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Performance and Aliasing Analysis of Multi-rate Digital Controllers with Interlacing

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***Abstract*— This paper is concerned with multi-rate digital control with interlacing. The multi-rate controller is designed from a single fast-rate controller by implementing slow-mode components of the controller at a slow rate. The slow mode part of the controller is further decomposed to several elements to interlace: i.e. different elements are updated at different fast sampling time instances. The objective of multi-rate control with interlacing is to reduce the amount of real-time computation. This paper addresses two issues due to slow rate implementation of the slow-mode components: performance degradation and signal aliasing. The multi-rate control system is a linear periodically time-varying (LPTV) system. Using a lifting approach, the frequency response of the multi-rate system is represented by a frequency domain matrix. This matrix provides the information to quantify the aliasing and the distortion effect in the multi-rate system. Finally, as an example of multi-rate control with interlacing, track-following control for hard disk drive is considered. In particular, two different multi-rate controller structures will be studied: series structure and parallel structure. Performance and aliasing analysis is presented for both the series and parallel structures.**

I. Introduction

Multi-rate digital control has been a popular subject of research in the last two decades [1]–[4]. The digital control system may become multi-rate because it naturally involves more than one sampling period. The control system becomes multi-rate if the control input is updated faster than the measurement sampling rate. Motivation for such multi-rate control is to make the control input smooth when the measurement sampling rate cannot be arbitrarily increased because of the sensor bandwidth or any other hardware constraint. In [5], a multi-rate design method has been proposed from the viewpoint of saving real time computation. In the implementation of control algorithms on consumer products such as hard disk drives, the amount of computation for real time control should be kept minimal in order not to overload the digital signal processor or microprocessor, which may be a low end processor and may be performing various tasks in addition to real time control. This multi-rate controller is designed from a single fast-rate controller. By slowing down the updating rate of the slow-mode component of the controller, computational saving can be achieved. Furthermore, slow modes are updated by interlacing them so that the amount of computation remains uniform from one fast sampling time instance to another. There are two different structures to implement such multi-rate control: one structure, series structure, is obtained by placing the slow and fast parts in series, the other, parallel structure, is obtained by placing them in parallel.

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The multi-rate control system is a linear periodically time-varying (LPTV) system. One method to describe LPTV systems is to use a frequency domain lifting technique such that the frequency response of the LPTV system can be represented by a frequency domain matrix [6]. Based on the information in the frequency response matrix, a measure can be obtained to quantify the aliasing effect. If aliasing is negligible, we can approximate it by a linear time invariant (LTI) system [7].

In this paper, the aliasing in the multi-rate controller with interlacing is studied. Aliasing is presented due to the down sampling operation in the slow-rate implementation. It is shown that the aliasing level is small base on the assumption that the measurement sampling rate is fast enough so that the down sampling will not introduce serious aliasing. The performance analysis of the multi-rate controller is also considered.

The remainder of this paper is organized as follows. In Section II, some key results of the frequency response of LPTV systems are reviewed. The multi-rate control schemes are analyzed in Section III in frequency domain. Finally, in Section IV, the analytical results are applied to a multi-rate track following problem for hard disk drives (HDD), and some simulation results are presented. Conclusions are given in Section V.

II. Frequency Domain Representation of LPTV Systems

The concept of frequency response for LPTV systems is defined in [6], [8]. Let an LPTV system with input $x(k)$, output $y(k)$ and period P be given by

$$y(k) = \sum_{l=-\infty}^{\infty} f(k,l)x(l) \tag{1}$$

where $f(k,l)$ is the response of the system at time k to a unite impulse at time l and $f(k,l) = f(k+P, l+P)$. By defining $h(l,k) = f(k+l,l)$, the P-periodic condition gives $h(l,k) = h(l+P,k)$. Equation (1) can be rewritten

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as

$$y(k) = \sum_{i=-\infty}^{k} h(i, k-i)x(i) = \sum_{l=0}^{P-1} \sum_{r=-\infty}^{\infty} h(l, k-(Pr+l))x(Pr+l) \quad (2)$$

Define the LTI system $H^l(z) = \sum_{k=0}^{\infty} h(l,k)z^{-k}$, and due to periodicity P, $H^l = H^{l+P}$. Then the LPTV system can be represented by P LTI systems H^l with a switch in the output as shown in Fig. 1 [9]. The switch is connected to H^l at sampling time $Pk + l$.

By lifting the input and output sequences into vectors of length P

$$\underline{y}(k) = \begin{bmatrix} y(Pk) & y(Pk+1) & \cdots & y(Pk+P-1) \end{bmatrix}^T$$
$$\underline{x}(k) = \begin{bmatrix} x(Pk) & x(Pk+1) & \cdots & x(Pk+P-1) \end{bmatrix}^T$$

The system can be represented as a multi-input, multi-output LTI system H described by

$$\underline{Y}(z^P) = H(z^P)\underline{X}(z^P) \quad (3)$$

where

$$H(z^P) = \begin{bmatrix} H_0^0(z^P) & z^{-P}H_{P-1}^1(z^P) & \cdots & z^{-P}H_1^{P-1}(z^P) \\ H_1^0(z^P) & H_0^1(z^P) & \cdots & z^{-P}H_2^{P-1}(z^P) \\ \vdots & \vdots & \ddots & \vdots \\ H_{P-1}^0(z^P) & H_{P-2}^1(z^P) & \cdots & H_0^{P-1}(z^P) \end{bmatrix} \quad (4)$$

where $\underline{Y}(z^P) = \begin{bmatrix} Y_0(z^P) \; Y_1(z^P) \; \cdots \; Y_{P-1}(z^P) \end{bmatrix}^T$ is the $\mathcal{Z}$-transform of the vector sequence $\underline{y}(k)$, and $\underline{X}(z^P)$ is similarly defined for $\underline{x}(k)$. $H_k^l(z^P)$'s are the polyphase components of $H^l(z)$, i.e.

$$H^l(z) = \sum_{k=0}^{P-1} z^{-k} H_k^l(z^P) \quad (5)$$

Equations (3) and (4) are based on lifting in the time domain.

As shown in [8], the expression for the input-output relation in frequency domain can be written as

$$Y(z) = G_0(z)X(z) + \sum_{n=1}^{P-1} G_n(z)X(z\phi^n) \quad (6)$$

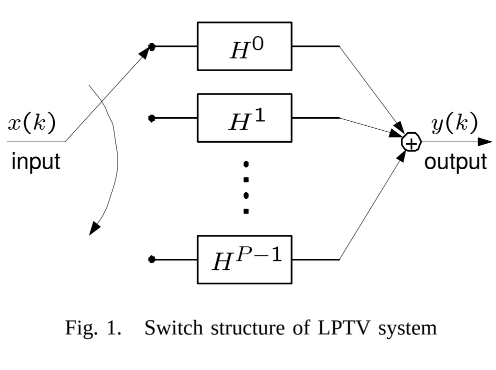

Fig. 1. Switch structure of LPTV system

where $\phi = e^{j2\pi/P}$. The lifted system in time domain and that in frequency domain are connected through

$$G_n(z) = \frac{1}{P} \sum_{i=0}^{P-1} \sum_{j=0}^{P-1} \phi^{jn} z^{j-i} H_{(i,j)}(z^P) \quad (7)$$

where $H_{(i,j)}(z^P)$ denotes the entry in row i, column j of the matrix $H(z^P)$. Equation (6) implies that the output spectrum is the sum of the shaped versions of frequency shifted input spectra, $X(z\phi^n)$. If $G_1(z) = \cdots = G_{P-1}(z) = 0$, then the aliasing is absent. Therefore, $G_n(z)|_{n=1,\cdots,P-1}$ give a measure of the aliasing effect in an LPTV system [7]. Moreover, the frequency spectrum of the input signal also affects the level of aliasing.

Equation (6) can be written in a matrix form.

$$\hat{\underline{Y}}(z) = \begin{bmatrix} G_0(z) & G_1(z) & \cdots & G_{P-1}(z) \\ G_{P-1}(z\phi) & G_0(z\phi) & \cdots & G_{P-2}(z\phi) \\ \vdots & \vdots & \ddots & \vdots \\ G_1(z\phi^{P-1}) & G_2(z\phi^{P-1}) & \cdots & G_0(z\phi^{P-1}) \end{bmatrix} \hat{\underline{X}}(z) = G^{FR}(z) \; \hat{\underline{X}}(z) \quad (8)$$

where $\hat{\underline{Y}}(z) = \begin{bmatrix} Y(z) \; Y(z\phi) \; \cdots \; Y(z\phi^{p-1}) \end{bmatrix}^T$, and $\hat{\underline{X}}(z)$ is similarly defined. In Eq. (8), $G^{FR}(z)$ is called the frequency response matrix of the LPTV system [6], [7]. It is obvious that the frequency response matrix for a LTI system, $Y(z) = G_0(z)U(z)$, is in diagonal form

$$\hat{\underline{Y}}(z) = \begin{bmatrix} G_0(z) & 0 & \cdots & 0 \\ 0 & G_0(z\phi) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_0(z\phi^{P-1}) \end{bmatrix} \hat{\underline{X}}(z) \quad (9)$$

The frequency response matrix transforms the LPTV system to a multi-input, multi-output LTI system. This description is used as an analysis tool in the following sections.

III. MULTI-RATE DIGITAL CONTROL WITH INTERLACING

The main purpose of the multi-rate controller discussed is to save real-time computation. The multi-rate controller is designed from a single-rate digital controller. The slow-mode and fast-mode components of the single-rate controller are separated according to the locations of poles. The slow-mode controller is implemented at slow rate. Furthermore, the slow-mode controller is decomposed into several elements and updated at different fast sampling time instances. Therefore, the amount of computation is uniformly reduced among all time instances. There are two kinds of multi-rate controller structures. In the parallel structure, the slow part and fast part are connected in parallel while in the serial structure the slow part is followed by the fast part. In this section, the multi-rate digital control systems are represented using frequency response matrices. The aliasing effect presents in the multi-rate controller due to down sampling is evaluated. The system performance for disturbance rejection is also considered. To simplify the discussion, we consider a multi-rate controller with a multi-rate ratio of 2. Furthermore, the slow part of the controller has been decomposed into two blocks and are interlaced.

A. Parallel Decomposition

Fig. 2 shows the multi-rate controller implemented in parallel form [5]. The output of the slow mode controller at even sampling time $2k$ is related to its input by

$$\begin{aligned} u_s(2k) &= u_1(2k) + u_2(2k-1) \\ &= C_{Ds1}(z^2)e(2k) + C_{Ds2}(z^2)e(2k-1) \end{aligned} \quad (10)$$

On the other hand, at odd sampling time $2k+1$

$$\begin{aligned} u_s(2k+1) &= u_1(2k) + u_2(2k+1) \\ &= C_{Ds1}(z^2)e(2k) + C_{Ds2}(z^2)e(2k+1) \end{aligned} \quad (11)$$

Equations (10) and (11) show that the multi-rate controller is a periodic system with a period of 2. Since the multi-rate controller makes the system an LPTV system, the analysis method discussed in the Section II can be used to analyze this multi-rate system.

From Eqs. (10) and (11), the lifted input and output signal for the slow part are related by

$$\underline{U_s}(z^2) = \begin{bmatrix} C_{Ds1}(z^2) & z^{-2}C_{Ds2}(z^2) \\ C_{Ds1}(z^2) & C_{Ds2}(z^2) \end{bmatrix} \underline{E}(z^2) \quad (12)$$

Notice that the slow-mode controller input $\underline{E}(z^2)$ and output $\underline{U_s}(z^2)$ correspond to $\underline{X}(z^P)$ and $\underline{Y}(z^P)$ respectively in Eq. (3).

The overall controller can be expressed in switch structure as shown in Fig. 1. $H^0(z)$ and $H^1(z)$ are obtained from Eqs. (4), (5) and (12).

$$H^0(z) = \left(C_{Ds1}(z^2) + z^{-1}C_{Ds1}(z^2)\right) + C_{Df}(z) \quad (13)$$

$$H^1(z) = \left(C_{Ds2}(z^2) + z^{-1}C_{Ds2}(z^2)\right) + C_{Df}(z) \quad (14)$$

The additional term $C_{Df}(z)$ comes from the fast-mode controller. The controller input error signal is connected to $H^0(z)$ at even time instants and to $H^1(z)$ at odd time instants. This gives us a better insight of how the multi-rate controller works.

For the frequency analysis, the controller may be represented in lifted form in frequency domain. According to Eqs. (7) and (12), the frequency response matrix for the slow part is

$$C_s^{FR}(z) = \begin{bmatrix} G_{s0}(z) & G_{s1}(z) \\ G_{s1}(-z) & G_{s0}(-z) \end{bmatrix} \quad (15)$$

where

$$\begin{aligned} G_{s0}(z) &= \frac{1}{2}(H_{(0,0)}(z^2) + zH_{(0,1)}(z^2) \\ &\quad + z^{-1}H_{(1,0)}(z^2) + H_{(1,1)}(z^2)) \\ &= \frac{1}{2}(1+z^{-1})(C_{s1}(z^2) + C_{s2}(z^2)) \quad (16) \\ G_{s1}(z) &= \frac{1}{2}(H_{(0,0)}(z^2) - zH_{(0,1)}(z^2) \\ &\quad + z^{-1}H_{(1,0)}(z^2) - H_{(1,1)}(z^2)) \\ &= \frac{1}{2}(1+z^{-1})(C_{s1}(z^2) - C_{s2}(z^2)) \quad (17) \end{aligned}$$

For the fast-mode component, the LTI transfer function $C_{Df}(z)$ is written in matrix form

$$C_f^{FR}(z) = \begin{bmatrix} C_{Df}(z) & 0 \\ 0 & C_{Df}(-z) \end{bmatrix} \quad (18)$$

Thus, the overall controller is expressed as

$$C^{FR} = \begin{bmatrix} C_0(z) & C_1(z) \\ C_1(-z) & C_0(-z) \end{bmatrix} = C_s^{FR}(z) + C_f^{FR}(z) \quad (19)$$

Notice that the aliasing component in the controller is obtained from

$$C_1(z) = G_{s1}(z) \quad (20)$$

B. Serial Decomposition

Figure 3 shows the multi-rate controller implemented in series form. If the controller is represented in switch structure in Fig. 1, H^0 and H^1 are given by

$$H^0(z) = \left(C_{Ds1}(z^2) + z^{-1}C_{Ds1}(z^2)\right) C_{Df}(z) \quad (21)$$

$$H^1(z) = \left(C_{Ds2}(z^2) + z^{-1}C_{Ds2}(z^2)\right) C_{Df}(z) \quad (22)$$

For the frequency domain analysis, the fast part and slow part expressions are the same as that of the parallel case except that there is an additional constant gain K_D in $G_{s0}(z)$. The overall controller is expressed as

$$C^{FR} = \begin{bmatrix} C_0(z) & C_1(z) \\ C_1(-z) & C_0(-z) \end{bmatrix} = C_f^{FR}(z)\ C_s^{FR}(z) \quad (23)$$

Also the aliasing component is

$$C_1(z) = \frac{1}{2}(1+z^{-1})C_{Df}(z)\left(C_{Ds1}(z^2) - C_{Ds2}(z^2)\right) \quad (24)$$

Notice that the multi-rate implementation results in a low-pass filter $0.5(1+z^{-1})$ in $C_1(z)$. This low-pass filter helps to reduce the signal aliasing at high frequencies.

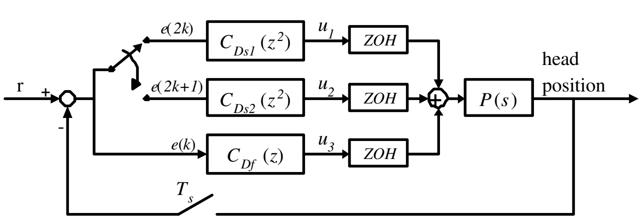

Fig. 2. Multi-rate controller with parallel decomposition

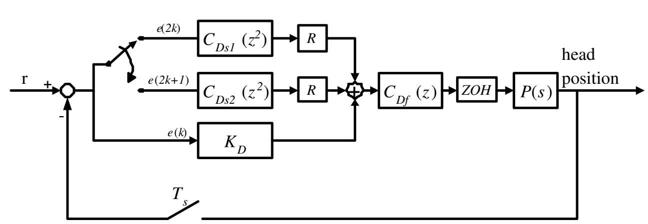

Fig. 3. Multi-rate controller with serial decomposition

C. Performance of the Multi-rate Control System

As long as the aliasing effect is small, the frequency response of the multi-rate controller can be best approximated by the LTI system $C_0(z)$ and it can be used for the computation of other quantities such as the output sensitivity function,

$$S(z) = (I + P(z)\ C_0(z))^{-1} \tag{25}$$

where $P(z)$ is the zero order hold equivalent of the controlled plant. If the aliasing effect is not negligible, we can use the frequency response matrix to model the controller and the plant. Hence, the overall system becomes a enlarged MIMO system. Its performance can be evaluated using MIMO system analysis methods.

IV. Example: HDD Track Following Control

As an example, we consider a digital controller designed for track following in a HDD application. The overall control system is as depicted in Fig. 4. The zero order hold equivalent of the controlled plant (a suspension/carriage assembly driven by a voice coil motor (VCM)) is characterized by the frequency response in Fig.5. The measurement sampling time is $20\mu s$. The track following controller is described by

$$C_D(z) = 1.088\frac{(z-0.9813)(z-0.9681)(z+1)}{(z-0.9999)(z-0.8496)(z-0.6137)} \tag{26}$$

Note that this third order controller possesses two slow modes characterized by two poles at 0.9999 and 0.8496 and one fast mode characterized by a pole at 0.6137

For parallel implementation, the multi-rate controller consists of

$$\begin{aligned} C_{Ds1}(z^2) &= \frac{0.04484}{z^2-0.9999} \\ C_{Ds2}(z^2) &= \frac{-1.637}{z^2-0.7218} \\ C_{Df}(z) &= \frac{1.088z+1.842}{z-0.6137} \end{aligned} \tag{27}$$

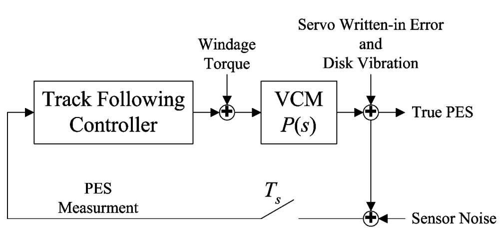

Fig. 4. HDD track-following servo system with disturbances

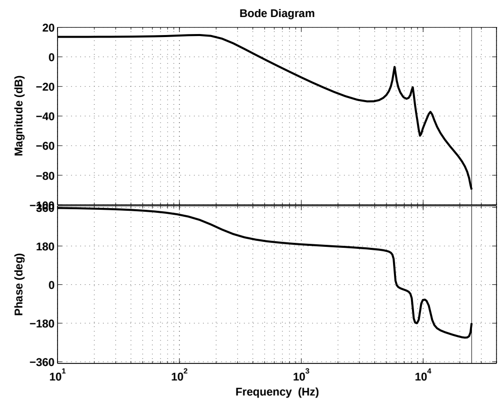

Fig. 5. Frequency response of VCM $P(z)$

while for serial implementation,

$$\begin{aligned} C_{Ds1}(z^2) &= \frac{0.008662}{z^2-0.9999} \\ C_{Ds2}(z^2) &= \frac{-0.2088}{z^2-0.7218} \\ C_{Df}(z) &= \frac{z+1}{z-0.6137} \\ K_D &= 1.0882 \end{aligned} \tag{28}$$

A. Frequency Domain Analysis

1) Aliasing effect in the controller: From the frequency response matrix, the input-output relation of the controller is given by

$$U(z) = C_0(z)E(z) + C_1(z)E(-z) \tag{29}$$

To quantify the aliasing in the controller, we use the ratio of $C_1(z)$ to $C_0(z)$ as a measure of aliasing.

$$A(z) = C_1(z)/C_0(z) \tag{30}$$

Fig. 6 shows the aliasing indicator $A(z)$ for the parallel decomposition and serial decomposition of the multi-rate controller. The plots indicate that the serial decomposition

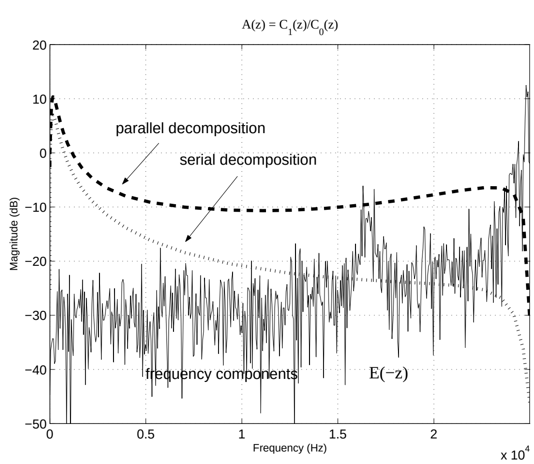

Fig. 6. Frequency response of aliasing indicator $A(z)$

is slightly better in terms of aliasing than the parallel decomposition. The effect of aliasing depends on $C_1(z)$ as well as $E(-z)$. The figure also shows that the indicator is larger than 0dB at low frequencies. This is not serious since $E(-z)$ is very small at low frequencies. It is important to check these two aspects in the multi-rate implementation and to make it sure the aliasing term $C_1(z)E(-z)$ is negligible.

2) Performance analysis: If the aliasing effect is negligible, the sensitivity function of the multi-rate system can be calculated from Eq. (25). The frequency responses of the sensitivity functions for the parallel decomposition and serial decomposition are shown in Fig. 7 and Fig. 8. From the performance comparison in the figures, it is observed that the performance of the multi-rate systems in this example is close to that of the original single fast rate system.

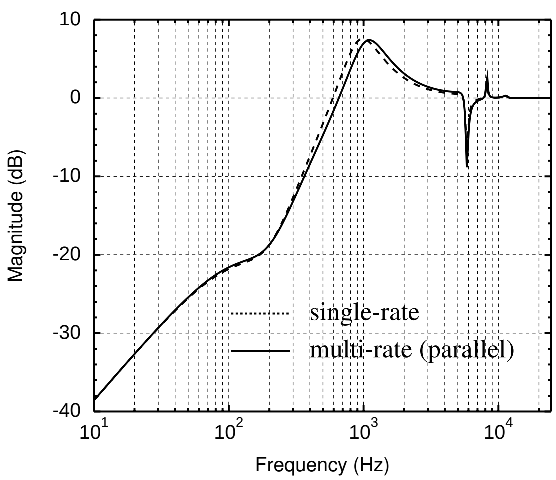

Fig. 7. Sensitivity function for parallel decomposition

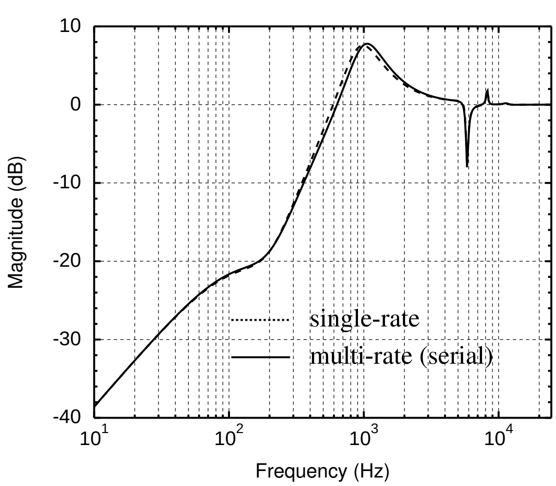

Fig. 8. Sensitivity function for serial decomposition

B. Time Domain Simulation

Using the simulated hard disk drive servo system and the controller described in Section IV, time domain responses are simulated by taking into account disturbances such as windage torque and sensor noise, which enter the system as depicted in Fig. 4. Fig. 9 shows the PES before and after the single-rate controller is applied. Time plots of PES for the systems using single-rate controller, multi-rate controller with parallel decomposition and serial decomposition are given in Fig. 10 (a), (b) and (c). These simulation results are used as a reference. In the next set of simulations, we introduced a fictitious sinusoidal disturbance above the slow rate Nyquist frequency, 12.5kHz. These simulations were performed to investigate the effect of aliasing and show that the performance is indeed deteriorated by multi-rate control. The frequency and amplitude of the fictitious disturbance $d(k)$ were 24kHz and 6nm. Note that this disturbance does not exist in real HDD systems. As observed in Figure 10 (d), (e) and (f), the disturbance effect is more visible in the case of parallel decomposition.

The performance of a controller in time domain is evaluated based on the TMR which is the 3 times standard deviation (3σ) of PES. In Table I, case 1 is the simulation without $d(k)$. The multi-rate controllers can achieve nearly the same performance of the single-rate controller. When the additional disturbance is added in case 2, the 3σ value is increased by 25.78% using parallel multi-rate controller and by 12.98% using series multi-rate controller. The performance deterioration in the multi-rate system is caused by the serious aliasing effect which comes from $d(k)$. The results are consistent to the analytical predication shown in Fig. 6.

V. CONCLUSIONS

Performance and aliasing effect have been studied for multi-rate control for saving of computations. Using the frequency domain lifting technique, frequency response matrices were used to analyze the multi-rate system. The

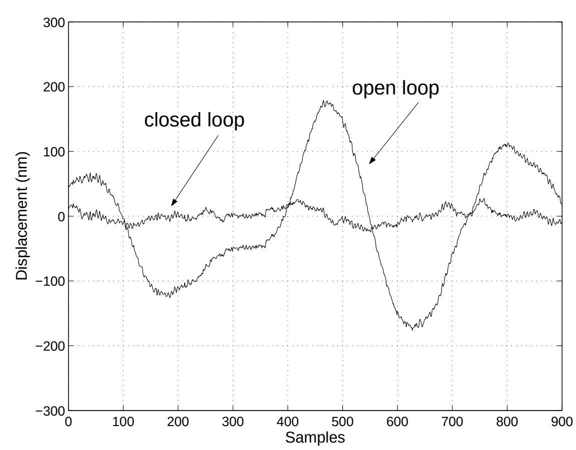

Fig. 9. PES before and after control is applied

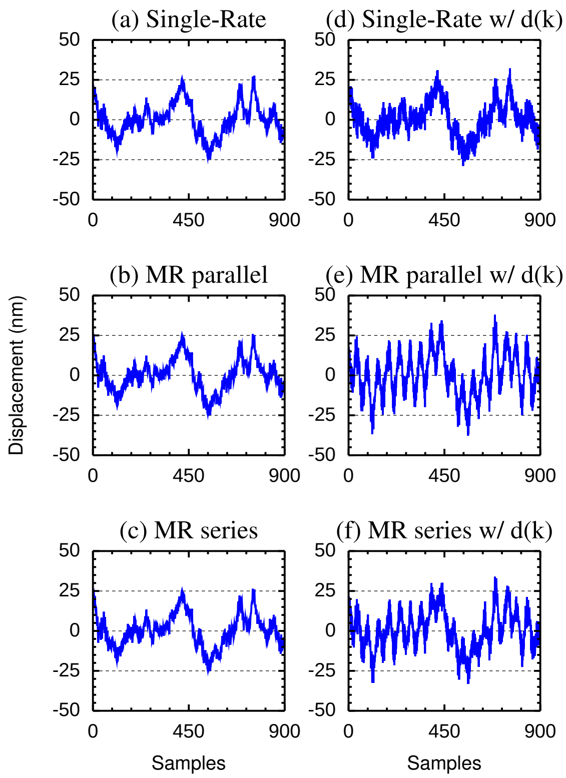

Fig. 10. Time plots of PES: (a) single-rate, (b) parallel multi-rate, (c) series multi-rate, (d) single-rate with $d(k)$, (e) parallel multi-rate with $d(k)$, (f) series multi-rate with $d(k)$

TABLE I

PERFORMANCE COMPARISONS

Scheme	Case 1: 3σ (nm)	Case 2: 3σ (nm)
Single-rate	32.34	34.78
MR parallel	32.16	43.75
Degradation	-	25.78%
MR series	32.29	39.29
Degradation	-	12.98%

aliasing effect due to down sampling in the multi-rate controller depends on off-diagonal elements of the frequency response matrices and the error signal, which is the input to the controller. Analysis methods were applied to a HDD example to quantify the aliasing effect and performance. In this example, the effect of aliasing was not serious for assumed disturbances.

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