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Robust actuator placement in flexible plates subject to worst-case spatial distribution of disturbances

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***Abstract*— In this study we consider the problem of actuator placement in the vibration control of flexible plates. Unlike earlier approaches based on enhanced controllability and/or improved performance measures, we consider the case of actuator placement based on robustness-with-respect to spatial variability of disturbances. As predicted from our extensive numerical investigation, when the spatial distribution of disturbances is known, then the resulting optimal location differs from the one predicted from controllability/LQR approaches. When the distribution is not known, then one may introduce an element of spatial robustness by considering the worst-case spatial distribution. Extensive numerical results are reported here which provide an insight on the judicious actuator placement when spatial distributions of disturbances are taken into consideration for actuator placement and controller design.**

I. INTRODUCTION

The objective of this investigation is to present a methodology for obtaining the optimal location of actuators for a special class of distributed parameter systems when the spatial distribution of disturbances is taken into consideration. This class of systems describes the dynamics of flexible structures and, in general, of second order distributed parameter systems.

The issue of sensor and actuator placement has been addressed in earlier investigations [20], and especially for flexible space structures, in the context of controllability and observability indices [12]. The algorithms for this placement along with references on prior work may be found, for example, in the books [12], [14], [19]. Similar work which considered performance measures was presented in [11] and [8], by studying only the problem of actuator placement and utilized LQR measures for actuator placement. For the case of placing collocated actuator/sensor pairs [4] consider mostly open loop approaches where the controllability of certain modes was enhanced via optimization of the associated controllability Gramian, and [13] consider a spatial $\mathcal{H}^2$ norm as a placement measure.

The treatment here is somewhat different, in the sense that it is based on *minimizing the effects of the spatial distribution of disturbances on a certain performance output*. We follow closed loop techniques that take into consideration the distribution of disturbances. In fact, we expand on prior work for a flexible beam in which the actuators considered were piezoceramic patches [10], [6], and the goal was to place them in locations that provide the additional spatial robustness. The contribution of the current work is twofold: (i) to incorporate spatial distribution of disturbances in the search for optimal actuator placement in 2D spatial domains and (ii) to integrate the actuator placement and the controller design in order to arrive at a system that exhibits both a *temporal* and *spatial* robustness for the worst-case scenario of disturbance distributions.

The mathematical model of the flexible plate under consideration is presented in Section II along with conditions on the locations of piezoceramic patches that would provide approximate controllability. The finite dimensional representation of the associated partial differential equation along with both open loop and closed loop measures for actuator placement are presented in Section III. Integrated actuator placement and controller design which exhibit robustness with respect to *both* spatial and temporal components of disturbances are also proposed in Section III. Implementation issues along with a description of the various spatial distribution of disturbances are summarized in Section IV and the numerical results follow in Section V. Conclusions along with future works are presented in Section VI.

II. MATHEMATICAL MODEL

The equation of motion describing the forced vibration of a thin rectangular plate of dimension $a \times b \times h$, as shown in Figure 1, is given by

$$\rho v_{tt}(\xi,\zeta,t) + D_E \nabla^4 v(\xi,\zeta,t) + c_d \nabla^4 v_t(\xi,\zeta,t) = \frac{\partial^2 M_{p\xi}}{\partial \xi^2} + \frac{\partial^2 M_{p\zeta}}{\partial \zeta^2} + d(\xi,\zeta)w(t), \tag{1}$$

where the *plate flexural rigidity* D_E is defined in terms of the Poisson's ratio ν and the plate thickness h as $D_E = Eh^3/(12(1-\nu^2))$. The plate transverse deflection at the spatial point $(\xi,\zeta) \in \Omega = [0,a] \times [0,b]$ and at time t is denoted by $v(\xi,\zeta,t)$. For the rectangular coordinates under consideration, the *biharmonic* operator ∇^4 is given by

$$\nabla^4 \phi(\xi,\zeta) = \frac{\partial^4 \phi(\xi,\zeta)}{\partial \xi^4} + 2\frac{\partial^4 \phi(\xi,\zeta)}{\partial \xi^2 \partial \zeta^2} + \frac{\partial^4 \phi(\xi,\zeta)}{\partial \zeta^4},$$

and the clamped boundary conditions are $v(0,\zeta,t) = v(a,\zeta,t) = 0 = v(\xi,0,t) = v(\xi,b,t)$, $v_\xi(0,\zeta,t) = v_\xi(a,\zeta,t) = 0 = v_\zeta(\xi,0,t) = v_\zeta(\xi,b,t)$. Here ρ represents the mass per unit area and the moments per unit length $M_{p\xi}, M_{p\zeta}$ exerted by the piezoceramic patch in the ξ and ζ directions with dimensions $L_\xi \times L_\zeta$, represent the form of actuation used in the vibration suppression of the plate. Following [1], [16], we thus consider viscoelastic damping with a damping parameter denoted by c_d. In addition to the control signal in the right hand side of (1), generated by the voltage $V_p(t)$ supplied to the piezoceramic patch, a

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disturbance signal $w(t)$ is also included whose spatial distribution is denoted by $d(\xi,\zeta)$ and which describes the manner at which disturbances (exogenous forcing, fluid/structure coupling) are affecting the plate dynamics at each spatial point (ξ,ζ) of the domain Ω.

Following a rather standard approach for approximation [14], the solution is assumed to have the form $v(\xi,\zeta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn}(\xi,\zeta)X_{mn}(t)$, where $X_{mn}(t)$ denotes the generalized coordinate and the spatial function $V_{mn}(\xi,\zeta)$ may represent either the eigenfunction or the approximating function as used in the finite element approximation [14], [17]. In our numerical study we considered cubic b-splines [15], modified for the clamped-clamped boundary conditions.

While the goal is to find the best actuator location via optimization of a certain measure, one must restrict the search in a subset of the spatial domain Ω in order to (i) avoid search over an infinite set of candidate locations with the obvious computational advantages, (ii) avoid regions where the system looses controllability and (iii) incorporate design restrictions and hardware limitations that dictate performance requirements. Thus, one must first find the locations of a subset of Ω that provide controllability; this translates to actuator locations that ensure that the control term

$$\frac{\partial^2 M_{p\xi}}{\partial \xi^2} + \frac{\partial^2 M_{p\zeta}}{\partial \zeta^2} \tag{2}$$

is not near zero. By considering the weak form of the control term (2) with a piezo patch placed at (ξ_k,ζ_k), we have

$$\begin{aligned}
&\int_0^a\int_0^b \Big(\frac{\partial^2 M_{p\xi}}{\partial \xi^2} + \frac{\partial^2 M_{p\zeta}}{\partial \zeta^2}\Big)U_{ij}(\xi,\zeta)\,d\xi d\zeta = \\
&\Big(\Phi_i'(\xi_k + L_\xi/2) - \Phi_i'(\xi_k - L_\xi/2)\Big) \\
&\qquad\times \int_{\zeta_k - L_\zeta/2}^{\zeta_k + L_\zeta/2} \Psi_j(\zeta)\,d\zeta d_{31} V_p + \\
&\int_{\xi_k - L_\xi/2}^{\xi_k + L_\xi/2} \Phi_i(\xi)\,d\xi \\
&\qquad\times \Big(\Psi_j'(\zeta_k + L_\zeta/2) - \Psi_j'(\zeta_k - L_\zeta/2)\Big) d_{32} V_p,
\end{aligned}$$

where the piezoelectric charge constants d_{31} and d_{32} denote the mechanical strain experienced by a piezoelectric element per unit of electrical energy applied in the ξ and ζ directions [16], and where it was implicitly assumed that $U_{ij}(\xi,\zeta) = \Phi_i(\xi)\Psi_j(\zeta)$. It should be noted that the above expression may become zero (or assume a "very small" value) for different combinations of the mode number (i,j), the piezo location (ξ_k,ζ_k), and the piezo length L_ξ and width L_ζ. It suffices to impose that either of the double integrals is not zero (or near zero) for a given (ξ_k,ζ_k) and up to certain finite values of the integers i and j. For example, if the above is imposed for a given location (ξ_k,ζ_k) and $i = 1,\dots,M$ and $j = 1,2,\dots,N$, it means that up to the (M,N) mode, the system will be controllable by the piezoceramic patch placed at location (ξ_k,ζ_k). For brevity, we denote

$$\begin{aligned}
C_1(\xi_k,\zeta_k,i,j) &\triangleq d_{31}\int_{\zeta_k - L_\zeta/2}^{\zeta_k + L_\zeta/2} \Psi_j(\zeta)\,d\zeta \\
&\quad\times\Big(\Phi_i'(\xi_k + L_\xi/2) - \Phi_i'(\xi_k - L_\xi/2)\Big), \\
C_2(\xi_k,\zeta_k,i,j) &\triangleq d_{32}\int_{\xi_k - L_\xi/2}^{\xi_k + L_\xi/2} \Phi_i(\xi)\,d\xi \\
&\quad\times\Big(\Psi_j'(\zeta_k + L_\zeta/2) - \Psi_j'(\zeta_k - L_\zeta/2)\Big).
\end{aligned} \tag{3}$$

Approximate controllability of a location (ξ_k,ζ_k) translates to imposing

$$\Big|C_1(\xi_k,\zeta_k,i,j) + C_2(\xi_k,\zeta_k,i,j)\Big| > 0 \tag{4}$$

for all $i = 1,2,\dots,N$ and all $j = 1,2,\dots,M$. Since we are interested in the controllability of certain modes, we therefore define the *set of admissible locations* that provide approximate controllability via

$$\Theta = \Big\{(\xi,\zeta)\in\Omega \,:\, (4) \text{ valid } \forall i = 1,\dots,M, j = 1,\dots,N\Big\}.$$

Notice that the above set Θ gives an infinite number of

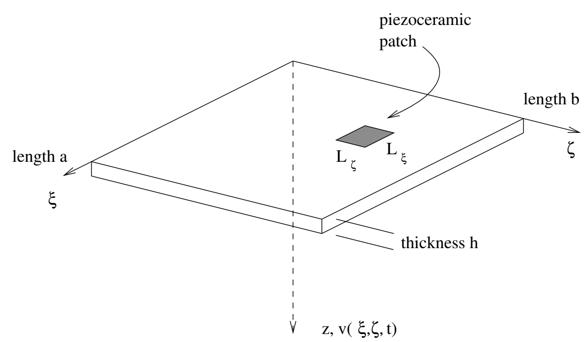

Fig. 1. Thin flexible plate with a piezoceramic laminate.

candidate actuator locations. In practice, one has but a few locations for consideration. This then prompts us to define the set of *practical admissible locations* via

$$\Theta_p = \Big\{(\xi_\ell,\zeta_\ell)\in\Theta \,:\, \ell = 1,2,\dots,p\Big\} \tag{5}$$

which consists of only p candidate locations dictated by design considerations. The fact that the actuator search will be restricted to the set Θ_p would address any concerns on computational burdens due to the otherwise heavy optimization search over an infinite set.

III. Approximation and actuator placement

When the above model is approximated (truncation of the infinite sum, or projection to a slow manifold), it results in the vector second order system [14]

$$M\ddot{z}(t) + D\dot{z}(t) + Kz(t) = B(\xi,\zeta)u(t) + E(\xi,\zeta)w(t) \tag{6}$$

where $E(\xi, \zeta)$ is the finite dimensional representation of the spatial distribution of the disturbance $d(\xi, \zeta)$. The resulting first order formulation is given by

$$\begin{aligned} \dot{x} &= Ax + B_1(\xi, \zeta)w(t) + B_2(\xi, \zeta)u(t) \\ z &= C_1 x + D_{12} u, \end{aligned} \tag{7}$$

where

$$x = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}, \ A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ M^{-1}E(\xi, \zeta) \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ M^{-1}B(\xi, \zeta) \end{bmatrix},$$

and C_1, D_{12} with $C_1^T D_{12} = 0$, are appropriately chosen matrices that reflect weights of the state and input signals on the performance output z.

We consider various optimality and robustness measures in order to place the actuator(s) in the best possible locations.

A. Enhanced controllability

Similar to the flexible beam case [7], [9], one searches in the set of admissible actuator locations Θ_p for the location that would rendered the system "more" controllable; thus one considers the *location-parameterized* controllability Gramian $W_c(\xi, \zeta)$ given by the solution of

$$AW_c(\xi, \zeta) + W_c(\xi, \zeta)A' = -B_2(\xi, \zeta)B_2'(\xi, \zeta). \tag{8}$$

The placement criterion then becomes that of maximizing a norm, or measure, of $W_c(\xi, \zeta)$, as for example the smallest eigenvalue $\lambda_{min}(\cdot)$, the trace$(\cdot)$, or the smallest singular value $\sigma_{min}(\cdot)$ of $W_c(\xi, \zeta)$. For ease of computation we use

$$(\xi^{opt}, \zeta^{opt}) = \arg \max_{(\xi,\zeta)\in\Theta_p} \sigma_{min}\Big[W_c(\xi, \zeta)\Big]. \tag{9}$$

Equivalently, by considering the system $\dot{x} = Ax + B_2(\xi, \zeta)u$, $z = Ix$, one uses the transfer function

$$T_{zu}(s; \xi, \zeta) = I\Big(sI - A\Big)^{-1} B_2(\xi, \zeta)$$

as a measure for actuator placement. Its $\mathcal{H}^2$ norm as defined in [21], is given by $\|T_{zu}(s; \xi, \zeta)\|_2^2 = \text{tr}\Big[W_c(\xi, \zeta)\Big]$. Maximization of the $\mathcal{H}^2$ norm with respect to all admissible locations $(\xi, \zeta) \in \Theta_p$ would then yield the optimal actuator location. The equivalent of the optimization (9) above is

$$\begin{aligned} (\xi^{opt}, \zeta^{opt}) &= \arg \max_{(\xi,\zeta)\in\Theta_p} \|T_{xu}(s; \xi, \zeta)\|_2^2 \\ &= \arg \max_{(\xi,\zeta)\in\Theta_p} \text{tr}\Big[W_c(\xi, \zeta)\Big]. \end{aligned} \tag{10}$$

In a similar fashion, one may consider the $\mathcal{H}^\infty$ norm of $T_{xu}(s; \xi, \zeta)$ and attempt to choose the optimal actuator location (ξ^{opt}, ζ^{opt}) so that its effect on the output z is maximized (maximize impulse response). This then translates to maximizing the smallest singular value of T_{xu} and thus the actuator placement criterion now becomes

$$(\xi^{opt}, \zeta^{opt}) = \arg \max_{(\xi,\zeta)\in\Theta_p} \sigma_{min}\Big[T_{xu}(s; \xi, \zeta)\Big] \tag{11}$$

In (10), (11) above, either the $\mathcal{H}^2$ or the $\mathcal{H}^\infty$ norm of the system $(A, B_2(\xi, \zeta), I)$ was maximized with respect to the candidate locations in the set Θ_p. However, *neither* approach considers the effects of the spatial distribution of the disturbances B_1. When a placement measure incorporates the worst distribution $d(\xi, \zeta)$ via $B_1(\xi, \zeta)$, it is then expected that the actuator location would provide *spatial robustness*.

B. Improved performance/robustness

Using a state feedback gain $K(\xi, \zeta)$, parameterized by an actuator location $(\xi, \zeta) \in \Theta_p$, the closed loop system, with $u = K(\xi, \zeta)x$, becomes

$$\begin{aligned} \dot{x} &= \Big(A + B_2(\xi, \zeta)K(\xi, \zeta)\Big)x + B_1(\xi, \zeta)w(t) \\ z &= \Big(C + D_{12}K(\xi, \zeta)\Big)x. \end{aligned} \tag{12}$$

The resulting closed loop transfer function T_{zw} is given by

$$T_{zw}(s; \xi, \zeta) = C_{cl}\Big(sI - A_{cl})\Big)^{-1} B_1(\xi, \zeta), \tag{13}$$

where $C_{cl} = C_{cl}(\xi, \zeta) \triangleq C_1 + D_{12}K(\xi, \zeta)$ and $A_{cl} = A_{cl}(\xi, \zeta) \triangleq A + B_2(\xi, \zeta)K(\xi, \zeta)$. The goal now is to minimize the effects of disturbances w on the measured output z and thus one may consider an appropriate norm of T_{zw} as a measure for location optimization. To improve robustness with respect to disturbances, one may consider the *worst distribution* of disturbances $B_1(\xi, \zeta)$ in order to reach on an actuator location that provides a *truly spatial robustness*. Furthermore, to require that every single state can be affected by the disturbance, one may take as the performance output the entire state and thus $z = Ix$. This would ensure that the *worst possible spatial distribution of disturbances* would affect all states and thus the optimization would produce an actuator location which provides robustness to each and every state against the worst possible distribution of disturbances. In fact, this was also pointed out in another study in [5] for parabolic distributed parameter systems. The $\mathcal{H}^2$ norm of the system $(A_{cl}(\xi, \zeta), B_1(\xi, \zeta), C_{cl}(\xi, \zeta))$ is thus given by

$$\|T_{zw}\|_2^2 = \text{tr}\ \Big[B_1'(\xi, \zeta)X_{ob}B_1(\xi, \zeta)\Big], \tag{14}$$

where $X_{ob} = X_{ob}(\xi, \zeta)$ is the observability Gramian of the pair $(C_{cl}(\xi, \zeta), A_{cl}(\xi, \zeta))$ and is given by the solution to the Lyapunov equation

$$A_{cl}'X_{ob}(\xi, \zeta) + X_{ob}(\xi, \zeta)A_{cl} + C_{cl}'C_{cl} = 0. \tag{15}$$

In a similar fashion as in the open-loop (enhanced controllability) method, the optimal actuator location is found via

$$(\xi^{opt}, \zeta^{opt}) = \arg \min_{(\xi,\zeta)\in\Theta_p} \text{tr}\ \Big[B_1'(\xi, \zeta)X_{ob}B_1(\xi, \zeta)\Big]. \tag{16}$$

If instead, one considers the $\mathcal{H}^\infty$ norm of (12), the optimal actuator location is found via

$$(\xi^{opt}, \zeta^{opt}) = \arg \min_{(\xi,\zeta)\in\Theta_p} \|T_{zw}(s;\xi,\zeta)\|_\infty. \tag{17}$$

The above (17) can be converted to an LMI optimization via the solution of an associated matrix Riccati equation by using the bounded real lemma, [3], [18].

Remark 3.1: In (16) or (17), the best location is optimal/robust for a wide range of frequencies. If one has a priori knowledge on the frequency range of interest, then the above transfer functions may be weighted by an appropriate transfer function that reflects the relevant frequency band. Alternatively, one may utilize projection methods on the finite dimensional model and project on the manifold that contains the frequency band under consideration.

Remark 3.2: It should be noted that the feedback gain $K(\xi,\zeta)$ was not necessarily designed using optimality criteria. In fact, for a given actuator location (ξ,ζ) with a corresponding input distribution vector $B_2(\xi,\zeta)$ a pole-placing method may be employed to find such a feedback gain. Optimization of the resulting closed loop transfer function T_{zw} with respect to the admissible actuator locations would certainly produce a location that provides robustness with respect to spatial distribution, but would not necessarily produce an optimal gain. What this means is that the feedback gain is only indirectly affected by the disturbance distribution B_1 in the sense that it is only the actuator location that is being optimized with respect to disturbances, and in turn, a feedback gain is found for that location. To include this "double" optimality, one may integrate the controller design and the robust-with-respect-to spatial disturbance distributions placement as summarized below.

1) $\mathcal{H}^2$-integrated actuator/controller optimization: In this case, both the actuator location $B_2(\xi,\zeta)$ and its associated feedback gain $K(\xi,\zeta)$ will be optimized. In a similar fashion as in (16), one seeks an internally stabilizing controller that minimizes the performance criterion $\|z\|_2^2$. In this case, one solves, for each of the p candidate locations in the set Θ_p, for the following *location-parameterized* Riccati equation solution $X_2(\xi,\zeta)$ (*cf.* (15))

$$\begin{aligned}&A^{'}X_2(\xi,\zeta) + X_2(\xi,\zeta)A + C_1^{'}C_1\\ &\qquad -X_2(\xi,\zeta)B_2(\xi,\zeta)B_2^{'}(\xi,\zeta)X_2(\xi,\zeta) = 0.\end{aligned} \tag{18}$$

The location optimization is simply given via

$$\begin{aligned}&(\xi^{opt},\zeta^{opt})_{\mathcal{H}^2} =\\ &\arg \max_{(\xi,\zeta)\in\Theta_p} \operatorname{tr}\left[B_1^{'}(\xi,\zeta)X_2(\xi,\zeta)B_1(\xi,\zeta)\right].\end{aligned} \tag{19}$$

Once the optimal actuator location is found, the corresponding optimal feedback gain is therefore given by

$$\begin{aligned}K^{opt}_{\mathcal{H}^2} &= K(\xi,\zeta)\Big|_{(\xi,\zeta)=(\xi^{opt},\zeta^{opt})_{\mathcal{H}^2}}\\ &= -B_2^{'}(\xi^{opt},\zeta^{opt})X_2(\xi^{opt},\zeta^{opt}).\end{aligned} \tag{20}$$

Unlike (15) or (16) in which the feedback gain was parameterized by the actuator location and which was not necessarily found using optimality/performance criteria, the approach presented in (19), (20) integrates the actuator placement and feedback design by finding *both* the internally stabilizing optimal gain *and* actuator location that provide the smallest bound of the $\mathcal{H}^2$ norm of T_{zw}.

2) $\mathcal{H}^\infty$-integrated actuator/controller optimization: Finally, we consider the case in which the optimal actuator location provides robustness with respect to *both* the temporal and spatial components of disturbances. To do so, one parameterizes the associated $\mathcal{H}^\infty$ Riccati equation by the candidate actuator locations and seeks an admissible state feedback that minimizes the $\mathcal{H}^\infty$ norm of T_{zw}. Similarly, one solves for the parameterized solution $X_\infty(\xi,\zeta)$ of

$$\begin{aligned}&A^{'}X_\infty(\xi,\zeta) + X_\infty(\xi,\zeta)A + C_1^{'}C_1\\ &\quad -X_\infty(\xi,\zeta)B_2(\xi,\zeta)B_2^{'}(\xi,\zeta)X_\infty(\xi,\zeta)\\ &\quad +\frac{1}{\gamma^2}X_\infty(\xi,\zeta)B_1(\xi,\zeta)B_1^{'}(\xi,\zeta)X_\infty(\xi,\zeta) = 0.\end{aligned} \tag{21}$$

Notice that in the Riccati equation (21) above, the spatial distribution of disturbance enters explicitly, which also directly affects the feedback gain which is given by, *cf.* (20)

$$\begin{aligned}K^{opt}_{\mathcal{H}^\infty} &= K_\infty(\xi,\zeta)\Big|_{(\xi,\zeta)=(\xi^{opt},\zeta^{opt})_{\mathcal{H}^\infty}}\\ &= -B_2^{'}(\xi^{opt},\zeta^{opt})X_\infty(\xi^{opt},\zeta^{opt}).\end{aligned} \tag{22}$$

The corresponding optimal actuator location is given by

$$(\xi^{opt},\zeta^{opt})_{\mathcal{H}^\infty} = \arg \max_{(\xi,\zeta)\in\Theta_p} \gamma(\xi,\zeta) \tag{23}$$

where $\gamma(\xi,\zeta)$ is the smallest γ that yields a positive definite solution to (21) for each pair $(\xi,\zeta)\in\Theta_p$.

IV. Implementation issues for numerical study

For our numerical study, we consider the enhanced controllability method given by (10) to extract the set of locations (ξ,ζ) that make the system "more" controllable. Specifically, in this two-stage procedure, we consider a larger set of candidate locations and choose those p locations that give enhanced controllability, i.e. the first p maxima of $\operatorname{tr}\left[W_c(\xi,\zeta)\right]$. This set, which comprises Θ_p, will then in turn be used in both the $\mathcal{H}^2$ approach (18) - (20) and the $\mathcal{H}^\infty$ approach (21) - (23). While additional optimization is required to find what is the worst spatial distribution of disturbances and then find the associated actuator location and its corresponding feedback gain, we consider few representative "worst" spatial disturbance distributions in order to gain an insight of the effects of spatial components of disturbances on actuator/controller optimization. For that, four different distributions $d(\xi,\zeta)$ are considered here that are normalized such that their volume is unity. These four distributions, depicted in Figure 2, are $d_1(\xi,\zeta) = \frac{1}{ab}$[1], $d_2(\xi,\zeta) = G(0.5a, 0.5b, a/8, b/8)$, $d_3(\xi,\zeta) = G(0.85a, 0.15b, a/56, b/56)$, and $d_4(\xi,\zeta) = \frac{\phi_2(\xi)\psi_2(\zeta)}{\int_0^a \phi_2(\xi)\,d\xi \int_0^b \psi_2(\zeta)\,d\zeta}$, where G denotes the 2-D gaussian distribution with means μ_ξ and μ_ζ and standard deviations σ_ξ and σ_ζ and given by

$$G(\mu_\xi,\mu_\zeta,\sigma_\xi,\sigma_\zeta) = \frac{\exp\left[-\frac{1}{2}\left(\left(\frac{\xi-\mu_\xi}{\sigma_\xi}\right)^2 + \left(\frac{\zeta-\mu_\zeta}{\sigma_\zeta}\right)^2\right)\right]}{2\pi\sigma_\xi\sigma_\zeta},$$

and where $\phi_2(\xi), \psi_2(\zeta)$ are the second eigenfunctions [2], resulting in the $(2,2)$ mode.

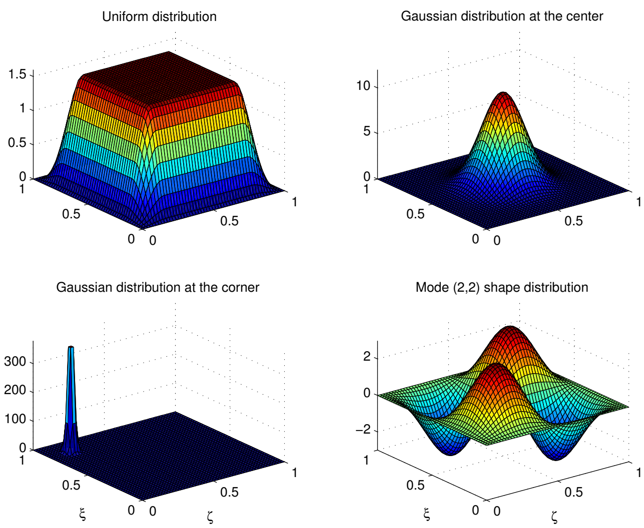

Fig. 2. Distribution of the four normalized disturbances, $d(\xi,\zeta)$.

V. NUMERICAL RESULTS

Using cubic b-spline elements modified to account for clamped-clamped boundary conditions with $n_\xi = 15$ and $n_\zeta = 14$ elements, we constructed the associated system matrices and calculated the (ξ,ζ)-parameterized $\mathcal{H}^2$ cost given in (18), (19). Using a uniform grid with 23×23 points, we computed the associated $\|T_{zw}(s;\xi,\zeta)\|_2$ corresponding to each of the four disturbance distributions. Figure 3 depicts[2] the distribution of $\|T_{zw}(s;\cdot,\cdot)\|_2$ in which one may observe that optimal location for one type of spatial disturbance distribution might be the worst possible location for another; for example when the $(2,2)$ mode distribution $d_2(\xi,\zeta)$ is assumed, the best locations are at $(0.291a, 0.291b)$, $(0.291a, 0.709b)$, $(0.709a, 0.291b)$ and $(0.709a, 0.709b)$ and one of the non-optimal locations is at the center $(0.5a, 0.5b)$. This non-optimal location in fact serves as the optimal location for both the uniform and Gaussian (at center) distributions. To demonstrate the effects of the disturbance distribution on the optimal actuator location, we have simulated the closed loop system with $d = d_4$, an optimal actuator at $(0.291a, 0.291b)$ and a non-optimal actuator at $(0.5a, 0.5b)$. The closed loop gains were computed via (18). The evolution of the L_2 norm of the state, given by $\sqrt{\dot{z}'(t)M\dot{z}(t) + z'(t)Kz(t)}$ and which represents the sum of kinetic and potential energy, is plotted in Figure 4.

[1]In fact, the uniform distribution was smoothed out to attain a value of zero at the boundaries.

[2]For ease of visualization, we've plotted the difference of the normalized absolute value of $\|T_{zw}(s;\xi,\zeta)\|_2$ from 1, so that the 3-D distribution would show as its maxima the optimal actuator locations.

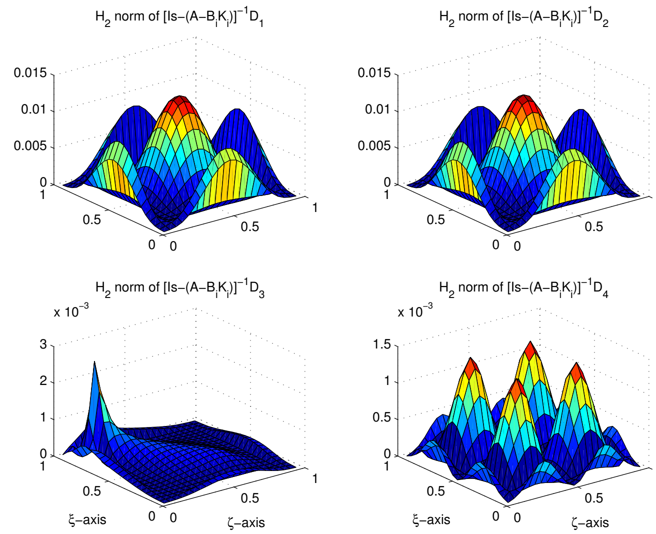

Fig. 3. Distribution of the $\mathcal{H}^2$ costs (19) for each of the four normalized disturbances, $d(\xi,\zeta)$.

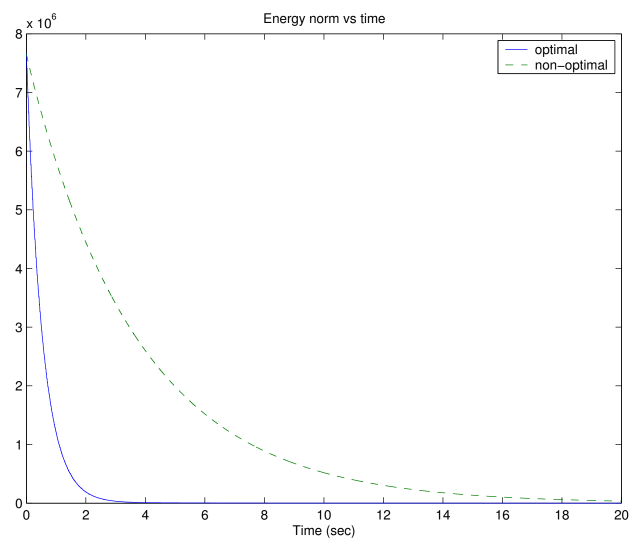

Fig. 4. Evolution of L_2 system norm.

VI. CONCLUSIONS AND FUTURE WORKS

An actuator location scheme for the control of flexible structures in which the placement had the additional element of incorporating the distribution of disturbances has been presented. When the spatial distribution of disturbances is known, then the proposed scheme allows one to place actuating devices that can better address external disturbances. When the spatial distribution is not known, then the placement scheme can account for the worst spatial distribution of disturbances thus incorporating *spatial robustness* in the actuator placement. The proposed scheme has practical applications in vibration control of flexible structures that interact with the environment and which allow for reduced control effort, since the actuating devices can better address external disturbances.

The immediate extension would be the incorporation of both the collocated actuator/sensor and the non-collocated actuator/sensor placement that incorporates information on the spatial distribution of disturbances along with the integrated placement and compensator design.

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