Hamiltonian matrix is given by  $H_i = \begin{bmatrix} A_i & R_i \\ -Q_i & -A_i^\top \end{bmatrix}$  where  $R_i = 2(I - B_i(B_i^\top B_i)^{-1} B_i^\top) \geq 0$ ,  $Q_i = (\gamma^2 + \eta_i)I > 0$ . The following lemma gives a condition under which  $H_i$  is hyperbolic; thus, by lemma 1, it gives a sufficient condition for the existence of a unique  $P_i = P_i^\top > 0$  to the ARE (29).

*Lemma 4:*  $H_i$  is hyperbolic if and only if

$$\delta(A_i^\top, \sqrt{2(\gamma^2 + \eta_i)} (B_i^\top B_i)^{-1/2} B_i^\top) > \sqrt{2(\gamma^2 + \eta_i)}. \quad (33)$$

*Proof:* Consider the determinant of the matrix  $(sI - H_i)$

$$\begin{aligned} \det(sI - H_i) &= \det \begin{bmatrix} sI - A_i & -2(I - B_i(B_i^\top B_i)^{-1} B_i^\top) \\ (\gamma^2 + \eta_i)I & sI + A_i^\top \end{bmatrix} \\ &= (-1)^{n_i} \det \begin{bmatrix} (\gamma^2 + \eta_i)I & sI + A_i^\top \\ sI - A_i & -2(I - B_i(B_i^\top B_i)^{-1} B_i^\top) \end{bmatrix} \end{aligned}$$

Since  $(\gamma^2 + \eta_i)I$  is non-singular, using the formula for determinant of block matrices [20, p. 650], we obtain

$$\begin{aligned} \det(sI - H_i) &= (-1)^{n_i} \det \left[ -2(\gamma^2 + \eta_i) \left( I - B_i(B_i^\top B_i)^{-1} B_i^\top \right) - (sI - A_i)(sI + A_i^\top) \right] \\ &\triangleq (-1)^{n_i} \det(G(s)). \end{aligned}$$

From the above equation,  $s$  is an eigenvalue of  $H_i$  if and only if  $G(s)$  is singular. Hence, to prove that  $H_i$  is

hyperbolic, one can prove that  $G(-i\omega)$  is non-singular for all  $\omega \in \mathbb{R}$ . Notice that

$$\begin{aligned} \Delta_c(-i\omega) &\triangleq -(-i\omega I - A_i)(-i\omega I + A_i^\top) + 2(\gamma^2 + \eta_i)B_i(B_i^\top B_i)^{-1} B_i^\top \\ &= \left[ \frac{i\omega I - A_i^\top}{\sqrt{2(\gamma^2 + \eta_i)} (B_i^\top B_i)^{-1/2} B_i^\top} \right]^H \left[ \frac{i\omega I - A_i^\top}{\sqrt{2(\gamma^2 + \eta_i)} (B_i^\top B_i)^{-1/2} B_i^\top} \right]. \end{aligned}$$

Therefore, if

$$\delta(A_i^\top, \sqrt{2(\gamma^2 + \eta_i)} (B_i^\top B_i)^{-1/2} B_i^\top) > \sqrt{2(\gamma^2 + \eta_i)},$$

then  $G(-i\omega) > 0$  for all  $\omega \in \mathbb{R}$ . Thus,  $H_i$  is hyperbolic. This completes the sufficiency part of the proof. The necessary part of the proof is similar to that of lemma 2. ■

The Hamiltonian matrix associated with the ARE (30) is

$$\tilde{H}_i = \begin{bmatrix} A_i & \tilde{R}_i \\ -\tilde{Q}_i & -A_i^\top \end{bmatrix} \quad (34)$$

where  $\tilde{R}_i = I > 0$  and  $\tilde{Q}_i = \tilde{Q}_{i1} + \tilde{\eta}_i I - \varepsilon_i C_i^\top C_i$ . Choose  $\tilde{\eta}_i > 0$  and  $\varepsilon_i > 0$  such that  $\tilde{Q}_i > 0$ . The following lemma gives a condition under which  $\tilde{H}_i$  is hyperbolic; thus, by lemma 1, it gives a sufficient condition for the existence of a symmetric positive definite solution to the ARE (30).

*Lemma 5:*  $\tilde{H}_i$  is hyperbolic if and only if

$$\sqrt{\lambda_{\max}(\tilde{Q}_{i1}) + \tilde{\eta}_i} < \delta(A_i, C_i). \quad (35)$$

*Proof:* Similar to lemma 4. ■

*Theorem 1:* For the large-scale system given by (3), the decentralized controller and observer as given by (15) and (16) will result in exponential stabilization of the overall system, if (33) and (35) are satisfied for all  $i = 1 : N$ .

*Proof:* If (33) and (35) are satisfied for all  $i = 1 : N$ , then from lemmas 4, 5 and 1, the AREs (29) and (30) have symmetric positive definite solutions,  $P_i$  and  $\tilde{P}_i$ , respectively. Thus, one can choose  $V(x, \tilde{x})$  given by (20) as the Lyapunov function of the overall system (3). Exponential stabilization of the overall closed-loop system is achieved. ■

*Remark 3:* Since  $\delta(A_i^\top, \sqrt{2\gamma^2} (B_i^\top B_i)^{-1/2} B_i^\top)$

is a continuous function of  $\gamma$ ,  $f(\gamma) \triangleq \sqrt{2\gamma^2} - \delta(A_i^\top, \gamma(B_i^\top B_i)^{-1/2} B_i^\top)$  is also a continuous function of  $\gamma$ . Therefore, if  $f(\gamma) < 0$ , then there exists a  $\gamma_1 > \gamma$  such that  $f(\gamma_1) < 0$ . Hence, if  $f(\gamma) < 0$ , then there exists an  $\eta_i > 0$  such that  $f(\sqrt{2(\gamma^2 + \eta_i)}) < 0$ , that is, (33) holds. Same arguments hold for condition (35). Hence, instead of checking conditions given by (33) and (35), one can respectively check the following two conditions

$$\sqrt{2} \gamma < \delta(A_i^\top, \sqrt{2} \gamma (B_i^\top B_i)^{-1/2} B_i^\top), \quad (36a)$$

$$\sqrt{\lambda_{\max}(\tilde{Q}_{i1})} < \delta(A_i, C_i). \quad (36b)$$

Notice that the conditions given by (36) guarantees the existence of  $\eta_i > 0$  and  $\tilde{\eta}_i > 0$ , but not their values. Conditions (33) and (35) with specified  $\eta_i$  and  $\tilde{\eta}_i$  give the rate of convergence of the controller and observer, respectively.

*Remark 4:* If the sufficient condition given in lemma 4 is satisfied, then there exists a  $P_i = P_i^\top > 0$  satisfying (29)

and a  $K_i$  can be obtained by (27). As a special case, when the matrix  $B_i$  is invertible, we have the following result.

*Lemma 6:* If  $B_i$  is invertible, then there always exists a symmetric positive definite solution  $P_i$  to the ARE (29).

*Proof:* When  $B_i$  is invertible,  $I - B_i(B_i^\top B_i)^{-1}B_i^\top = 0$ , as a result, the ARE (29) reduces to Lyapunov equation

$$A_i^\top P_i + P_i A_i + (\gamma^2 + \eta_i)I = 0. \quad (37)$$

Since  $(\gamma^2 + \eta_i)I > 0$  and  $A_i$  is stable, there always exists a  $P_i = P_i^\top > 0$  satisfying (37) for any  $\gamma, \eta_i > 0$ . ■

Similarly, we have the following lemma on  $C_i$ .

*Lemma 7:* If  $C_i$  is invertible, there always exists a symmetric positive definite matrix  $\tilde{P}_i$  to the ARE (30).

*Proof:* Because  $\tilde{Q}_{i1} + \tilde{\eta}_i I$  is a constant matrix,  $\varepsilon_i$  can be chosen large enough such that

$$\tilde{Q}_i \triangleq \tilde{Q}_{i1} + \tilde{\eta}_i I - \varepsilon_i C_i^\top C_i < 0. \quad (38)$$

Notice that  $-\tilde{Q}_i = \tilde{G}_i \tilde{G}_i^\top > 0$ . The ARE (30) reduces to

$$(-A_i)^\top \tilde{P}_i + \tilde{P}_i (-A_i) - \tilde{P}_i \tilde{P}_i + \tilde{G}_i \tilde{G}_i^\top = 0. \quad (39)$$

Since  $(I, -A_i^\top)$  is observable and  $(-A_i^\top, \tilde{G}_i)$  is controllable, ARE (39) has a unique positive definite solution  $\tilde{P}_i$  [21]. ■

It is possible that a matrix  $C_i$  is invertible, i.e., all state variables of the  $i$ -th subsystem are available for feedback, then the condition for existence of a positive definite solution to the ARE (30) is given by (38) instead of (35).

*Remark 5:* Since the constant  $\varepsilon_i$  affects the convergence rate of the observation error and the stability of the overall system, a natural question to ask is what happens if we increase/decrease the value of  $\varepsilon_i$ . The following lemma gives a result related to this.

*Lemma 8:* If the sufficient condition (35) is satisfied for a particular  $\varepsilon_i$ , then there exists a symmetric positive definite solution to the ARE (29) for any  $\varepsilon'_i \geq \varepsilon_i$  instead of  $\varepsilon_i$ . Moreover, the solution corresponding to  $\varepsilon'_i$  for the ARE (30),  $\tilde{P}'_i$ , satisfies  $\tilde{P}'_i \geq \tilde{P}_i$ .

*Proof:* Lemma 8 is the direct result of lemma 3. ■

*Remark 6:* The convergence rate of each subsystem observer can be increased by amplifying the observer gain matrix  $L_i$  obtained from (27) by  $\varepsilon'_i/\varepsilon_i$ . Let  $L_i = \varepsilon'_i \tilde{P}_i^{-1} C_i^\top / 2$ , where  $\tilde{P}_i = \tilde{P}_i^\top > 0$  is the solution to the ARE (30) obtained with  $\varepsilon_i$ . Then the inequality (31) becomes

$$\dot{V}(x, \tilde{x}) \leq -\sum_{i=1}^N \left[ \eta_i x_i^\top x_i + \tilde{\eta}_i \tilde{x}_i^\top \tilde{x}_i + (\varepsilon'_i - \varepsilon_i) \tilde{x}_i^\top C_i^\top C_i \tilde{x}_i \right]. \quad (40)$$

Since  $\varepsilon'_i - \varepsilon_i > 0$ , the convergence rate of  $\tilde{x}_i$  to zero is increased.

*Remark 7:* The inequality (22) used in separating the terms can be quite conservative. Instead of (22), one can use the following inequality

$$X^\top Y + Y^\top X \leq X^\top X / \varepsilon + \varepsilon Y^\top Y, \quad \varepsilon > 0 \in \mathbb{R}. \quad (22')$$

The disadvantage of this approach is that one has to choose the constants  $\varepsilon$  in the design also. Using (22') for the terms with under braces in (21), we obtain

$$-\tilde{x}_i^\top A_{Bi}^\top P_i x_i - x_i^\top P_i A_{Bi} \tilde{x}_i \leq \tilde{x}_i^\top A_{Bi}^\top A_{Bi} \tilde{x}_i / \varepsilon_{i1} + \varepsilon_{i1} x_i^\top P_i P_i x_i, \quad (23'a)$$

$$h_i^\top P_i x_i + x_i^\top P_i h_i \leq h_i^\top h_i / \varepsilon_{i2} + \varepsilon_{i2} x_i^\top P_i P_i x_i, \quad (23'b)$$

$$h_i^\top \tilde{P}_i \tilde{x}_i + \tilde{x}_i^\top \tilde{P}_i h_i \leq h_i^\top h_i / \varepsilon_{i3} + \varepsilon_{i3} \tilde{x}_i^\top \tilde{P}_i \tilde{P}_i \tilde{x}_i. \quad (23'c)$$

Then by choosing same observer matrix as given by (27) and the controller gain matrix as follows

$$K_i = -(\varepsilon_{i1} + \varepsilon_{i2})(B_i^\top B_i)^{-1} B_i^\top P_i / 2, \quad (27')$$

the two AREs give by (29) and (30), respectively, become

$$A_i^\top P_i + P_i A_i + (\varepsilon_{i1} + \varepsilon_{i2})P_i(I - B_i(B_i^\top B_i)^{-1}B_i^\top)P_i + \gamma^2 I + \eta_i I = 0, \quad (29')$$

$$A_i^\top \tilde{P}_i + \tilde{P}_i A_i + \varepsilon_{i3} \tilde{P}_i \tilde{P}_i + \tilde{Q}_{i1} + \tilde{\eta}_i I - \varepsilon_i C_i^\top C_i = 0. \quad (30')$$

The sufficient conditions for the existence of the positive definite solutions to (29') and (30'), respectively, are

$$\delta(A_i^\top, \sqrt{(\gamma^2 + \eta_i)(\varepsilon_{i1} + \varepsilon_{i2})} (B_i^\top B_i)^{-1/2} B_i^\top) > \sqrt{(\gamma^2 + \eta_i)(\varepsilon_{i1} + \varepsilon_{i2})}. \quad (33')$$

$$\sqrt{\varepsilon_{i3}(\lambda_{\max}(\tilde{Q}_{i1}) + \tilde{\eta}_i)} < \delta(A_i, \sqrt{\varepsilon_{i3} \varepsilon_i} C_i). \quad (35')$$

where  $\tilde{Q}_{i1} \triangleq (\varepsilon_{i1} + \varepsilon_{i2})^2 P_i B_i (B_i^\top B_i)^{-1} B_i^\top P_i / (4\varepsilon_{i1})$ . Notice that with this approach, one has to also choose three more constants,  $\varepsilon_{i1}, \varepsilon_{i2}, \varepsilon_{i3}$ , for each subsystem in the design of the decentralized controller and observer.

## V. SIMULATION

Consider the following large-scale system:

$$\begin{aligned} \dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ -125 & -22.5 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1 + h_1(x), \quad x_1(0) = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \\ y_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_1 \\ \dot{x}_2 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -37.5 & -50 & -13.5 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2 + h_2(x)x, \quad x_2(0) = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} \\ y_2 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_2 \end{aligned}$$

where  $x_1^\top = [x_{11} \ x_{12}]$ ,  $x_2^\top = [x_{21} \ x_{22} \ x_{23}]$ ,  $x^\top = [x_1^\top \ x_2^\top]$ ,  $h_1(x) = \alpha_1 \cos(x_{22})H_1 x$ ,  $h_2(x) = \alpha_1 \cos(x_{11})H_2 x$ ,  $\alpha_1 = \alpha_2 = 0.2$ ,  $H_1 = I_{2,5}/\sqrt{10}$  and  $H_2 = I_{3,5}/\sqrt{15}$  are normalized matrices.  $I_{i,j}$  denotes an  $i \times j$  dimensional matrix with all its elements being 1. The gain  $\gamma$  is computed based on the values of  $\alpha_1, \alpha_2, H_1$  and  $H_2$  as  $\gamma = 0.4$ . The following constant gains are chosen:  $\varepsilon_1 = 0.5$ ,  $\varepsilon_2 = 0.125$ ,  $\eta_1 = 0.1$ ,  $\tilde{\eta}_1 = 0.5$ ,  $\eta_2 = 0.01$  and  $\tilde{\eta}_2 = 0.2$ . It is checked that the conditions given by (33) and (35) are satisfied for both subsystems. Thus, there exist positive definite solutions to (29) and (30). These solutions are

$$\begin{aligned} P_1 &= \begin{bmatrix} 0.9317 & 0.0080 \\ 0.0080 & 0.0061 \end{bmatrix}, \quad \tilde{P}_1 = \begin{bmatrix} 1.6912 & 0.0114 \\ 0.0114 & 0.0116 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} 0.7895 & 0.6622 & 0.0306 \\ 0.6622 & 0.8394 & 0.0378 \\ 0.0306 & 0.0378 & 0.0093 \end{bmatrix}, \quad \tilde{P}_2 = \begin{bmatrix} 0.4309 & 0.3852 & 0.0055 \\ 0.3852 & 0.6109 & 0.0149 \\ 0.0055 & 0.0149 & 0.0085 \end{bmatrix}. \end{aligned}$$

The control and observer gain matrices from (27) are

$$K_1^\top = \begin{bmatrix} -0.0080 \\ -0.0061 \end{bmatrix}, \quad K_2^\top = \begin{bmatrix} -0.0306 \\ -0.0378 \\ -0.0093 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0.1488 \\ -0.1465 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0.3358 \\ -0.2157 \\ 0.1625 \end{bmatrix}.$$

To increase the convergence rate of the observers, we choose  $100L_1$  and  $10L_2$  as the observer gain matrices for the first and second subsystem, respectively, in the simulation.

The simulation results are shown in Figures 1 and 2. In Fig. 1, the state  $x_{11}$  and its estimate  $\hat{x}_{11}$ , the state  $x_{12}$  and its estimate  $\hat{x}_{12}$ , and the control  $u_1$  are shown in the first, second and third plot, respectively. Fig. 2 shows the states  $x_2$ , their estimates  $\hat{x}_2$ , and the control  $u_2$ . It can be observed from both the figures that the state of the overall system,  $x$ , and their estimates,  $\hat{x}$ , converge to zero.

## VI. CONCLUSION

In this paper, we proposed a decentralized control and observer for a class of large-scale interconnected nonlinear systems. The interconnecting nonlinearity of each subsystem was assumed to be bounded by a quadratic form of states of the overall system. Local output signals from each subsystem are required to generate the local feedback controller and exact knowledge of the nonlinear interconnection is not required for designing the proposed decentralized controller and observer. Sufficient conditions for the existence of the decentralized controller and observer are given via the analysis of two AREs. Simulation results on a numerical example verify the proposed design.

There are some challenging problems related to the quantity  $\delta$ . The quantities  $\delta(A, C)$  or  $\delta(A^T, B^T)$  are realization dependent. The properties of  $\delta$  as a function of various state-space realizations is of importance. In particular, finding the state-space realization that maximizes the value of  $\delta$  will be useful.

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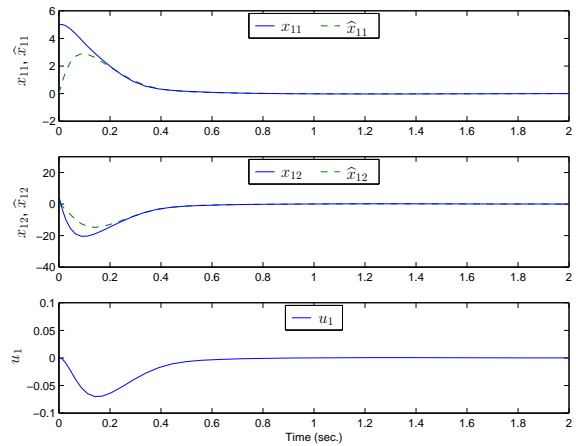


Fig. 1. Simulation results for the first sub-system.

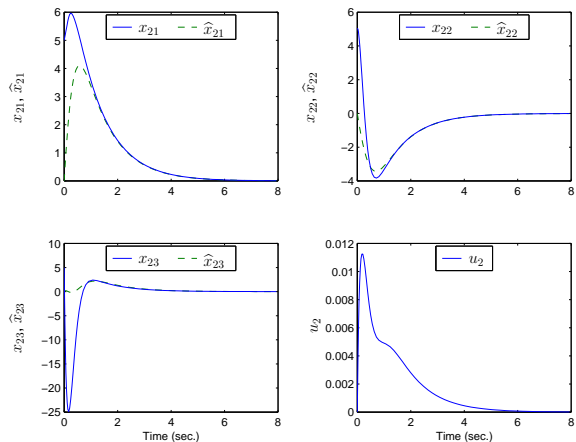


Fig. 2. Simulation results for the second sub-system.