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# Stability of Modern Guidance Laws with Model Mismatch

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*Abstract*—**It is well known that modern guidance laws are capable of better performance than the traditional proportional navigation laws, provided that the missile model in the guidance law matches that of the actual missile. In this paper we study the stability of modern guidance laws when the missile's actual model differs from the model used in the design. The analysis is performed by means of Lyapunov functions and by means of the multivariable circle criterion.**

## I. Introduction

It is well known that modern guidance laws are capable of better performance than the traditional proportional navigation laws, if the missile model in the guidance law matches that of the actual missile. The purpose of this paper is to provide a partial answer to the question: To what extent may the dynamics of the actual missile differ from that used in the derivation of the guidance law without compromising the superior performance of the modern guidance laws? An attempt is made to provide an answer to this question by studying the stability properties of the guidance loop.

Both time and frequency domain tools are used to study the stability of the guidance loop. The time domain analysis is based on Lyapunov functions generated by modifications of the cost to go associated with the solution of the guidance law optimization problem. The frequency domain analysis is based on the use of the circle criterion. Using the time domain analysis, explicit analytic terms are derived and ensure that: 1. the missile states remain bounded throughout the time interval studied; 2. the zero effort miss is a decreasing function. The first condition implies that the missile acceleration is bounded.

The time domain and frequency methods complement each other, the total stability region derived is the union of the stability regions. Since proportional navigation can be derived as a special case of the optimal guidance law, the stability analysis derived in this paper is also applicable to the study of a Proportional Navigation Guidance (PNG) loop.

Stability of guidance systems has been investigated only for the proportional navigation law. In the case of a non-ideal dynamics the PNG system tends to diverge as the time to go approaches zero [1], [2]. Therefore the relevant stability for a PNG system is the stability associated with a finite time interval. Guelman [2] and Gurfil et al [3] supply sufficient conditions for the finite time stability of a PNG system. In their work the length of the time interval, for which the system is globally absolutely asymptotically stable, is defined in terms of a lower bound on the time to go. Guelman [2] uses the Popov criterion to derive the lower bound. Gurfil et al [3] use the circle criterion in their derivation. Rew et al [4] apply practical stability methods to derive a lower bound on the time to go for a PNG system with a single time constant dynamics. Tanaka and Hirofumi [5] use absolute stability methods to determine the stable range of a PNG system.

The paper is organized as follows. In the next section we present the notation and the required background material about optimal guidance laws. Sections 3 and 4 respectively present the time domain (Lyapunov) stability and the frequency domain (circle criterion) analyses. The detailed stability domain calculations for first and second order missile models, using both time and frequency domain methods are carried out in section 5.

## II. Problem Statement

### A. Scenario

Interceptor missiles are usually skid to turn, roll stabilized and have two independent perpendicular guidance channels in the lateral planes. Hence the guidance problem can be treated as planar. The equations are linearized around the initial line sight. The state variables are the distance, velocity, and the acceleration perpendicular to the line of sight.

### B. Dynamic Equations

The equations of the relative motion between the interceptor and its target, in a direction normal to the initial line of sight, are

$$\begin{aligned}\dot{y} &= v \\ \dot{v} &= a_T - a\end{aligned}. \tag{1}$$

Based on the linearization assumption and assuming that in the endgame the closing velocity is constant, the interception time $t_f$ is taken as fixed.

Let the guidance law be derived on the basis of the missile model

$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{F}\mathbf{z} + \mathbf{g}a_c \\ a &= \mathbf{h}\mathbf{z} + da_c\end{aligned}. \tag{2}$$

Since stability is unaffected by the target acceleration we henceforth assume that $a_T = 0$. Then, in matrix vector form the dynamic equation is

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$$\begin{bmatrix} \dot{y} \\ \dot{v} \\ \dot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\mathbf{h} \\ 0 & 0 & \mathbf{F} \end{bmatrix} \begin{bmatrix} y \\ v \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ -d \\ \mathbf{g} \end{bmatrix} a_c , \tag{3}$$

or more compactly

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}a_c . \tag{4}$$

### C. Optimal Guidance Problem and its Solution

The optimal guidance law is the solution of the optimization problem

$$\min_{a_c} \left\{ \frac{1}{2}\mathbf{x}_f^T \mathbf{K}_f \mathbf{x}_f + \frac{1}{2}\int_0^{t_f} a_c^2 dt \right\} \tag{5}$$

subject to Eq. (4) where $\mathbf{K}_f$ is a square matrix with a single non-zero entry in the $(1,1)$ position. The solution of this optimization problem is well known and is given by

$$a_c = -\mathbf{B}^T\mathbf{K}\mathbf{x} \tag{6}$$

where $\mathbf{K}$ is the solution of the Riccati differential equation

$$\dot{\mathbf{K}} = -\mathbf{A}^\mathrm{T}\mathbf{K} - \mathbf{K}\mathbf{A} + \mathbf{K}\mathbf{B}\mathbf{B}^\mathrm{T}\mathbf{K}; \quad \mathbf{K}(t_f) = \mathbf{K}_f \tag{7}$$

In the following we make use of the following well known facts [6], [7] about the optimal law. The optimal law may be expressed as

$$a_c = \Lambda(t_g) Zem \tag{8}$$

where $t_g$ is the time to go and $Zem$ is the zero effort miss distance, namely the miss that would be obtained by setting $a_c$ to zero from the current to the final time.

Now suppose that the real missile differs from the model used to generate the guidance, and is instead given by

$$\begin{aligned} \dot{z}_m &= \mathbf{F}_m z_m + \mathbf{g}_m a_c \\ a &= \mathbf{h}_m z_m + d_m a_c \end{aligned} \tag{9}$$

where we assume that $\mathbf{F}_m, \mathbf{g}_m, \mathbf{h}_m, d_m$ is a minimal realization and $\mathbf{F}_m$ is stable.

The actual system is defined by replacing $\mathbf{F}, \mathbf{g}, \mathbf{h}, d$ in Eq.(3) with, $\mathbf{F}_m, \mathbf{g}_m, \mathbf{h}_m, d_m$ , and may be expressed as

$$\dot{\mathbf{x}}_m = \mathbf{A}_m \mathbf{x}_m + \mathbf{B}_m a_c \tag{10}$$

where $\mathbf{x}_m = [y \; v \; \mathbf{z}_m]^T$. If the size of the state $\mathbf{z}$ differs from that of $\mathbf{z}_m$, and this is known, then we modify the optimal law as

$$a_c = -\mathbf{B}^T\mathbf{K}\mathbf{G}\mathbf{x}_m \tag{11}$$

where the matrix $\mathbf{G}$ is

$$\mathbf{G} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_r \end{bmatrix} \tag{12}$$

and $\mathbf{G}_r$ satisfies

$$\mathbf{z} \approx \mathbf{G}_r \mathbf{z}_m \tag{13}$$

that is; $\mathbf{G}_r\mathbf{z}_m$ is an approximation to $\mathbf{z}$ .

### D. Guidance Loop Stability

The guidance problem is inherently a finite time problem. Hence the concept of stability of this problem differs from that of time invariant problems. For the stability of the guidance system we require that:

1. The $Zem$ is a decreasing function of time.
2. The norm of the missile state vector $\mathbf{z}_m$ is a non-increasing function of time.

In this paper we restrict our attention to studying whether stability is retained over the entire problem interval. If the one non-zero entry in $\mathbf{K}_f$ is unbounded then the two stability requirements ensure that the missile hits (with zero miss distance) the target with bounded values of its state vector.

The next sections deal with the stability of the guidance loop where the dynamics of the actual missile differ from that used in the guidance derivation.

## III. TIME DOMAIN ANALYSIS

In this section we define the Lyapunov function [8] that we use for the time-domain analysis, and derive some of its properties.

### A. The Stability Condition

We seek a Lyapunov function of the form

$$V = \mathbf{x}_m^T \mathbf{G}^T \mathbf{K}\mathbf{G}\mathbf{x}_m + \mathbf{z}_m^T \mathbf{R}\mathbf{z}_m \tag{14}$$

where $\mathbf{R}$, is an as yet undetermined, positive definite matrix. The matrix $\mathbf{K}$ has the following special form

$$\mathbf{K} = \gamma\, \mathbf{n}\mathbf{n}^T \tag{15}$$

where $\gamma = \mathbf{K}(1,1)$, $\mathbf{n}^T = \begin{bmatrix} 1 & t_g & \mathbf{n}_r^T(t_g) \end{bmatrix}$ and $\mathbf{n}_r(t_g)$ is the lateral displacement response due to the missile's initial acceleration states. Thus the Lyupanov function can be written as

$$V = \gamma\, Zem^2 + \mathbf{z}_m^T \mathbf{R}\mathbf{z}_m . \tag{16}$$

where $Zem$ is an approximation of the zero effort miss based on the missile model (2) used in deriving the optimal guidance law. Note that this approximation relies on the relation $\mathbf{x} \approx \mathbf{G}\mathbf{x}_m$ obtained from Eqs.(12) and (13).

This Lyapunov function is only positive semi-definite, but the negativity of its derivative is a sufficient condition for the stability of the optimal law. Note also that $\gamma \to \infty$ as the weight on the terminal miss grows without bound.

Using Eq.(7) we obtain that the time derivative of the Lyapunov function (14) is

$$\dot{V} = \begin{aligned} & Zem^2\gamma^2\mathbf{n}^T\left\{-\mathbf{B}\mathbf{B}_m^T\mathbf{G} + \mathbf{B}\mathbf{B}^T - \mathbf{G}\mathbf{B}_m\mathbf{B}^T\right\}\mathbf{n} + \\ & 2\gamma\mathbf{x}_m^T\left[\mathbf{A}_m^T\mathbf{G}^T - \mathbf{G}^T\mathbf{A}^T\right]\mathbf{n}Zem + \\ & \mathbf{z}_m^T\left[\mathbf{F}_m^T\mathbf{R} + \mathbf{R}\mathbf{F}_m\right]\mathbf{z}_m - 2\gamma\,\mathbf{z}_m^T\mathbf{R}\mathbf{g}_m\mathbf{B}^T\mathbf{n}\, Zem \end{aligned} \tag{17}$$

Using the structure of the $\mathbf{G}, \mathbf{A}, \mathbf{A}_m$ matrices $\dot{V}$ may be rewritten as

$$\dot{V} = -\begin{bmatrix} Zem & \mathbf{z}_m^T \end{bmatrix} \mathbf{S} \begin{bmatrix} Zem \\ \mathbf{z}_m \end{bmatrix} \tag{18}$$

where,

$$\mathbf{S} = \begin{bmatrix} \gamma^2 \mathbf{n}^T \left[ \mathbf{B}\mathbf{B}_m^T \mathbf{G}^T - \mathbf{B}\mathbf{B}^T + \mathbf{G}\mathbf{B}_m \mathbf{B}^T \right] \mathbf{n} & \gamma \left[ t_g (\mathbf{h}_m - \mathbf{h}\mathbf{G}_r) - \mathbf{n}_r^T (\mathbf{G}_r \mathbf{F}_m - \mathbf{F}\mathbf{G}_r) \right] + \gamma\, \mathbf{n}^T \mathbf{B}\mathbf{g}_m^T \mathbf{R} \\ \gamma \left[ t_g (\mathbf{h}_m^T - \mathbf{G}_r^T \mathbf{h}^T) - (\mathbf{F}_m^T \mathbf{G}_r^T - \mathbf{G}_r \mathbf{F}^T) \mathbf{n} \right] + \gamma\, \mathbf{R}\mathbf{g}_m \mathbf{B}^T \mathbf{n}_r & -\left( \mathbf{F}_m^T \mathbf{R} + \mathbf{R}\mathbf{F}_m \right) \end{bmatrix} \tag{19}$$

Then a sufficient condition for stability is the existence of $\mathbf{R}$ , such that

$$\mathbf{R} > \mathbf{0}, \quad \mathbf{S} > \mathbf{0} \tag{20}$$

Because the functions $\gamma$ and $\mathbf{n}$ depend on time, equations (19) and (20) describe an infinite system of *Linear Matrix Inequalities* [9], [10].

### B. *Conditions for Small Variations Stability*

We now show that when the missile is stable and the actual missile dynamics are equal to the dynamics assumed in the optimal guidance problem then there always exists an $\mathbf{R}$ satisfying $\mathbf{R} > 0$ . Further, by continuity, the stability is retained for values of the missile dynamics near its assumed values.

When the actual missile dynamics equals that of the optimization problem then by substituting $\mathbf{F}_m = \mathbf{F}$ , $\mathbf{B}_m = \mathbf{B}$ , $\mathbf{h}_m = \mathbf{h}$ , $\mathbf{G}_r = \mathbf{I}$ in equation (19) we obtain,

$$\mathbf{S} = \begin{bmatrix} \gamma^2 \mathbf{n}^T \mathbf{B}\mathbf{B}^T \mathbf{n} & \gamma \mathbf{n}^T \mathbf{B}\mathbf{g}^T \mathbf{R} \\ \gamma\, \mathbf{R}\mathbf{g}\mathbf{B}^T \mathbf{n} & -\mathbf{F}^T \mathbf{R} - \mathbf{R}\mathbf{F} \end{bmatrix} \tag{21}$$

Using Schur complements [9] this is equivalent to

$$-\mathbf{F}^{\mathbf{T}} \mathbf{R} - \mathbf{R}\mathbf{F} > \mathbf{0} \tag{22}$$

$$\mathbf{n}^T \mathbf{B} \left[ 1 - \mathbf{g}^T \mathbf{R} (-\mathbf{F}^T \mathbf{R} - \mathbf{R}\mathbf{F})^{-1} \mathbf{R}\mathbf{g} \right] \mathbf{B}^T \mathbf{n} > 0 \tag{23}$$

Since we assume that the missile's transfer function is stable, there exists a positive definite solution $\mathbf{R}$ of Eq.(22) for <u>any</u> positive definite $\mathbf{Q}$

$$\mathbf{F}^T \mathbf{R} + \mathbf{R}\mathbf{F} = -\mathbf{Q} \tag{24}$$

Substituting(24) into(23) we obtain

$$\mathbf{n}^T \mathbf{B} \left[ 1 - \mathbf{g}^T \mathbf{R}\mathbf{Q}^{-1} \mathbf{R}\mathbf{g} \right] \mathbf{B}^T \mathbf{n} > 0 \tag{25}$$

which can be satisfied by choosing $\mathbf{Q} > \mathbf{0}$ sufficiently small. Therefore the nominal closed loop system is stable. Further, because the left hand side of inequality (25) is a continuous function, stability is retained in the face of small parameter variations.

### C. *Specialization of the LMI*

In this section we derive the special form of the LMI defined by Eqs.(19) and (20) for the case that: 1. the missile model used in the design of the optimal law is a first order system; 2. the transfer function of the actual missile is strictly proper. We now use these assumptions to simplify the system of LMI's from the previous section. From the assumptions we conclude that

$$\mathbf{F} = -\frac{1}{\tau_p}, \mathbf{g} = \frac{1}{\tau_p}, \mathbf{h} = 1, d = 0 . d_m = 0, \mathbf{G}_r = \mathbf{h}_m \tag{26}$$

Substituting Eqs. (26) into Eq. (19) and using Schur complements and the fact that $\gamma$ and $\mathbf{n}_\mathbf{r}$ are non-negative scalars we find that the inequality

$$\mathbf{S} > \mathbf{0} \tag{27}$$

is equivalent to

$$\mathbf{T} > \mathbf{0} \tag{28}$$

where $\mathbf{T}$ is

$$\mathbf{T} = \begin{bmatrix} \frac{1}{\tau_p} \left( \mathbf{g}_m^T \mathbf{h}_m^T - 1 + \mathbf{h}_m \mathbf{g}_m \right) & \left[ \mathbf{h}_m \left( \mathbf{F}_m - \frac{1}{\tau_p} \mathbf{I} \right) - \frac{1}{\tau_p} \mathbf{g}_m^T \mathbf{R} \right] \\ \left[ \left( \mathbf{F}_m^T - \frac{1}{\tau_p} \mathbf{I} \right) \mathbf{h}_m^T - \frac{1}{\tau_p} \mathbf{R}\mathbf{g}_m \right] & -\mathbf{F}_m^T \mathbf{R} - \mathbf{R}\mathbf{F}_m \end{bmatrix} \tag{29}$$

Thus we arrive at a set of 2 LMI's $\mathbf{R} > \mathbf{0}, \quad \mathbf{S} > \mathbf{0}$ instead of an infinite set, as a sufficient condition for stability.

## IV. FREQUENCY DOMAIN ANALYSIS

### A. *Stability Conditions for First Order Guidance Law*

We recall the actual missile model described by Eq. (9). Assuming that the missile model used for the design of the guidance law is a first order model of the form

$$\frac{a}{a_c} = \frac{1 + s\tau_z}{1 + s\tau_p} \tag{30}$$

then the acceleration command, as defined by the guidance law, is

$$a_c = \Lambda\, Zem \tag{31}$$

where,

$$Zem = y + vt_g - \tau_p g_p(t_g)\, a \tag{32}$$

and $\Lambda$ is defined by

$$\Lambda(t_g) = \frac{g_{pz}(t_g)}{\int_0^{t_g} g_{pz}^2(\lambda) d\lambda} \tag{33}$$

The variables $\tau_p g_p$ , $g_{pz}$ are the lateral displacements due to initial acceleration and due to impulse input applied to $a(s)/a_c(s)$, respectively [7].

Note that $Zem$ is the estimated zero effort miss calculated by the guidance law model.

Let us define a new state variable $w$ by

$$w = (\tau_p d_m + \frac{1}{g_p \Lambda}) a_c \tag{34}$$

Using Eq. (34) the time derivative of $w$ is

$$\dot{w} = (\tau_p d_m + \frac{1}{g_p \Lambda})\dot{a}_c - (\frac{g_p \dot{\Lambda} + \dot{g}_p \Lambda}{g_p^2 \Lambda^2}) a_c \tag{35}$$

The following intermediate results can be used to derive a compact expression for $\dot{w}$ :

$$\dot{a}_c = \dot{\Lambda} Zem + \Lambda Z\dot{e}m \tag{36}$$

$$\dot{\Lambda} = \frac{\dot{g}_{pz}}{g_{pz}} \Lambda + g_{pz} \Lambda^2 \tag{37}$$

$$Z\dot{e}m = -g_p a - \tau_p g_p \dot{a} \tag{38}$$

$$\dot{a} = h_m \dot{z}_m + d_m \dot{a}_c \tag{39}$$

$$\begin{aligned} (\tau_p d_m + \frac{1}{g_p \Lambda})\dot{a}_c &= -\mathbf{h}_m (I + \tau_p \mathbf{F}_m)\mathbf{z}_m - \\ (\tau_p \mathbf{h}_m \mathbf{g}_m + d_m) a_c &+ \frac{1}{g_p \Lambda}(\frac{\dot{g}_{pz}}{g_{pz}} + g_{pz}\Lambda) a_c \end{aligned} \tag{40}$$

Using equations (36) to (40) $\dot{w}$ can be expressed as

$$\dot{w} = -\mathbf{h}_m (I + \tau_p \mathbf{F}_m)\mathbf{z}_m - (\tau_p \mathbf{h}_m \mathbf{g}_m + d_m) a_c - \frac{\dot{g}_p / g_p}{g_p \Lambda} a_c \tag{41}$$

Equation (41) enables us to define the following set of state equations

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{F}_m & 0 \\ -\mathbf{h}_m (I + \tau_p \mathbf{F}_m) & 0 \end{bmatrix}, \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{g}_m & 0 \\ -(\tau_p \mathbf{h}_m \mathbf{g}_m + d_m) & -1 \end{bmatrix} \tag{42}$$

$$\bar{\mathbf{C}} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \bar{\mathbf{D}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{43}$$

where the state $x$ is $[z_m w]^T$ and the control $u$ satisfies

$$u_1 = a_c = -\frac{g_p \Lambda}{1 + d_m \tau_p g_p \Lambda} w; \quad u_2 = \frac{\dot{g}_p / g_p}{g_p \Lambda} a_c \tag{44}$$

The control can also be described by

$$u = -\Phi(t, \mathbf{y}) = -\begin{bmatrix} \dfrac{g_p \Lambda}{1 + d_m \tau_p g_p \Lambda} & 0 \\ 0 & \dfrac{\dot{g}_p / g_p}{1 + d_m \tau_p g_p \Lambda} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{45}$$

Assuming that the time varying gains $g_p \Lambda / (1 + d_m \tau_p g_p \Lambda)$ and $(\dot{g}_p / g_p)/(1 + d_m \tau_p g_p \Lambda)$ are sector bounded, the stability of the system (42) to (45) can be analyzed using a multivariable version of the Circle Criterion, [8] Corollary 5.6.28, p.225, which we repeat here,

*Multivariable circle criterion*

Consider the system

$$\begin{aligned} \dot{\mathbf{x}} &= \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{B}}\mathbf{u} \\ \mathbf{y} &= \bar{\mathbf{C}}\mathbf{x} + \bar{\mathbf{D}}\mathbf{u} \end{aligned} \tag{46}$$

where

$$\mathbf{u} = -\mathbf{\Phi}(t, \mathbf{y}) \tag{47}$$

Suppose that (i) $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ is controllable (ii) $(\bar{\mathbf{C}}, \bar{\mathbf{A}})$ is observable (iii) $\mathbf{\Phi}$ belongs to the sector $[\varepsilon, \mu]$.

Define

$$\mathbf{W}_\varepsilon(s) = \mathbf{W}(s)[I + \varepsilon \mathbf{W}(s)]^{-1} \tag{48}$$

where,

$$\mathbf{W}(s) = \bar{\mathbf{C}}(sI - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}} + \bar{D} \tag{49}$$

and suppose that

$$\tilde{\mathbf{W}}(s) = \mathbf{W}_\varepsilon(s) + \frac{1}{\mu - \varepsilon} \tag{50}$$

is strictly positive real, and all poles of $\mathbf{W}_\varepsilon(s)$ have negative real parts. Under these conditions the system (46), (47) is exponentially stable.

We constrain the time interval $[t_1, t_2]$ where we expect to establish stability so that the following are valid

$$0 < \varepsilon \le \frac{g_p \Lambda}{1 + d_m \tau_p g_p \Lambda} \le \mu \; ; 0 < \varepsilon \le \frac{\dot{g}_p / g_p}{1 + d_m \tau_p g_p \Lambda} \le \mu \tag{51}$$

where $\mu > \varepsilon$. We choose $\varepsilon$ to be an arbitrarily small positive number, for example $\mu = 1/\varepsilon$. For large values of $t_g$ the feedback gains tend to zero, while for small values of $t_g$ they tend to infinity. We have here implicitly assumed that the missile transfer function (30) used to design the guidance law is a minimum phase, or alternatively we restrict $t_1$ and $t_2$ so that (51) is valid. Under minimum phase conditions we may take $t_1$ to be any finite number and $t_2$ to be arbitrarily close to the terminal time.

The poles of $\mathbf{W}(s)$ consist of the poles of the missile transfer function associated with (9), and a simple pole at the origin. To apply the theorem we still have to prove that all the poles of $\mathbf{W}_\varepsilon(s)$ have negative real parts, and $\tilde{\mathbf{W}}(s)$ is strictly positive real. Now suppose that $\mathbf{W}(s)$ is positive real. Then (a) $\mathbf{W}_\varepsilon(s)$ is positive real, since it represents a positive real system with a strictly positive gain $\varepsilon$ in a negative feedback configuration. Further, by considering the properties of the root locus of $W_\varepsilon(s)$ (with $\varepsilon$ as the parameter) we may conclude that for sufficiently small but positive values of $\varepsilon$ all the poles of $\mathbf{W}_\varepsilon(s)$ have negative real parts. (b) $\tilde{\mathbf{W}}(s)$ is a strictly positive real transfer function since $W_\varepsilon(s)$ is positive real.

It is then sufficient to show that $\mathbf{W}(s)$ is positive real in order to show that the guidance loop is exponentially stable on any interval where $\mathbf{\Phi}$ in Eq. (45) is in the sector $[\varepsilon, 1/\varepsilon]$. For the class of control laws studied in this paper this implies stability up to a time arbitrarily close to the terminal time.

Note that the positive real condition can be tested directly in terms of the state matrices or as a linear matrix inequality [11].

*B. Conditions for Stability Near $t_g = 0$*

Consider the case where the missile model used in the guidance law is described by Eq. (26), and the actual missile model is described by Eq. (9) with $d_m = 0$.

Conditions for stability near $t_{go} = 0$ can be derived using approximations for the time varying functions $-\frac{\dot{g}_p / g_p}{g_p \Lambda}$ in Eq. (41) and $g_p \Lambda$ in Eq. (44) assuming $d_m = 0$.

Using the following approximations near $t_g = 0$,

$$\frac{\dot{g}_p}{g_p} \approx -\frac{2}{t_g}; g_p \Lambda \approx \frac{5}{t_g}; \frac{\dot{g}_p / g_p}{g_p \Lambda} \approx -\frac{2}{5} \tag{52}$$

then system (42), (43), and (44) is reduced to

$$\begin{bmatrix} \dot{\mathbf{z}}_m \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_m & 0 \\ -\mathbf{h}_m (I + \tau_p \mathbf{F}_m) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_m \\ w \end{bmatrix} + \begin{bmatrix} -\mathbf{g}_m \\ (\tau_p \mathbf{h}_m \mathbf{g}_m - \frac{2}{5}) \end{bmatrix} a_{cc} \tag{53}$$

where,

$$a_{cc} = -a_c = -g_p \Lambda w \tag{54}$$

The minus sign in Eq.(54) guarantees that the transfer function $W(s) = \bar{\mathbf{c}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1} \bar{\mathbf{b}}$; $\bar{\mathbf{c}} = [0 \quad 1]$ is associated with negative feedback as required in the circle criterion.

Note that $g_p \Lambda \approx \frac{5}{t_g} \in [0, \infty]$. In this case the stability is guaranteed if the transfer function $W(s)$ is positive real.

## V. EXAMPLES

In the following we analyze three cases: a first order missile, a second order missile without a zero and a second order system including a zero using the time domain and the frequency domain methods. In all the examples the transfer function used in the derivation of the optimal guidance law is described by Eq. (26)

*A. Example 1: First Order Missile*

In this section we study the stability properties of the optimal guidance law when the actual missile is described by a first order system

$$\dot{z} = -\frac{1}{\tau_m} z + \frac{1}{\tau_m} a_c \; ; \; a = z \tag{55}$$

*1) Time domain analysis*

Using Eq. (29) we obtain that $\mathbf{T}$ is

$$\mathbf{T} = \begin{bmatrix} \frac{2}{\tau_p \tau_m} - \frac{1}{\tau_p^2} & \frac{1}{\tau_m} - \frac{1}{\tau_p} + \frac{1}{\tau_p \tau_m} \mathbf{R} \\ \frac{1}{\tau_m} - \frac{1}{\tau_p} + \frac{1}{\tau_p \tau_m} \mathbf{R} & \frac{2}{\tau_m} \mathbf{R} \end{bmatrix} \tag{56}$$

Choosing $\mathbf{R} = \tau_p$ we conclude that if $\tau_m < 2\tau_p$ both conditions $\mathbf{R} > \mathbf{0}$, $\mathbf{T} > \mathbf{0}$ are fulfilled and stability is retained.

*2) Frequency domain analysis*

We obtain that the transfer function $W(s)$ associated with the stability near $t_g = 0$ has the form

$$W(s) = \frac{3/5}{s} + \frac{(\tau_p - \tau_m)}{1 + s\tau_m} \tag{57}$$

Hence if $\tau_m \le \tau_p$ then $W(s)$ is positive real and the guidance loop is stable.

Note that in this case the frequency domain analysis generates a stability region which is a subset of the region generated by the time domain analysis.

*3) Time responses*

The system was simulated for a number of values of $\tau_m$, and it was noted that the condition $\tau_m < 2\tau_p$ properly delimits the regions of stability and instability.

*B. Example 2: Second Order Missile*

In this section we show that for the case of a second order missile transfer function a zero is required in order to obtain a stable guidance loop.

*1) Case 1: Second Order Missile without a zero*

Let the missile state equation be defined by

$$\mathbf{F}_m = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\omega_0 \xi \end{bmatrix}; \mathbf{g}_m = \begin{bmatrix} 0 \\ \omega_0^2 \end{bmatrix}; \mathbf{h_m} = [1 \quad 0]; d_m = 0 \tag{58}$$

Substituting these values into Eq.(29) we obtain that $\mathbf{T}(1,1) = -1/\tau_p{}^2$.

Hence the stability test fails. The system was simulated and it was found that the states indeed diverge near the final time.

*2) Case 2: Second order missile including a zero*

We now consider a missile with a second order transfer function which includes zero at $-1/\mathrm{T}_z$.

The associated system matrices are

$$\mathbf{F}_m = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\xi\omega_0 \end{bmatrix}; \mathbf{g}_m = \begin{bmatrix} 0 \\ \omega_0^2 \end{bmatrix}; \mathbf{h_m} = [1 \quad T_z]; d_m = 0 \tag{59}$$

*a) Time domain analysis*

Substituting these values into equation Eq. (29) the inequality $\mathbf{T} > \mathbf{0}$ can be derived. We also need $\mathbf{R} > \mathbf{0}$. The last two inequalities define a system of Linear Matrix Inequalities. Although an analytic solution was not found a set of parameters satisfying both inequalities was found by means of the Matlab® LMI toolbox [10]. The results of the calculation for $\tau_p = 1\sec$ are shown in Figure 1.

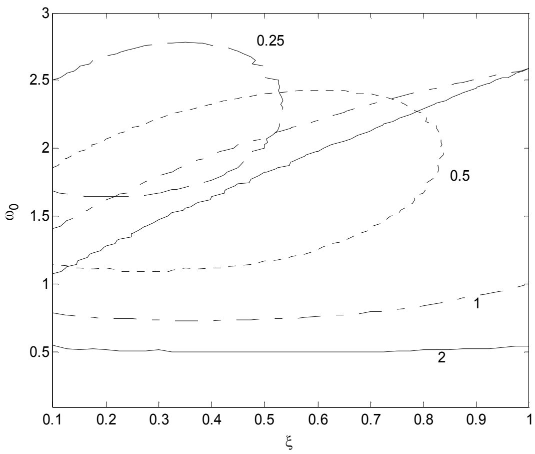


**Figure 1: Regions of stability, time domain results for $T_z = 2,\ 1,\ 0.5,\ 0.25$ seconds**

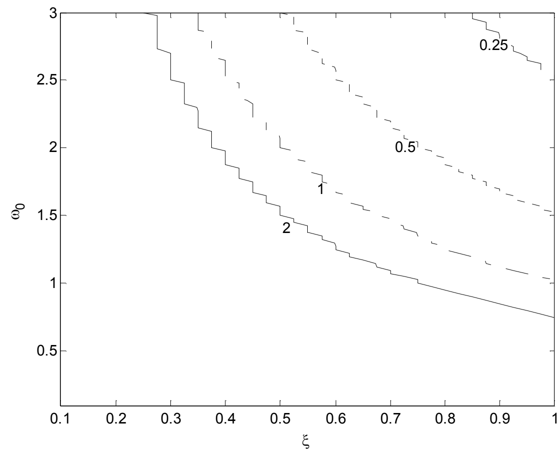


**Figure 2: Regions of stability, frequency domain results for $T_z = 2,\ 1,\ 0.5,\ 0.25$ seconds**

*b) Frequency domain analysis*

The system (59) was used to calculate the transfer function $W(s)$ associated with the stability near $t_g = 0$.

Testing the transfer function $W(s)$ for the positive real condition using its LMI version [11] and the Matlab® LMI toolbox [10], we obtain the stability region in Figure 2 where stability is verified in the region above and to the right of the lines.

In contrast to the first order case there is no simple relationship between the regions predicted by the time domain and the frequency domain methods. In fact both sets of stability regions are subsets of the actual stability region as we demonstrate in the next section.

*c) Time Responses*

Again we take $\tau_p = 1\sec$ and study the acceleration commands generated in the closed loop guidance system.

Three cases are examined. **Case a:** $\omega_0 = 1 rad/\sec$, $\xi = 0.6$, $T_z = 2\sec$. In this case the time domain criterion predicts stability, while the frequency domain criterion fails to predict stability. **Case b:** $\omega_0 = 0.3\ rad/\sec$ $\xi = 0.6$, $T_z = 2$ sec. In this case neither criterion predicts stability. **Case c:** $\omega_0 = 3 rad/\sec$, $\xi = 0.6$, $T_z = 2$ sec. In this case the time domain criterion does not predict stability, while the frequency domain criterion does predict stability. The missile state responses for each have been investigated, and it was found that cases **a** and **c** are stable while case **b** results in an unstable system.

## VI. SUMMARY AND CONCLUSIONS

A frequency domain method and a time domain method were used to study the properties of a missile guidance system based on an optimal guidance law. Both of these methods provide sufficient conditions for the stability of the closed loop system. We have shown that stability is retained in the face of significant missile parameter departure from its nominal values. Nevertheless, by means of simple simulations we have shown that the derived stability regions are subsets of the actual region of stability.

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