Robustness of CDMA Power Control Against Disturbances and Time-Delays

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***Abstract*— This paper studies robustness of a gradient-type CDMA power control algorithm with respect to disturbances and time-delays. This problem is of practical importance because unmodeled secondary interference effects from neighboring cells play the role of disturbances, and propagation delays are ubiquitous in wireless data networks. We first show $L_{\mathbf{p}}$-stability, for $\mathbf{p} \in [1, \infty]$, with respect to additive disturbances. Next, using the L_∞ property and a loop transformation, we prove that global asymptotic stability is preserved for sufficiently small time-delays in forward and return channels. For larger delays, we achieve global asymptotic stability by scaling down the step-size in the gradient algorithm.**

I. Introduction

Power control has been a significant research topic for wireless communication networks [1], [2], [3], [5]. Increased power ensures longer transmission distance and higher data transfer rate, but it also consumes battery and produces greater amount of interference to neighboring users. In code division multiple access (CDMA) systems, this problem has been formulated as a noncooperative game by Alpcan *et al.* [4], [5], in which each user tries to maximize

$$\max_i \; J_i = U_i\left(\gamma_i\left(p\right)\right) - P_i\left(p_i\right), \tag{1}$$

where U_i is a utility function for the i^{th} user, and P_i represents the cost of power. The function $\gamma_i\left(p\right)$ in (1) is the signal-to-interference ratio (SIR) of the i^{th} user, defined as

$$\gamma_i\left(p\right) := \frac{Lh_ip_i}{\sum\limits_{k\neq i} h_ip_k + \sigma^2}, \tag{2}$$

where L is the spreading gain of the CDMA system, h_i is the channel gain between the i^{th} mobile and the base station, and σ^2 is the noise variance containing the contribution of the secondary background interference. The authors then propose the gradient-type power control law

$$\dot{p}_i = -\lambda_i \frac{\partial J_i}{\partial p_i} = \frac{dU_i}{d\gamma_i} \frac{L\lambda_i h_i}{\sum\limits_{k\neq i} h_k p_k + \sigma^2} - \lambda_i \frac{dP_i(p_i)}{dp_i}, \quad \lambda_i > 0, \tag{3}$$

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and prove asymptotic stability of the Nash equilibrium under several assumptions on the functions $U_i\left(\cdot\right)$ and $P_i\left(\cdot\right)$, and on the number of users.

In this paper, we study the robustness of this control law against additive disturbances and time-delays. This study is important because of modelling errors, power noise, secondary interference effects, such as those from neighboring cells, and propagation delays. Our starting point is a passivity-based stability proof for the algorithm (3), presented in the companion paper [6]. Using the Lyapunov functions obtained from this passivity analysis, in this paper we first show that the controller (3) is robust to additive $L_{\mathbf{p}}$-disturbances. In particular, L_∞-disturbances are pursued here within the input-to state stability (ISS) framework of Sontag [7], which makes explicit the vanishing effect of initial conditions. We then proceed to the study of delays using this ISS property. We first represent the delayed system as a feedback interconnection of the nominal delay-free model, and a perturbation block, the ISS-gain of which depends on the amount of delay. Then we prove global asymptotic stability (GAS) for sufficiently small delays using the ISS Small-Gain Theorem of Teel *et al.* [8], [9]. For larger delays, we achieve GAS by scaling down the stepsize λ_i.

The paper is organized as follows. Section 2 gives the notation and definitions used in this paper, and reviews the first-order gradient power control algorithm and its nominal stability properties. Section 3 considers additive disturbances and proves an $L_{\mathbf{p}}$-stability property. Section 4 derives bounds for time-delays that the system can tolerate without losing stability. For larger delays, it proposes a scaling of the step-size λ_i in (3). Conclusions are given in Section 5.

II. Notations and Preliminaries

We will use *projection functions* to ensure nonnegative values for physical quantities, such as power. Given a function $f\left(x\right)$, its positive projection is defined as

$$\left(f\left(x\right)\right)_x^+ := \begin{cases} f\left(x\right) & \text{if } x > 0, \text{ or } x = 0 \text{ and } f\left(x\right) \geq 0 \\ 0 & \text{if } x = 0 \text{ and } f\left(x\right) < 0. \end{cases}$$

If x and $f\left(x\right)$ are vectors, then $\left(f\left(x\right)\right)_x^+$ is interpreted in the component-wise sense. When $\left(f\left(x\right)\right)_x^+ = 0$, we say that

the projection is *active*. When $(f(x))_x^+ = f(x)$, we say that the projection is *inactive*. We denote by $\|x\|$ the vector norm of x, and by $\|x\|_{L_\mathbf{p}}$ the $L_\mathbf{p}$-norm of $x(t)$, $p \in [0,\infty]$. For $d \in L_\infty$, we define $\|d\|_a = \lim_{t\to\infty} \sup \|d(t)\|$. A system $\dot{x} = f(x,u)$ is said to be *input-to state stable* (ISS) if there exist class-K functions[1] $\gamma_0(\cdot)$ and $\gamma(\cdot)$ such that, for any input $u(\cdot) \in L_\infty^m$ and $x_0 \in R^n$, the response $x(t)$ from the initial state $x(0) = x_0$ satisfies

$$\|x\|_{L_\infty} \le \gamma_0(\|x_0\|) + \gamma\left(\|u\|_{L_\infty}\right), \quad \|x\|_a \le \gamma(\|u\|_a).$$

We now review the stability properties of the gradient-type power control law (3). As shown in Alpcan *et al.* [4], [5], the following assumption ensures that a unique Nash equilibrium p^* exists for the game (1).

Standing Assumption: *The function $P(\cdot)$ in (1) is twice continuously differentiable, nondecreasing, and strictly convex in p_i, i.e.,*

$$\frac{\partial P_i(p_i)}{\partial p_i} \ge 0, \quad \frac{\partial^2 P_i(p_i)}{\partial p_i^2} > 0, \quad \forall p_i, \tag{4}$$

and

$$U_i(\gamma_i) = u_i \log(\gamma_i + L), \tag{5}$$

where u_i is a constant, and γ_i and L are as in (2).

The choice of the logarithmic utility function in (5) is meaningful because it represents the maximum achievable bandwidth as in Shannon's Theorem [**?**]. Substituting this $U_i(\gamma_i)$ in (3) and adding projection $(\cdot)_{p_i}^+$ to ensure positivity of p_i, we obtain

$$\dot{p}_i = \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} + \frac{u_i\lambda_i h_i}{\sum_k h_k p_k + \sigma^2}\right)_{p_i}^+. \tag{6}$$

Note that in this derivation, the term $\sum_{k\neq i} h_k p_k + \sigma^2$ in (3) has been cancelled by the derivative of the logarithmic U_i, and replaced by $\sum_k h_k p_k + \sigma^2$. This means that we can represent (6) as in Figure 1, in which the diagonal entries Σ_i of the forward block are given by

$$\Sigma_i: \qquad \dot{p}_i = \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} + u_i\lambda_i w_i\right)_{p_i}^+, \quad i = 1,\cdots,M, \tag{7}$$

where

$$w := -h \cdot q, \tag{8}$$

$$h := \begin{bmatrix} h_1 & h_2 & \cdots & h_M \end{bmatrix}^T, \tag{9}$$

[1]A function $\gamma(\cdot)$ is defined to be class-K if it is continuous, zero at zero, and strictly increasing.

$$q := \varphi(y) = -\frac{1}{y+\sigma^2}, \tag{10}$$

$$y := h^T p. \tag{11}$$

In this representation the forward block corresponds to the mobiles and the feedback path corresponds to the base station. Stability of the equilibrium p^* is proved in [6], using passivity properties of both the feedforward and feedback paths:

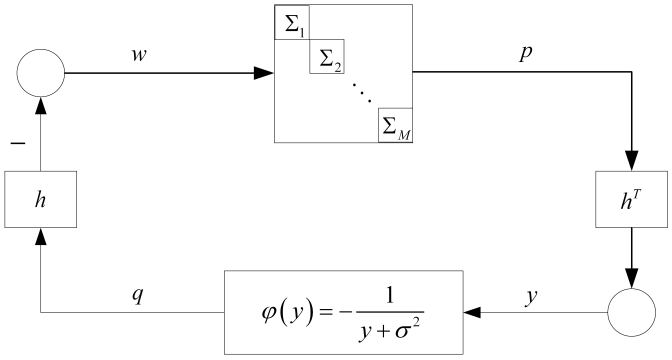

Fig. 1. First-order gradient algorithm of CDMA power control.

Proposition 1: Consider the feedback system (9)-(7), represented as in Figure 1. The equilibrium $p = p^*$ is globally asymptotically stable.

III. Robustness to Disturbances

In this section, we prove $L_\mathbf{p}$ and input-to-state stability of the first-order gradient power control algorithm (6) with respect to additive disturbances, such as secondary interference effects from neighboring cells. Denoting by d_{1i} and d_{2i} disturbances acting on the i^{th} mobile, we replace (6) with the perturbed model,

$$\dot{p}_i = \left(\frac{u_i\lambda_i h_i}{\sum_k h_k p_k + d_{2i} + \sigma^2} - \lambda_i \frac{dP_i(p_i)}{dp_i} + d_{1i}\right)_{p_i}^+, \tag{12}$$

and prove an $L_\mathbf{p}$-stability property ($\mathbf{p} \in [1,\infty]$):

Theorem 1: Consider the power control system (12), where $P_i(p_i)$ satisfies, for all $p_i \ge 0$, $i = 1,2,\cdots M$,

$$P_i''(p_i) \ge \eta$$

where η is a positive constant. If $d_1 = [d_{11}, d_{12}, \cdots d_{1M}]$ and $d_2 = [d_{21}, d_{22}, \cdots d_{2M}]$are $L_\mathbf{p}$-disturbances, $\mathbf{p} = [0,\infty)$, then (12) guarantees

$$\|p - p^*\|_{L_\mathbf{p}} \le \bar{u}\bar{\lambda}(\alpha p)^{-\frac{1}{p}} \sqrt{\sum_i \frac{1}{u_i\lambda_i}(p_i(0) - p_i^*)^2} + \sqrt{2}\bar{u}\bar{\lambda}(\alpha_1 q)^{-\frac{1}{q}} \|\beta\|_{L_\mathbf{p}} \tag{13}$$

where

$$\alpha = \frac{u\lambda\eta}{\bar{u}}, \quad \beta = \frac{\bar{u}\bar{\lambda}}{\sqrt{2}u\lambda}\left\|d_1\right\| + \frac{\bar{u}\bar{\lambda}\bar{h}}{\sqrt{2}\sigma^4}\left\|d_2\right\| \tag{14}$$

$$\begin{array}{l} \bar{u} = \max\limits_i \{u_i\}, \ u = \min\limits_i \{u_i\}, \\ \bar{\lambda} = \max\limits_i \{\lambda_i\}, \ \lambda = \min\limits_i \{\lambda_i\}, \\ \bar{h} = \max\limits_i \{h_i\}, \ h = \min\limits_i \{h_i\} \end{array} \tag{15}$$

and $\mathbf{q}$ and $\mathbf{p}$ are complementary indices, that is

$$\mathbf{p}^{-1} + \mathbf{q}^{-1} = 1. \tag{16}$$

When $\mathbf{p} = \infty$, the system satisfies the ISS estimate

$$\left\|p - p^*\right\| \leq \bar{u}\bar{\lambda}e^{-\alpha t}\sqrt{\sum_i \frac{1}{u_i\lambda_i}\left(p_i(0) - p_i^*\right)^2} + \frac{\sqrt{2}\bar{u}\bar{\lambda}}{\alpha}\left\|\beta_1\right\|_{L_\infty}. \tag{17}$$

Proof: The derivative of the storage function

$$V_1\left(p - p^*\right) = \frac{1}{2}\sum_i \frac{1}{u_i\lambda_i}\left(p_i - p_i^*\right)^2 \tag{18}$$

along the solution of (12) is

$$\dot{V}_1 = \sum_i \frac{1}{u_i\lambda_i}\left(p_i - p_i^*\right)\left(-\lambda_i\frac{dP_i(p_i)}{dp_i} + u_i\lambda_i w_i + d_{1i}\right)_{p_i}^{+}. \tag{19}$$

We first note

$$\begin{array}{l} \frac{1}{u_i\lambda_i}\left(p_i - p_i^*\right)\left(-\lambda_i\frac{dP_i(p_i)}{dp_i} + u_i\lambda_i w_i + d_{1i}\right)_{p_i}^{+} \\ \leq \frac{1}{u_i\lambda_i}\left(p_i - p_i^*\right)\left(-\lambda_i\frac{dP_i(p_i)}{dp_i} + u_i\lambda_i w_i + d_{1i}\right) \end{array}$$

because, if the projection is inactive then both sides of the inequality are equal, and if the projection is active, $p_i = 0$ and $-\lambda_i\frac{dP_i(p_i)}{dp_i} + u_i\lambda_i w_i + d_{1i} < 0$, then the left hand side is zero, and the right hand side is non-negative. By adding and subtracting $u_i\lambda_i w_i^*$ and $\frac{u_i\lambda_i h_i}{\sum_k h_k p_k + \sigma^2}$, we obtain

$$\begin{array}{l} \dot{V} \leq \sum\limits_i \frac{1}{u_i\lambda_i}(p_i - p_i^*)\left(-\lambda_i\frac{dP_i(p_i)}{dp_i} + \underbrace{u_i\lambda_i w_i^*}_{\lambda_i\frac{dP_i\left(p_i^*\right)}{dp_i}}\right) \\ \quad + \sum\limits_i \frac{1}{u_i\lambda_i}(p_i - p_i^*)\left(-\underbrace{u_i\lambda_i w_i^*}_{\frac{u_i\lambda_i h_i}{\sum_k h_k p_k^* + \sigma^2}} + \frac{u_i\lambda_i h_i}{\sum_k h_k p_k + \sigma^2}\right) \\ \quad + \sum\limits_i \frac{1}{u_i\lambda_i}(p_i - p_i^*)\left(-\frac{u_i\lambda_i h_i}{\sum_k h_k p_k + \sigma^2} + u_i\lambda_i w_i + d_{1i}\right) \\ = \sum\limits_i \frac{\left(p_i - p_i^*\right)}{u_i}\left(-\frac{dP_i(p_i)}{dp_i} + \frac{dP_i\left(p_i^*\right)}{dp_i}\right) \\ \quad + \sum\limits_i \frac{\left(p_i - p_i^*\right)}{u_i\lambda_i}\left(\frac{u_i\lambda_i h_i}{\sum_k h_k p_k + \sigma^2} - \frac{u_i\lambda_i h_i}{\sum_k h_k p_k^* + \sigma^2}\right) \\ \quad + \sum\limits_i \frac{\left(p_i - p_i^*\right)}{u_i\lambda_i}\left(\frac{u_i\lambda_i h_i}{\sum_k h_k p_k + d_{2i} + \sigma^2} - \frac{u_i\lambda_i h_i}{\sum_k h_k p_k + \sigma^2}\right) \\ \quad + \sum\limits_i \frac{1}{u_i\lambda_i}(p_i - p_i^*)d_{1i} \\ = \sum\limits_i \frac{\left(p_i - p_i^*\right)}{u_i}\left(-\frac{dP_i(p_i)}{dp_i} + \frac{dP_i\left(p_i^*\right)}{dp_i}\right) \\ \quad + \left(\frac{1}{y+\sigma^2} - \frac{1}{y^*+\sigma^2}\right)(y - y^*) \\ \quad + \sum\limits_i \left(\frac{1}{\sum_k h_k p_k + d_{2i} + \sigma^2} - \frac{1}{\sum_k h_k p_k + \sigma^2}\right) h_i(p_i - p_i^*) \\ \quad + \sum\limits_i \frac{1}{u_i\lambda_i}(p_i - p_i^*)d_{1i}. \end{array} \tag{20}$$

Since $\left(\frac{1}{y+\sigma^2} - \frac{1}{y^*+\sigma^2}\right)(y - y^*) \leq 0$ and $P_i'' \geq \eta$, we obtain

$$\begin{array}{l} \dot{V} \leq \sum\limits_i -\frac{\eta}{u_i}\left(p_i - p_i^*\right)^2 + \sum\limits_i \frac{1}{u_i\lambda_i}\left(p_i - p_i^*\right)d_{1i} \\ \quad + \sum\limits_i h_i\left|\frac{1}{y + d_{2i} + \sigma^2} - \frac{1}{y+\sigma^2}\right|\left|p_i - p_i^*\right| \\ \leq -\frac{\eta}{\bar{u}}\left\|p - p^*\right\|^2 + \frac{1}{u\lambda}\left\|p - p^*\right\|\left\|d_1\right\| \\ \quad + \sum\limits_i \frac{h_i}{\sigma^4}\left|d_{2i}\right|\left|p_i - p_i^*\right| \\ \leq -2\frac{u\lambda\eta}{\bar{u}}V + \sqrt{2}\frac{\bar{u}\bar{\lambda}}{u\lambda}\sqrt{V}\left\|d_1\right\| + \sqrt{2}\frac{\bar{u}\bar{\lambda}\bar{h}}{\sigma^4}\sqrt{V}\left\|d_2\right\| \\ \leq -2\alpha V + 2\beta\sqrt{V} \end{array}$$

which, from [**?**, Theorem 6.1], implies that

$$\left\|\sqrt{V}\right\|_{L_{\mathbf{p}}} \leq (\alpha p)^{-\frac{1}{p}}\left\|\sqrt{V(0)}\right\| + (\alpha_1 q)^{-\frac{1}{q}}\left\|\beta\right\|_{L_{\mathbf{p}}}, \tag{21}$$

and

$$\left\|\sqrt{V}\right\| \leq e^{-\alpha t}\left\|\sqrt{V(0)}\right\| + \frac{1}{\alpha}\left\|\beta\right\|_{L_\infty}. \tag{22}$$

Inequality (13) and (17) then follows from (21), (22), and

$$\left\|p - p^*\right\| \leq \sqrt{2}\bar{u}\bar{\lambda}\left\|W(t)\right\|.$$

□

IV. Robustness to Time-Delays

We now prove that global asymptotic stability is preserved for sufficiently small time-delays between mobiles and the base station. This study is important because wireless data networks exhibit significant propagation delays. Denoting by τ_i the round-trip delay for the i^{th} mobile, we represent the algorithm (6) as in Figure 3:

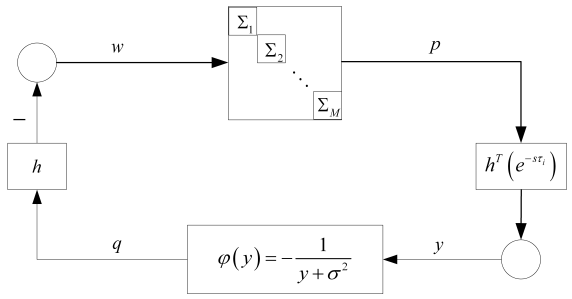

Fig. 2. First-order gradient algorithm of CDMA power control in the case of time-delay.

where $h^T\left(e^{-s\tau_i}\right) := \begin{bmatrix} h_1 e^{-s\tau_1} & h_2 e^{-s\tau_2} & \cdots & h_M e^{-s\tau_M} \end{bmatrix}$. To transform the delay robustness problem to the framework of Theorem 3, we add and subtract the term h^T from $h^T\left(e^{-s\tau_i}\right)$ in Figure 3, and represent it as in Figure 4, where the inner loop represents the nominal system without delay, and the outer loop is the perturbation due to delay.

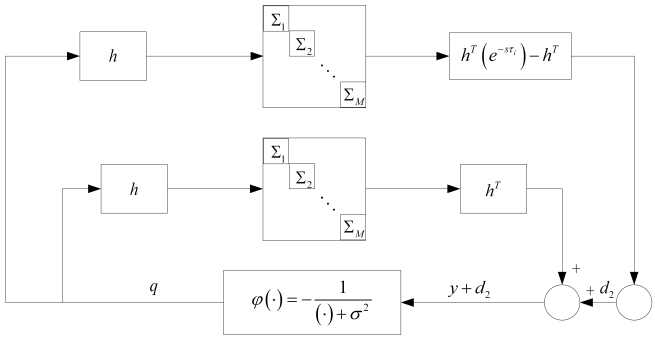

Fig. 3. Equivalent system of gradient algorithm of CDMA power control after loop-transformation.

With this representation we prove stability using a small-gain argument. From Theorem 3, it is not difficult to show that the ISS gain of the feedback path from d_2 to $\tilde{q}$ is

$$g_1 = \frac{\|h\|}{\sigma^4}\left(\frac{\sqrt{2}\bar{u}\bar{\lambda}}{\alpha}\frac{\bar{u}\bar{\lambda}\bar{h}}{\sqrt{2}\sigma^4}+\frac{1}{\sigma^4}\right). \tag{23}$$

In Theorem 4 below, we also show that the feedforward path from $\tilde{q}$ to d_2 has gain

$$g_2 = \sqrt{2M\bar{h}}\bar{\tau}\left(\frac{\eta_1\bar{u}^2\bar{\lambda}^2\|R\|}{u\lambda\eta_2}+\bar{u}\bar{\lambda}\bar{h}\right) \tag{24}$$

where

$$\bar{\tau} := \max_i\{\tau_i\}. \tag{25}$$

This means that for sufficiently small $\bar{\tau}$, the small-gain condition

$$g_1 g_2 < 1 \tag{26}$$

holds and GAS is preserved. If $\bar{\tau}$ is not sufficiently small, then we can scale down the stepsize λ_i in the power control (9) to recover GAS:

Theorem 2: Consider the feedback interconnection in Figure 3, and suppose that $P_i(p_i)$, $i = 1, 2, \cdots M$, are such that for all $p_i \geq 0$,

$$\eta_1 \geq P_i''(p_i) \geq \eta_2 \tag{27}$$

with $\eta_1 > \eta_2 > 0$. If either the delay $\bar{\tau}$ or the stepsize λ is small enough that (26) is satisfied, then the power control scheme (9)-(7) guarantees global asymptotic stability.

Proof: We first show that the feedforward path in Figure 4 has gain g_2 as in (24). We prove this in two steps, where the first step gives the gain from $q-q^*$ to $\dot{p}$, and the second step gives the gain from $\dot{p}$ to d_2.

Step 1: We let

$$V_1(p-p^*) = \frac{1}{2}\sum_i \frac{1}{u_i\lambda_i}(p_i-p_i^*)^2$$

as in (18). Following the same stapes as (20), we obtain

$$\begin{aligned}\dot{V} &\leq \sum_i \tfrac{1}{u_i}(p_i-p_i^*)\left(-\tfrac{dP_i(p_i)}{dp_i}+\tfrac{dP_i(p_i^*)}{dp_i}\right)+(q-q^*)(y-y^*)\\ &\leq -\tfrac{\eta_2}{\bar{u}}\|p-p^*\|^2+\|q-q^*\|\|R\|\|p-p^*\|\\ &\leq -2\tfrac{u\lambda\eta_2}{\bar{u}}V+\sqrt{2\bar{u}\bar{\lambda}}\|q-q^*\|\|R\|\sqrt{V}.\end{aligned}$$

From [**?**, Theorem 6.1], we have

$$\left\|\sqrt{V(t)}\right\| \leq e^{-\frac{u\lambda\eta_2}{\bar{u}}t}\left\|\sqrt{V(0)}\right\| + \frac{\bar{u}\sqrt{\bar{u}\bar{\lambda}}\|h\|}{\sqrt{2}u\lambda\eta_2}\|q-q^*\|_{L_\infty}$$

which, with $\|p(t)-p^*\| \leq \sqrt{2\bar{u}\bar{\lambda}}\left\|\sqrt{V(t)}\right\|$, yields

$$\|p(t)-p^*\|_{L_\infty} \leq \tfrac{\sqrt{\bar{u}\bar{\lambda}}}{\sqrt{u\lambda}}\|p(0)-p^*\| + \tfrac{\bar{u}^2\bar{\lambda}\|R\|}{u\lambda\eta_2}\|q-q^*\|_{L_\infty} \tag{28}$$

$$\|p(t)-p^*\|_a \leq \frac{\bar{u}^2\bar{\lambda}\|R\|}{u\lambda\eta_2}\|q-q^*\|_a. \tag{29}$$

Next, because

$$\left|\left(-\lambda_i\tfrac{dP_i(p_i)}{dp_i}+u_i\lambda_i w_i\right)_{x_i}^+\right| \leq \left|-\lambda_i\tfrac{dP_i(p_i)}{dp_i}+u_i\lambda_i w_i\right|,$$

$$\begin{aligned}\|\dot{p}\| &\leq \left\|-\lambda\tfrac{dP(p)}{dp}+\lambda\tfrac{dP(p^*)}{dp}+u_i\lambda_i w_i^*-u_i\lambda_i w_i\right\|\\ &\leq \left\|-\lambda\tfrac{dP(p)}{dp}+\lambda\tfrac{dP(p^*)}{dp}\right\|+\|u_i\lambda_i h_i q^*-u_i\lambda_i h_i q\|.\end{aligned}$$

Thus, from (27), we obtain

$$\|\dot{p}\| \leq \bar{\lambda}\eta_1 \|p - p^*\| + \bar{u}\bar{\lambda}\bar{h} \|q - q^*\|,$$

which implies, from (28) and (29)

$$\|\dot{p}\|_{L_\infty} \leq \frac{\bar{\lambda}\eta_1\sqrt{\bar{u}\bar{\lambda}}}{\sqrt{u\lambda}}\|p(0)-p^*\| + \left(\frac{\eta_1 \bar{u}^2\bar{\lambda}^2\|R\|}{u\lambda\eta_2} + \bar{u}\bar{\lambda}\bar{h}\right)\|q-q^*\|_{L_\infty} \quad (30)$$

$$\|\dot{p}(t)\|_a \leq \left(\frac{\eta_1 \bar{u}^2\bar{\lambda}^2\|R\|}{u\lambda\eta_2} + \bar{u}\bar{\lambda}\bar{h}\right)\|q-q^*\|_a. \quad (31)$$

Step 2: Next, we claim that the subsystem from $\dot{p}$ to d_2 satisfies

$$\|d_2\|_a \leq \sqrt{2M\bar{h}\bar{\tau}}\,\|\dot{p}(t)\|_a, \quad (32)$$

$$\|d_2\|_{L_\infty} \leq \sqrt{2M\bar{h}\bar{\tau}}\|\dot{p}\|_{L_\infty} + \sqrt{2M\bar{h}\bar{\tau}} \sup_{-\bar{\tau}<t\leq 0} \left\| -\lambda\frac{dP(p(t))}{dp} - diag\left\{ \begin{array}{cccc} u_1\lambda_1 & \cdots & u_M\lambda_M \end{array} \right\}^{w(t)} \right\|. \quad (33)$$

To prove this, we first note that

$$\begin{aligned} |d_2(t)| &= \left|\sum_{i=1}^M h_i p_i(t-\tau_i) - \sum_{i=1}^M h_i p_i(t)\right| \leq \sum_{i=1}^M h_i \int_{t-\tau_i}^t |\dot{p}_i(\sigma)|d\sigma \\ &\leq \sum_{i=1}^M h_i \int_{\max\{0,t-\tau_i\}}^t |\dot{p}_i(\sigma)|d\sigma + \sum_{i=1}^M h_i \int_{\min\{0,t-\tau_i\}}^0 |\dot{p}_i(\sigma)|d\sigma \end{aligned}$$

which implies by Young's Inequality

$$\begin{aligned} d_2(t)^2 &\leq 2\left(\sum_{i=1}^M h_i \int_{\max\{0,t-\tau_i\}}^t |\dot{p}_i(\sigma)|\,d\sigma\right)^2 \\ &\quad + 2\left(\sum_{i=1}^M h_i \int_{\min\{0,t-\tau_i\}}^0 |\dot{p}_i(\sigma)|\,d\sigma\right)^2 \\ &\leq 2M\sum_{i=1}^M \left(h_i \int_{\max\{0,t-\tau_i\}}^t |\dot{p}_i(\sigma)|\,d\sigma\right)^2 \\ &\quad + 2M\sum_{i=1}^M \left(h_i \int_{\min\{0,t-\tau_i\}}^0 |\dot{p}_i(\sigma)|\,d\sigma\right)^2 \\ &\leq 2M\bar{h}\sum_{i=1}^M \left(\int_{\max\{0,t-\tau_i\}}^t |\dot{p}_i(\sigma)|\,d\sigma\right)^2 \\ &\quad + 2M\bar{h}\sum_{i=1}^M \left(\int_{\min\{0,t-\tau_i\}}^0 |\dot{p}_i(\sigma)|\,d\sigma\right)^2. \end{aligned}$$

Applying Cauchy-Schwarz inequality to each term, we get

$$\begin{aligned} d_2(t)^2 &\leq 2M\bar{h}\bar{\tau}\sum_{i=1}^M \int_{\max\{0,t-\tau_i\}}^t |\dot{p}_i(\sigma)|^2\,d\sigma \\ &\quad + 2M\bar{h}\bar{\tau}\sum_{i=1}^M \int_{\min\{0,t-\tau_i\}}^0 |\dot{p}_i(\sigma)|^2\,d\sigma \end{aligned}$$

which implies the vector norm of d_2 is

$$\begin{aligned} \|d_2\| &\leq \sqrt{2M\bar{h}\bar{\tau}\sum_{i=1}^M \int_{\max\{0,t-\tau_i\}}^t |\dot{p}_i(\sigma)|^2 d\sigma} \\ &\quad + \sqrt{2M\bar{h}\bar{\tau}\sum_{i=1}^M \int_{\min\{0,t-\tau_i\}}^0 |\dot{p}_i(\sigma)|^2 d\sigma}. \end{aligned}$$

Because $\max\{0, t-\tau_i\} \geq \max\{0, t-\bar{\tau}\}$ and $\min\{0, t-\tau_i\} \geq \min\{0, t-\bar{\tau}\}$, we get

$$\begin{aligned} \|d_2\| &\leq \sqrt{2M\bar{h}\bar{\tau}\sum_{i=1}^M \int_{\max\{0,t-\bar{\tau}\}}^t |\dot{p}_i(\sigma)|^2 d\sigma} \\ &\quad + \sqrt{2M\bar{h}\bar{\tau}\sum_{i=1}^M \int_{\min\{0,t-\bar{\tau}\}}^0 |\dot{p}_i(\sigma)|^2 d\sigma}. \end{aligned}$$

By changing the sequence of the sum and integral, we obtain

$$\begin{aligned} \|d_2\| &\leq \sqrt{2M\bar{h}\bar{\tau}\int_{\max\{0,t-\bar{\tau}\}}^t \sum_{i=1}^M |\dot{p}_i(\sigma)|^2 d\sigma} \\ &\quad + \sqrt{2M\bar{h}\bar{\tau}\int_{\min\{0,t-\bar{\tau}\}}^0 \sum_{i=1}^M |\dot{p}_i(\sigma)|^2 d\sigma} \\ &\leq \sqrt{2M\bar{h}\bar{\tau}^2\|\dot{p}(\sigma)\|_{L_\infty}^2} \\ &\quad + \sqrt{2M\bar{h}\bar{\tau}\int_{\min\{0,t-\bar{\tau}\}}^0 \|\dot{p}(\sigma)\|^2 d\sigma}, \end{aligned}$$

from which (32) and (33) follows.

Combining (30)- (31) and (32)-(33) from Step 1 and 2, we conclude that the L_∞-gain and asymptotic gain of the feedforward path are:

$$\|d_2\|_a \leq g_2\|q - q^*\|_a, \quad (34)$$

$$\|d_2\|_{L_\infty} \leq g_2\|q-q^*\|_{L_\infty} + \sqrt{2M\bar{h}\bar{\tau}}\frac{\bar{\lambda}\eta_1\sqrt{\bar{u}\bar{\lambda}}}{\sqrt{u\lambda}}\|\tilde{p}(0)\| + \sup_{-\bar{\tau}<t\leq 0}\left\| -\lambda\frac{dP(p(t))}{dp} - diag\left\{ \begin{array}{cccc} u_1\lambda_1 & \cdots & u_M\lambda_M \end{array} \right\}^{w(t)} \right\|. \quad (35)$$

where g_2 is as in (24).

Step 3: Finally, we show that the feedback path has a complementary gain g_1 as in (23). For the inner loop in Figure 4, it follows from Theorem 3 that

$$\begin{aligned} \|q - q^*\| &= \left\|\frac{1}{y+d_2+\sigma^2} - \frac{1}{y^*+\sigma^2}\right\| \\ &\leq \frac{1}{\sigma^4}\|y - y^* + d_2\| \\ &\leq \frac{\|h\|}{\sigma^4}\|p - p^*\| + \frac{1}{\sigma^4}\|d_2\| \end{aligned}$$

and, thus

$$\|q - q^*\|_a \leq g_1\|d_2\|_a, \quad (36)$$

$$\|q-q^*\|_{L_\infty} \leq \frac{\|R\|}{\sigma^4}\bar{u}\bar{\lambda}e^{-\alpha t}\sqrt{\sum_i \frac{1}{u_i\lambda_i}(p(0)-p^*)^2} + g_1\|d_2\|_{L_\infty}. \quad (37)$$

Substituting (36) and (37) into (34) and(35), and using the small-gain condition (26), we conclude

$$\|d_2\|_a \leq 0, \quad (38)$$

$$\|d_2\|_{L_\infty} \leq \frac{\frac{\|h\|}{\sigma^4}g_2\|p(0)-p^*\| + \frac{\sqrt{2M\bar{h}\bar{u}\bar{\lambda}}\bar{\lambda}\eta_1}{\sqrt{u\lambda}}\bar{\tau}\|p(0)-p^*\|}{1-g_1g_2} + \frac{\sup_{-\bar{\tau}<t\leq 0}\left\|-\lambda\frac{dP(p(t))}{dp} - q(t)\right\|}{1-g_1g_2}. \quad (39)$$

Finally, from Theorem 3, we have

$$\|p-p^*\|_a \leq 0 \tag{40}$$

$$\begin{aligned}\|p-p^*\|_{L_\infty} \leq & \frac{(1-g_1g_2)\|p(0)-p^*\|+\frac{\sqrt{2}\bar{u}^2\bar{\lambda}^2\bar{h}\|h\|}{\sqrt{2}\alpha\sigma^8}g_2\|p(0)-p^*\|}{1-g_1g_2} \\ & + \frac{\frac{\sqrt{2}\bar{u}^2\bar{\lambda}^3\bar{h}}{\sqrt{2}\sigma^4\alpha}\frac{\sqrt{2M\bar{h}\bar{u}\bar{\lambda}\eta_1}}{\sqrt{u\bar{\lambda}}}\bar{\tau}\|p(0)-p^*\|}{1-g_1g_2} \\ & + \frac{\frac{\sqrt{2}\bar{u}^2\bar{\lambda}^2\bar{h}}{\sqrt{2}\sigma^4\alpha}\sup_{-\bar{\tau}<t\leq 0}\left\|-\lambda\frac{dP(p(t))}{dp}-q(t)\right\|}{1-g_1g_2},\end{aligned} \tag{41}$$

which proves global asymptotic stability as defined in [**?**]. If the small-gain condition violates (26), then we can scale down the user-dependent stepsize λ_i by $\kappa > 0$, and rewrite (26) as

$$\frac{\|h\|}{\sigma^4}\left(\frac{\kappa^2\sqrt{2}\bar{u}^2\bar{\lambda}^2\bar{h}}{\sqrt{2}\sigma^4\alpha}+\frac{1}{\sigma^4}\right)\sqrt{2M\bar{h}}\bar{\tau}\kappa\left(\frac{\eta_1\bar{u}^2\bar{\lambda}\|h\|}{u\eta_2}+\bar{u}\bar{h}\right)<1 \tag{42}$$

which is satisfied for sufficiently small κ. Thus, for any delay $\bar{\tau}$, the scaled controller

$$\dot{p}_i = \frac{u_i\kappa\lambda_i h_i}{\sum_k h_k p_k + \sigma^2} - \kappa\lambda_i\frac{dP_i\left(p_i\right)}{dp_i}, \tag{43}$$

where κ is as (**??**), achieves GAS. □

V. Conclusion

We have addressed robustness of the first-order gradient power control algorithm in [4] against disturbances and time-delay. Using an ISS property of the nominal, delay-free, system, and a small-gain argument, we showed that global asymptotic stability is preserved in the presence of small time-delays. For larger delays, we achieved GAS by scaling down the step-size of the gradient algorithm. One shortcoming of reducing the gains, however, is that it may cause degradation in performance. Our next research task will be to investigate how robustness and performance can be improved with the broader classes of controllers proposed in our companion paper [6].

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